2704: Signals and Systems

Homework #2
Solutions

1)

Homogeneity:
Let $x_1(t) = g(t)$. Then $y_1(t) = u(g(t))$.
Let $x_2(t) = K\,g(t)$. Then $y_2(t) = u(K\,g(t)) \neq K\,y_1(t) = K\,u(g(t))$.
Not homogeneous

Additivity:
Let $x_1(t) = g(t)$. Then $y_1(t) = u(g(t))$.
Let $x_2(t) = h(t)$. Then $y_2(t) = u(h(t))$.
Let $x_3(t) = g(t) + h(t)$.
Then $y_3(t) = u(g(t) + h(t)) \neq y_1(t) + y_2(t) = u(g(t)) + u(h(t))$
Not additive

Since it is not homogeneous and not additive, it is not linear.

It is also not incrementally linear because incremental changes in the excitation do not produce proportional incremental changes in the response.

It is statically non-linear because it is non-linear without memory (lack of memory proven below).

Time Invariance:
Let $x_1(t) = g(t)$. Then $y_1(t) = u(g(t))$.
Let $x_2(t) = g(t - t_0)$.
Then $y_2(t) = u(g(t - t_0)) = y_1(t - t_0)$.
Time Invariant

Stability:
The unit step function can only have the values, zero or one, therefore any bounded (or unbounded) excitation produces a bounded response.
Stable

Causality:
The response at any time, $t = t_0$, depends only on the excitation at time, $t = t_0$ and not on any future values.

Invertibility:
There are many value of the excitation that all cause a response of zero and there are many values of the excitation that all cause a response of one. Therefore the system is not invertible.

2)

Homogeneity:
Let $x_1(t) = g(t)$. Then $y_1(t) = g(t-5) - g(3-t)$.
Let $x_2(t) = K\,g(t)$. Then $y_2(t) = K\,g(t-5) - K\,g(3-t) = K\,y_1(t)$.
Homogeneous

Additivity:
Let $x_1(t) = g(t)$. Then $y_1(t) = g(t-5) - g(3-t)$.
Let $x_2(t) = h(t)$. Then $y_2(t) = h(t-5) - h(3-t)$.
Let $x_3(t) = g(t) + h(t)$. Then $y_3(t) = g(t-5) + h(t-5) - g(3-t) - h(3-t) = y_1(t) + y_2(t)$
Additive

Since it is both homogeneous and additive, it is also linear.

It is also incrementally linear, since any linear system is incrementally linear.

It is not statically non-linear because it is linear.

Causality:
At time, $t = 0$, $y(0) = x(-5) - x(3)$. Therefore the response at time, $t = 0$, depends on the excitation at a later time, $t = 3$.
Not Causal

Invertibility:
A counterexample will demonstrate that the system is not invertible. Let the excitation be a constant, K. Then the response is $y(t) = K - K = 0$. This is the response, no matter what K is. Therefore when the output is a constant zero, the input cannot be determined.
Not Invertible.

3)

Homogeneity:

Let $x_1(t) = g(t)$. Then $y_1(t) = g\left(\frac{t}{2}\right)$.

Let $x_2(t) = K\,g(t)$. Then $y_2(t) = K\,g\left(\frac{t}{2}\right) = K\,y_1(t)$.

Homogeneous

Additivity:

Let $x_1(t) = g(t)$. Then $y_1(t) = g\left(\frac{t}{2}\right)$.

Let $x_2(t) = h(t)$. Then $y_2(t) = h\left(\frac{t}{2}\right)$.

Let $x_3(t) = g(t) + h(t)$. Then $y_3(t) = g\left(\frac{t}{2}\right) + h\left(\frac{t}{2}\right) = y_1(t) + y_2(t)$

Additive

Since it is both homogeneous and additive, it is also linear.

It is also incrementally linear, since any linear system is incrementally linear.

It is not statically non-linear because it is linear.

3 cont)

Time Invariance:

Let $x_1(t) = g(t)$. Then $y_1(t) = g\left(\frac{t}{2}\right)$.

Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = g\left(\frac{t}{2} - t_0\right) \neq y_1(t - t_0) = g\left(\frac{t - t_0}{2}\right)$.

