2704: Signals and Systems

Midterm Exam II

March 20, 2006

SOLUTION

I pledge that I have neither given nor received any assistance on this exam.

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(signed)

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Name (print) Student Number

1. (20 points) Multiple Choice – Choose the answer which *best* completes the sentence

1.1 Parseval's Theorem states that

(a) $\frac{1}{T_{Fx}} \int_{T_{Fx}} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$

(b) $\int_{T_{Fx}} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$

(c) $\frac{1}{T_{Fx}} \int_{T_{Fx}} |x(t)| \, dt = \sum_{k=-\infty}^{\infty} |X[k]|$

(d) $\int_{T_{Fx}} |x(t)| \, dt = \sum_{k=-\infty}^{\infty} |X[k]|$

1.2 The impulse response of an LTI system can be found by

(a) finding the response of the system to a pulse of area α and taking the limit as α approaches zero.
(b) finding the response of the system to a pulse of height α and constant width and taking the limit as α approaches infinity.
(c) finding the unit step response and taking the derivative.
(d) All of the above
(e) None of the above

1.3 The output of a system can be determined from its impulse response and the system input provided that the system is

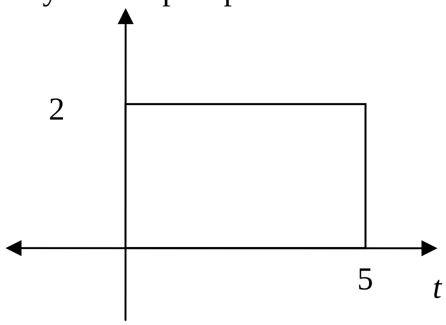
(a) linear
(b) time-invariant
(c) homogeneous
(d) linear and time-invariant
(e) linear and homogeneous

1.4 The Fourier Series representation for a signal is valid

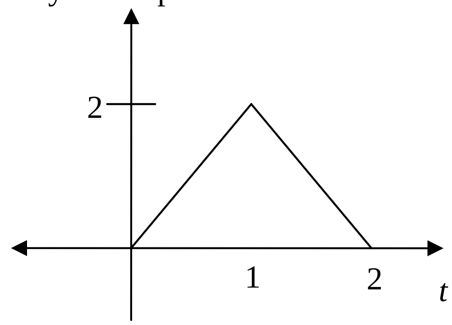
(a) over all time provided that the signal is periodic and the observation interval T_F is equal to one period T_o
(b) over all time provided that the signal is periodic and the observation interval T_F is equal to one period T_o or some integer multiple of T_o
(c) over all time provided that the signal is aperiodic
(d) only over the observation interval T_F
(e) All of the above

2. (25 points) Determine the output of an LTI system given the following information

System step response



System input



The output of a linear time-invariant system can be found as

$$y(t) = x(t) * h(t)$$

Where h(t) is the system impulse response. Further, the system impulse response is related to the system step response, $y_s(t)$ as:

$$h(t) = \frac{d}{dt}\{y_s(t)\}$$

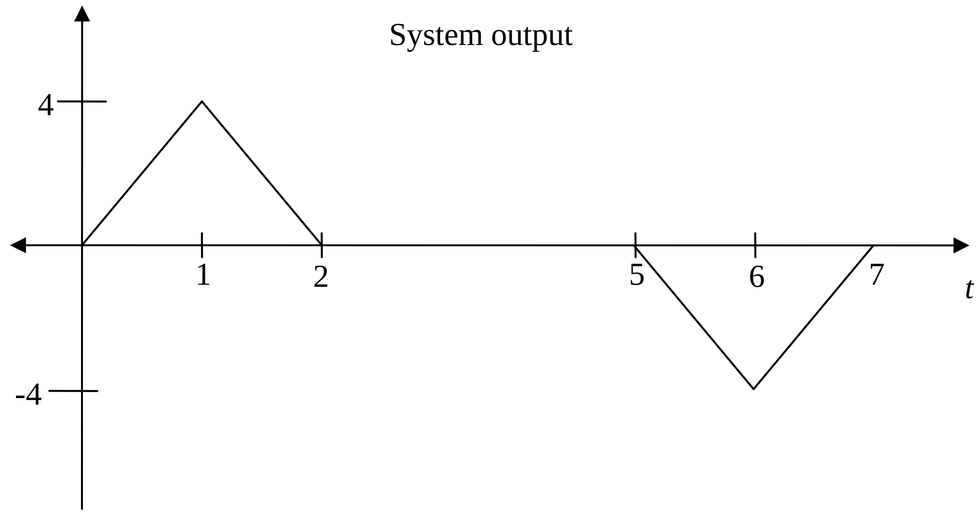
Taking the derivative of the system step response we get:

