# 2704: Signals and Systems

## Midterm Exam III
April 19, 2006

I pledge that I have neither given nor received any assistance on this exam.

______________________________________________
(signed)

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Name (print) Student Number

1. (20 points) Multiple Choice – Choose the answer which *best* completes the sentence

1.1 Parseval's Theorem states that

(a) $\frac{1}{T}\int_{T}\left|x(t)\right|^2 dt = \int_{-\infty}^{\infty}\left|X(f)\right|^2 df$

(b) $\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt = \int_{-\infty}^{\infty}\left|X(f)\right|^2 df$

(c) $\frac{1}{T}\int_{T}\left|x(t)\right| dt = \frac{1}{T}\int_{-\infty}^{\infty}\left|X(f)\right| df$

(d) $\int_{-\infty}^{\infty}\left|x(t)\right| dt = \int_{-\infty}^{\infty}\left|X(f)\right| df$

1.2 If a function *x(t)* is delayed in time, then its Fourier Transform *X(f)* experiences
(a) a frequency shift
(b) a frequency scaling
(c) a phase shift
(d) a phase scaling
(e) none of the above

1.3 The Fourier Transform of a constant is
(a) a constant
(b) an impulse at $f = f_o$
(c) an impulse at f = 0
(d) a sinusoid at $f = f_o$
(e) a sinusoid at f = 0
(f) none of the above

1.4 Dividing by frequency in the frequency domain is equivalent to what operation in the time domain?

(a) dividing by a complex sinusoid in the time domain
(b) multiplying by a complex sinusoid in the time domain
(c) integrating in the time domain
(d) taking the derivative in the time domain
(e) none of the above

2. (25 points) Determine the output of an LTI system given the following information

The system input is

$$x(t) = A + B\cos(2\pi f_o t)$$

and the system impulse response is

$$h(t) = 2W\text{sinc}(2Wt)$$

(a) (15 points) Assume that $f_o$ = 100 and $W$ = 1000.

*The output of an LTI system is*

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= 2W\text{sinc}(2Wt) * (A + B\cos(2\pi f_o t)) \end{aligned}$$

*However, using the Fourier Transform, this is simpler. Specifically,*

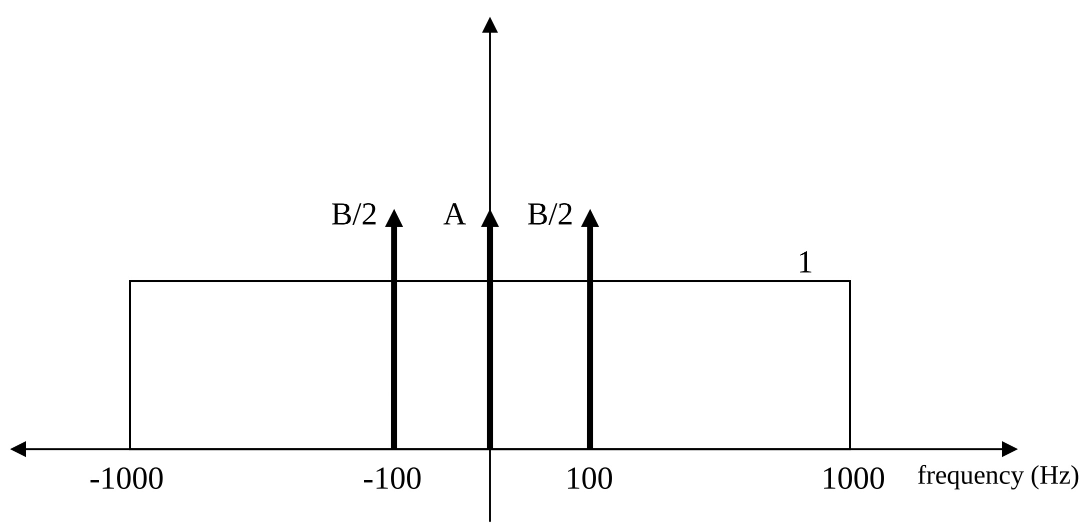
$$Y(f) = H(f)X(f)$$

*Further, taking the Fourier Transform of the input and the impulse response we have*

$$H(f) = rect\left(\frac{f}{2W}\right)$$

$$X(f) = A\delta(f) + \frac{B}{2}\left[\delta(f + f_o) + \delta(f - f_o)\right]$$

*With $f_o$ = 100Hz and W = 1kHz*
*A plot of the two transforms are plotted below.*