Time Variant

Causality:
At time, $t = -2$, $y(-2) = x(-1)$. Therefore the response at time, $t = -2$, depends on the excitation at a later time, $t = -1$.
Not Causal

4)

Homogeneity:
Let $x_1(t) = g(t)$. Then $y_1(t) = \cos(2\pi t)g(t)$.
Let $x_2(t) = K\,g(t)$. Then $y_2(t) = \cos(2\pi t)K\,g(t) = K\,y_1(t)$.
Homogeneous

Additivity:
Let $x_1(t) = g(t)$. Then $y_1(t) = \cos(2\pi t)g(t)$.
Let $x_2(t) = h(t)$. Then $y_2(t) = \cos(2\pi t)h(t)$.
Let $x_3(t) = g(t) + h(t)$. Then $y_3(t) = \cos(2\pi t)\left[g(t) + h(t)\right] = y_1(t) + y_2(t)$
Additive

Since it is both homogeneous and additive, it is also linear.

It is also incrementally linear, since any linear system is incrementally linear.

It is not statically non-linear because it is linear.

Time Invariance:
Let $x_1(t) = g(t)$. Then $y_1(t) = \cos(2\pi t)g(t)$.
Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = \cos(2\pi t)g(t - t_0) \neq y_1(t - t_0) = \cos\left(2\pi(t - t_0)\right)g(t - t_0)$.
Time Variant

Stability:
If $x(t)$ is bounded then $y(t)$ is bounded because it is multiplied by a cosine which is bounded.
Stable

Static:
$y(t_0)$ is determined entirely by $\cos(2\pi f t_0)x(t_0)$ and no other values of t.
Static

Invertibility:
This system is not invertible because when the cosine function is zero the unique relationship between x and y is lost; any x produces the same y, zero.
Not Invertible.

32)

We must first determine the mathematical expression that represents the block-diagram:

$$\frac{1}{4}x(t)-\left(\frac{1}{4}y'(t)+\frac{3}{4}y(t)\right)=y''(t)\rightarrow x(t)=4y''(t)+y'(t)+3y(t)$$

Homogeneity:

Let $x_1(t)=g(t)$. Then $4y_1''(t)+y_1'(t)+3y_1(t)=g(t)$.

Let $x_2(t)=K\,g(t)$. Then $4y_2''(t)+y_2'(t)+3y_2(t)=K\,g(t)$.

If we multiply the first equation by K, we get

$$4K\,y_1''(t)+K\,y_1'(t)+3K\,y_1(t)=K\,g(t)$$

Therefore

$$4K\,y_1''(t)+K\,y_1'(t)+3K\,y_1(t)=4y_2''(t)+y_2'(t)+3y_2(t)$$

This can only be true for all time if $y_2(t)=K\,y_1(t)$.

Homogeneous

Additivity:

Let $x_1(t)=g(t)$. Then $4y_1''(t)+y_1'(t)+3y_1(t)=g(t)$.

Let $x_2(t)=h(t)$. Then $4y_2''(t)+y_2'(t)+3y_2(t)=h(t)$.

Let $x_3(t)=g(t)+h(t)$. Then $4y_3''(t)+y_3'(t)+3y_3(t)=g(t)+h(t)$

Adding the first two equations,

$$4\left[y_1''(t)+y_2''(t)\right]+\left[y_1'(t)+y_2'(t)\right]+3\left[y_1(t)+y_2(t)\right]=g(t)+h(t)$$

Therefore

$$4y_3''(t)+y_3'(t)+3y_3(t)=4\left[y_1''(t)+y_2''(t)\right]+\left[y_1'(t)+y_2'(t)\right]+3\left[y_1(t)+y_2(t)\right]$$

$$4y_3''(t)+y_3'(t)+3y_3(t)=4\left[y_1(t)+y_2(t)\right]''+\left[y_1(t)+y_2(t)\right]'+3\left[y_1(t)+y_2(t)\right]$$

This can only be true for all time if $y_3(t)=y_1(t)+y_2(t)$.

Additive

Since the system is homogeneous and additive, it is linear.

Time Invariance:

Let $x_1(t)=g(t)$. Then $4y_1''(t)+y_1'(t)+3y_1(t)=g(t)$.

Let $x_2(t)=g(t-t_0)$.

Then $4y_2''(t)+y_2'(t)+3y_2(t)=g(t-t_0)$.

The first equation can be written as

$$4y_1''(t-t_0)+y_1'(t-t_0)+3y_1(t-t_0)=g(t-t_0)$$

Therefore

$$4y_1''(t-t_0)+y_1'(t-t_0)+3y_1(t-t_0)=4y_2''(t)+y_2'(t)+3y_2(t)$$

This can only be true for all time if $y_2(t)=y_1(t-t_0)$.