$$\begin{aligned} h(t) &= \frac{d}{dt}\left\{2rect\left(\frac{t-2.5}{5}\right)\right\} \\ &= 2\delta(t) - 2\delta(t-5) \end{aligned}$$

Thus, the output is:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= 2tri(t-1) * \{2\delta(t) - 2\delta(t-5)\} \\ &= 4tri(t-1) - 4tri(t-6) \end{aligned}$$

Plotting this we get:


System output
4
-4
1
2
5
6
7
t

3. (30 points) Fourier Series

Considering the following Fourier Series properties and representations,

$$x(t-t_o) \overset{FS}{\longleftrightarrow} e^{-j2\pi k f_{Fx} t_o} X[k]$$

$$e^{j2\pi f_o t} \overset{FS}{\leftrightarrow} \delta[k-1]$$

$$e^{j2\pi f_o t} \overset{FS}{\leftrightarrow} \delta[k-m]$$

$$\cos(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$$

$$e^{j2\pi k_o f_{Fx} t} x(t) \overset{FS}{\longleftrightarrow} X[k-k_o]$$

$$\cos(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-m] + \frac{1}{2}\delta[k+m]$$

$$\frac{d}{dt}\{x(t)\} \overset{FS}{\longleftrightarrow} (j2\pi k f_{Fx}) X[k] \qquad 1 \overset{FS}{\leftrightarrow} \delta[k]$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{FS}{\longleftrightarrow} \frac{1}{j2\pi k f_{Fx}} X[k] \qquad \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT_o-t_o}{T_w}\right) \overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o} f_o T_w \text{sinc}(k f_o T_w)$$

$$x(t)\,y(t) \overset{FS}{\longleftrightarrow} \sum_{q=-\infty}^{\infty} Y[q] X[k-q] \qquad \sum_{n=-\infty}^{\infty} tri\left(\frac{t-nT_o-t_o}{T_w}\right) \overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o} f_o T_w \text{sinc}^2(k f_o T_w)$$

Fourier Series with change in observation interval: $X_m[k] = \begin{cases} X\left[\frac{k}{m}\right] & \frac{k}{m} = \text{integer} \\ 0 & \text{else} \end{cases}$

(a) (10pts) Determine the Fourier Series of the following signal when the representation interval is T_F = 0.1s.

$$x(t) = \cos(20\pi t + \pi/5) + \sum_{i=-\infty}^{\infty} rect\left(\frac{t - i/10}{0.005}\right) - 0.5$$

First we note that:

$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

From the linearity property of the Fourier Series:

$$X[k] = X_1[k] + X_2[k] + X_3[k]$$

First, note that f_F = 10Hz. Now from the Fourier Series of a cosine with f_F=f_o and the time-shift property we can write:

$$X_1[k] = e^{-j2\pi k f_o t_o}\left\{\frac{1}{2}\delta(k-1) + \frac{1}{2}\delta(k+1)\right\}$$

Now in this case the delay can be found from the phase shift of π/5:

$$\begin{aligned} x_1(t) &= \cos(20\pi t + \pi/5) \\ &= \cos\left(20\pi\left(t + \frac{\pi/5}{20\pi}\right)\right) \\ &= \cos\left(20\pi\left(t + \frac{1}{100}\right)\right) \end{aligned}$$