*Clearly the system is an ideal low-pass filter with a bandwidth of 1kHz. Since the input is easily within the bandwidth of the filter, the input is passed without loss or distortion and we have:*

$$\begin{aligned} Y(f) &= H(f)X(f) \\ &= X(f) \end{aligned}$$

*and thus:*

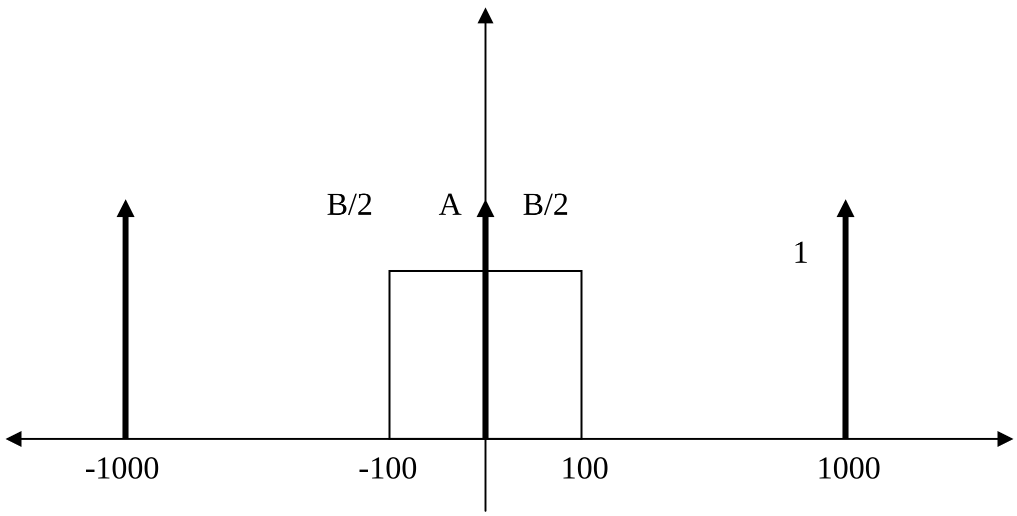
$$\begin{aligned} y(t) &= x(t) \\ &= A + B\cos(2\pi f_o t) \end{aligned}$$

(b) (10 points) Assume that $f_o$ = 1000 and *W* = 100.

*The output is still* $Y(f) = H(f)X(f)$ or with *$f_o$ = 1000Hz and W = 100Hz*

$$Y(f) = rect\left(\frac{f}{200}\right)\left\{A\delta(f) + \frac{B}{2}\left[\delta(f+1000) + \delta(f-1000)\right]\right\}$$

*which is shown in the figure below.*




*Thus, the output is*

$$Y(f) = A\delta(f)$$
$$y(t) = A$$

3. (30 points) Taking the Fourier Transform and Inverse Fourier Transform

(a) (15pts) Determine the Fourier Transform of the following signal

$$x(t) = \cos(20\pi t + \pi/5) + Ae^{-t}u(t) - 0.5$$

*We can use linearity and the following Fourier Transform pairs:*

$$\cos(2\pi f_o t) \xleftrightarrow{\mathcal{L}} \frac{1}{2}\left\{\delta(f - f_o) + \delta(f + f_o)\right\}$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{a + j2\pi f}$$

$$A \xleftrightarrow{\mathcal{L}} A\delta(f)$$

*Further, we can use the Fourier Transform Property:*

$$x(t - t_o) \xleftrightarrow{\mathcal{L}} e^{-j2\pi f t_o} X(f)$$

*In this case to = -1/100 and $f_o$ = 10. Thus, we have:*

$$\begin{aligned} X(f) &= \frac{e^{-j2\pi f t_o}}{2}\left\{\delta(f - f_o) + \delta(f + f_o)\right\} + \frac{A}{1 + j2\pi f} + B\delta(f) \\ &= \frac{e^{-j2\pi f(-1/100)}}{2}\left\{\delta(f - 10) + \delta(f + 10)\right\} + \frac{A}{1 + j2\pi f} - 0.5\delta(f) \\ &= \frac{e^{j\pi f/50}}{2}\left\{\delta(f - 10) + \delta(f + 10)\right\} + \frac{A}{1 + j2\pi f} - 0.5\delta(f) \end{aligned}$$

*Note that for some reason $\pi$ was printed p in the exam given. In this case* $f_o = \frac{10p}{\pi}$ *but otherwise the solution is the same.*

(b) (15pts) Find the inverse Fourier Transform of

$$X(f) = \frac{A}{5 + j2\pi f}$$

*From the previous problem, we have the pair*

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{a + j2\pi f}$$

*and thus the solution is*

$$x(t) = Ae^{-5t}u(t)$$

4. (25 points) Bode Diagrams

Consider the following filter

$$H(f) = \frac{100(10 + j2\pi f)}{(1000 + j2\pi f)(100 + j2\pi f)}$$

(a) (5pts) Determine the poles and zeros of this filter.