Time Invariant

Stability:
The eigenvalues are $\lambda_{1,2} = 0.125 \pm j0.85696$

Since the magnitude of the eigenvalues are less than 1, it hints that the system is stable. Also, if there is no excitation, but the zero-excitation response is not zero, the response will decay to zero as time increases.
BIBO stable

Causality & Memory:
The system can be rewritten as:

$$y(t) = \frac{1}{4}\left[\int_{-\infty}^{t}\int_{-\infty}^{\lambda_2} x(\lambda_1)d\lambda_1 d\lambda_2 - \int_{-\infty}^{t} y(\lambda_1)d\lambda_1 - 3\int_{-\infty}^{t}\int_{-\infty}^{\lambda_2} y(\lambda_1)d\lambda_1 d\lambda_2\right]$$

We can see that the response at time $t = t_0$ depends entirely on the excitation at and before t_0.
Causal & has Memory

Invertibility:
From the expression:

$$x(t) = 4y''(t) + y'(t) + 3y(t)$$

We see that the excitation is determined entirely by the response and its derivatives. Therefore the excitation can be uniquely determined by the response.
Invertible

33)
$y(t) = x^3(t)$
Linearity:
$A^3x_1^3(t) + B^3x_2^3(t) \neq Ay_1(t) + By_2(t)$
Not Linear.

Time Invariance:
$y(t) = x^3(t)$, $x^3(t-t_0) = y(t-t_0)$
Time Invariant.

Stability:
If $x(t)$ is bounded, then $y(t) = x^3(t)$ is also bounded.
BIBO stable.

Causality & Memory:
System response does not depend on future or past values of t, only the current value.
Causal and has no Memory

Invertibility:
$y(t) = x^3(t) \to x(t) = y^{1/3}(t)$. Cube root operation can obtain the same output value with multiple complex input values, therefore the function is not invertible. If the condition was given that only real-valued inputs would be used, the function would be invertible.

35)

Time Invariance:

Let $x_1(t) = g(t)$. Then $y_1(t) = \int_{-\infty}^{\frac{t}{3}} g(\lambda) d\lambda$.

Let $x_2(t) = g(t - t_0)$.

Then $y_2(t) = \int_{-\infty}^{\frac{t}{3}} g(\lambda - t_0) d\lambda = \int_{-\infty}^{\frac{t}{3} - t_0} g(u) du \neq y_1(t - t_0) = \int_{-\infty}^{\frac{t - t_0}{3}} g(\lambda) d\lambda$.

Time Variant

Stability:

If $x(t)$ is a constant, K, then $y(t) = \int_{-\infty}^{\frac{t}{3}} K d\lambda = K \int_{-\infty}^{\frac{t}{3}} d\lambda$ and, as $t \to \infty$, $y(t)$ increases without bound.

Unstable

Invertibility:

Differentiate both sides of $y(t) = \int_{-\infty}^{\frac{t}{3}} x(\lambda) d\lambda$ w.r.t. t yielding $y'(t) = x\left(\frac{t}{3}\right)$. Then it follows that $x(t) = y'(3t)$.

Invertible.

37)

Additivity:

Let $x_1(t) = g_1(t) + j h_1(t)$, where $g_1(t)$ and $h_1(t)$ are both real-valued functions.
Then $y_1(t) = \mathrm{Re}(g_1(t) + j h_1(t)) = g_1(t)$.
Let $x_2(t) = g_2(t) + j h_2(t)$, where $g_2(t)$ and $h_2(t)$ are both real-valued functions.
Then $y_2(t) = \mathrm{Re}(g_2(t) + j h_2(t)) = g_2(t)$.
Let $x_3(t) = g_1(t) + j h_1(t) + g_2(t) + j h_2(t)$.
Then $y_3(t) = \mathrm{Re}(g_1(t) + j h_1(t) + g_2(t) + j h_2(t)) = g_1(t) + g_2(t) = y_1(t) + y_2(t)$.

Additive

Homogeneity:

$y_1(t) = \mathrm{Re}\{a(t)\}$
$x_2(t) = ja(t), \qquad a(t) \quad real$
$y_2(t) = \mathrm{Re}\{x_2(t)\} = 0 \neq j y_1(t)$
Not homogeneous