Thus, $t_o = -1/100$. *We can thus write:*

$$\begin{aligned} X_1[k] &= e^{j2\pi k10/100}\left\{\frac{1}{2}\delta(k-1) + \frac{1}{2}\delta(k+1)\right\} \\ &= \frac{e^{j\pi k/5}}{2}\delta(k-1) + \frac{e^{j\pi k/5}}{2}\delta(k+1) \\ &= \frac{e^{j\pi/5}}{2}\delta(k-1) + \frac{e^{-j\pi/5}}{2}\delta(k+1) \end{aligned}$$

Secondly, we can write:

$$X_2[k] = f_o T_w \operatorname{sinc}(k f_o T_w)$$

where $f_o = 10\text{Hz}$ *and* $T_w = 0.005$. *Thus,*

$$X_2[k] = \frac{1}{20}\operatorname{sinc}\left(\frac{k}{20}\right)$$

Further,

$$X_3[k] = -0.5\delta[k]$$

Finally,

$$X[k] = \frac{e^{j\pi/5}}{2}\delta(k-1) + \frac{e^{-j\pi/5}}{2}\delta(k+1) + \frac{1}{20}\operatorname{sinc}\left(\frac{k}{20}\right) - 0.5\delta[k]$$

(b) (10pts) Using the derivative property of the Fourier Series, determine the Fourier Series for $\sin(2\pi f_o t)$ assuming that $T_F = T_o$. Simplify as much as possible.

Since we want to use the derivative property we can write

$$\sin(2\pi f_o t) = \frac{-1}{2\pi f_o}\frac{d}{dt}\left\{\cos(2\pi f_o t)\right\}$$

Now, using the derivative property we can write

$$\begin{aligned}
Y[k] &= \left(\frac{-1}{2\pi f_o}\right) j2\pi k f_o X[k] \\
&= \left(\frac{-1}{2\pi f_o}\right) j2\pi k f_o \left\{\frac{1}{2}\delta(k-1) + \frac{1}{2}\delta(k+1)\right\} \\
&= -jk\left\{\frac{1}{2}\delta(k-1) + \frac{1}{2}\delta(k+1)\right\} \\
&= \frac{-j}{2}\delta(k-1) + \frac{j}{2}\delta(k+1) \\
&= \frac{1}{2j}\left(\delta(k-1) - \delta(k+1)\right)
\end{aligned}$$

(c) (10pts) Using the time-shift property of the Fourier Series, determine the Fourier Series for $\sin(2\pi f_o t)$ assuming that $T_F = T_o$. Simplify as much as possible.

Now, to use the time-shift property we can write

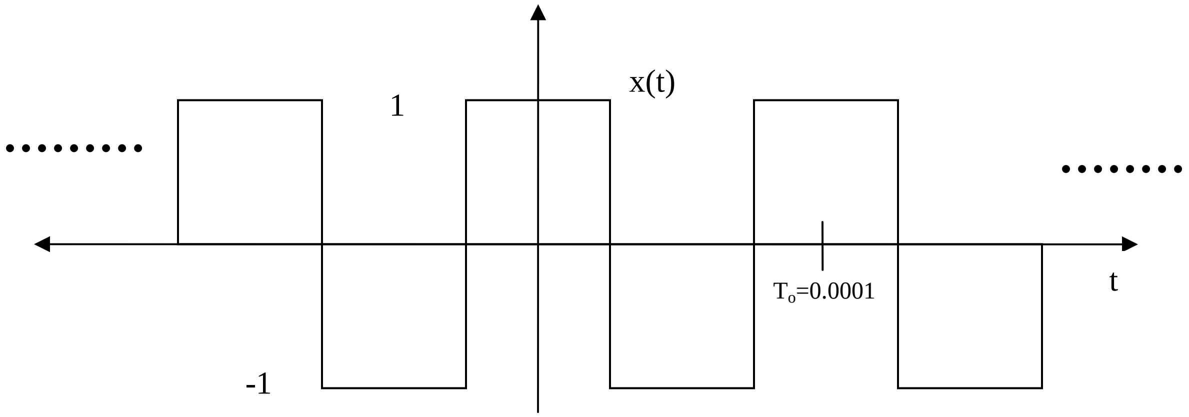
$$\begin{aligned}\sin(2\pi f_o t) &= \cos\left(2\pi f_o t - \frac{\pi}{2}\right) \\ &= \cos\left(2\pi f_o \left[t - \frac{1}{4 f_o}\right]\right)\end{aligned}$$