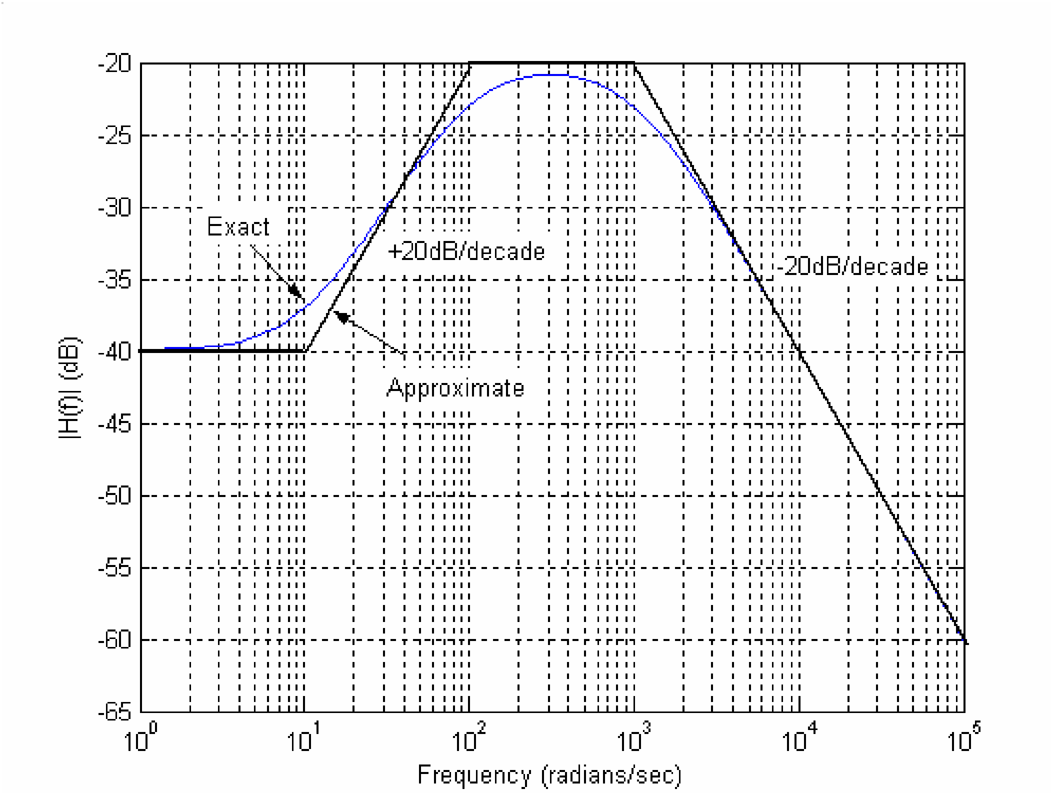
*We can write the poles and zeros either in terms of Hz (f) or radians/sec (ω):*

*There is one zero* $z_1 = -\frac{10}{2\pi}$ *Hz or* $z_1 = -10$ *radians/sec.*

*There are two poles at* $p_1 = -\frac{100}{2\pi}, p_2 = -\frac{1000}{2\pi}$ *Hz or* $p_1 = -100, p_2 = -1000$ *radians/sec*

(b) (15pts) Draw the approximate magnitude Bode diagram using a piece-wise linear approximation

*For very small values of f, the gain of the filter is* $|H(f)| = \frac{1000}{100,000} = \frac{1}{100}$ *or -40dB. The approximate plot starts with zero slope at -40dB. At the zero 10 radians/sec the slope increases to 20dB/decade until we reach a pole at 100 radians/sec. At this point the slope changes to 0 since we have the influence of one zero and one pole. The gain at this point is -20dB since we have gone one decade in frequency (factor of 10). The zero slope continues until we reach the second (and last) pole at 1000 radians/sec where the slope changes to -20dB/decade. This is shown in the figure below.*




(c) (5 points) Lightly sketch the exact magnitude response on the same diagram as the approximate Bode plot in part (b). Evaluate 2-3 points by hand to aid in you drawing.

*The most important points to examine are the pole/zero locations since that is where the approximation is the worst.*

*At $\omega = 10$:*

$$\left|H(f)\right|_{dB} = 20\log_{10}\left\{\left|\frac{100(10+j10)}{(1000+j10)(100+j10)}\right|\right\}$$
$$= -37dB$$

*At $\omega = 100$:*

$$\left|H(f)\right|_{dB} = 20\log_{10}\left\{\left|\frac{100(10+j100)}{(1000+j100)(100+j100)}\right|\right\}$$
$$= -23dB$$

*At $\omega = 1000$:*

$$\left|H\left(f\right)\right|_{dB} = 20\log_{10}\left\{\left|\frac{100\left(10 + j1000\right)}{\left(1000 + j1000\right)\left(100 + j1000\right)}\right|\right\}$$
$$= -23dB$$

*At $\omega$ = 10000:*

$$\left|H\left(f\right)\right|_{dB} = 20\log_{10}\left\{\left|\frac{100\left(10 + j10000\right)}{\left(1000 + j10000\right)\left(100 + j10000\right)}\right|\right\}$$
$$= -40dB$$

*The resulting sketch is plotted with the approximation above.*