Now, using the time shift property:

$$\begin{aligned}Y[k] &= e^{-j2\pi k f_o t_o} X[k] \\ &= e^{-j2\pi k f_o (1/(4 f_o))}\left\{\frac{1}{2}\delta(k-1) + \frac{1}{2}\delta(k+1)\right\} \\ &= e^{-jk\pi/2}\left\{\frac{1}{2}\delta(k-1) + \frac{1}{2}\delta(k+1)\right\} \\ &= \frac{e^{-jk\pi/2}}{2}\delta(k-1) + \frac{e^{-jk\pi/2}}{2}\delta(k+1) \\ &= \frac{e^{-j\pi/2}}{2}\delta(k-1) + \frac{e^{j\pi/2}}{2}\delta(k+1) \\ &= \frac{-j}{2}\delta(k-1) + \frac{j}{2}\delta(k+1) \\ &= \frac{1}{2j}\left(\delta(k-1) - \delta(k+1)\right)\end{aligned}$$

4. (25 points) Fourier Series

The following waveform is fed into an amplifier as a means of testing it. It is a 50% duty-cycle square wave with zero mean.



The amplifier passes all frequencies below 51kHz with an amplitude gain of G=10. All frequencies above 51kHz are not passed by the amplifier. (Note that this also applies to negative frequencies. In other words all frequencies between 0 and -51kHz are also passed while frequencies below-51kHz are not passed). The plot at the end of the test may be useful.

(a) (5pts) Determine the average power of the input signal.

The simplest way to determine the average power is to calculate it in the time domain:

$$
\begin{aligned}
P_x &= \frac{1}{T_o}\int_0^{T_o} |x(t)|^2 \, dt \\
&= \frac{1}{T_o}\int_0^{T_o} 1 * dt \\
&= \frac{1}{T_o} t\Big|_0^{T_o} \\
&= \frac{1}{T_o} T_o \\
&= 1
\end{aligned}
$$

(b) (15pts) Determine the average power of the output signal.

From Parseval's Theorem we can write:

$$P_x = \frac{1}{T_o}\int_0^{T_o} \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|X[k]\right|^2$$

Now, if we choose the fundamental period of the Fourier Series to be equal to one period $T_F = T_o = 0.0001$, the input signal has Fourier Series components:

$$\begin{aligned} X[k] &= 2 f_o T_w \operatorname{sinc}(k f_o T_w) - \delta[k] \\ &= \operatorname{sinc}\left(\frac{k}{2}\right) - \delta[k] \end{aligned}$$

where we have used $T_w = T_o/2$. Now, the amplifier provides a gain of 10 for all components below 51kHz. Since the fundamental frequency is $f_o = 10kHz$, this means that the gain of 10 applies to all Fourier Series components for which the frequency is below 51kHz. Specifically:

$$\begin{aligned} Y[k] &= \begin{cases} 10X[k] & kf_o \le 51kHz \\ 0 & else \end{cases} \\ &= \begin{cases} 10X[k] & k \le 5 \\ 0 & else \end{cases} \\ &= \begin{cases} 10\left\{\operatorname{sinc}\left(\frac{k}{2}\right) - \delta[k]\right\} & k \le 5 \\ 0 & else \end{cases} \\ &= \begin{cases} 10\operatorname{sinc}\left(\frac{k}{2}\right) - 10\delta[k] & k \le 5 \\ 0 & else \end{cases} \end{aligned}$$

Thus, the output power can be calculated as:

$$
\begin{aligned}
P_y &= \sum_{k=-\infty}^{\infty} \left|Y[k]\right|^2 \\
&= \sum_{k=-5}^{5} \left|10\left\{\operatorname{sinc}\left(\frac{k}{2}\right) - \delta[k]\right\}\right|^2 \\
&= \underbrace{100\left\{\operatorname{sinc}(0) - \delta[0]\right\}^2}_{=0} + \underbrace{2\sum_{k=1}^{5} \left|10\left\{\operatorname{sinc}\left(\frac{k}{2}\right) - \delta[k]\right\}\right|^2}_{\text{even terms}=0} \\
&= 200\left(\operatorname{sinc}^2\left(\frac{1}{2}\right) + \operatorname{sinc}^2\left(\frac{3}{2}\right) + \operatorname{sinc}^2\left(\frac{5}{2}\right)\right) \\
&= 93.3W
\end{aligned}
$$

(c) (5pts) If the amplifier passed all frequencies with an amplitude gain of 10, what would be the power of the output signal?

*Since the input power is 1 and the amplitude gain is ten, the output power would simply be 10^2*1 = 100W.*

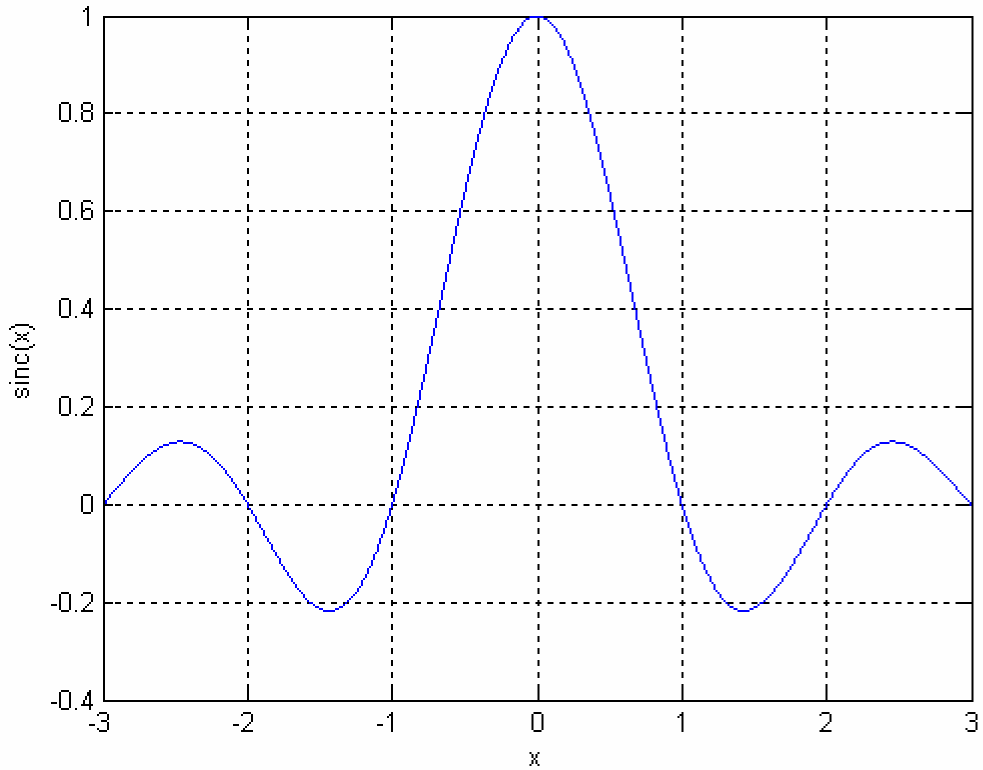


Figure 1 – Plot of sinc(x) versus x