2704: Signals and Systems

Homework #3
Solutions

1) (a) Determine the convolution of two unit triangle functions

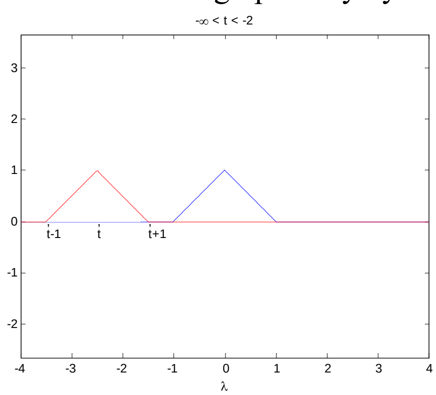
When using simple geometric shapes such as a triangle, graphical convolution lends itself well. We can see that there are six cases that need to be considered:

(1) When $t < -2$
(2) When $-2 \le t < -1$
(3) When $-1 \le t < 0$
(4) When $0 \le t < 1$
(5) When $1 \le t < 2$
(6) When $t \ge 2$

***t* < -2**

$$g(t < -2) = \int_{-\infty}^{-2} tri(\lambda)tri(t-\lambda)d\lambda = \int_{-\infty}^{-2} 0 d\lambda = 0$$

We can see this is 0 graphically by realizing the triangles never intersect (see below).



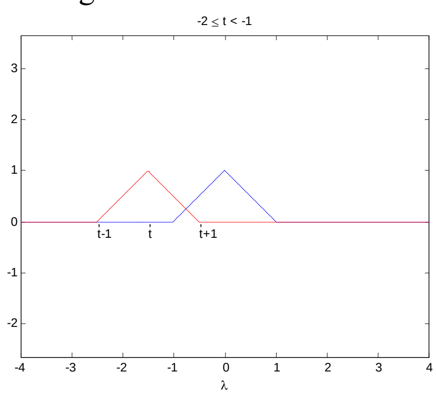
-2 ≤ *t* < -1

From here on, we must look at the lines that make up the triangles and how they intersect. For the red triangle we have the following equations:

Left side: $\lambda + 1 - t$
Right side: $1 + t - \lambda$

For the blue triangle, we have the following equations:

Left side: $\lambda + 1$
Right side: $1 - \lambda$



In this interval, only the right side of the red triangle and the left side of the blue triangle intersect, so we have:

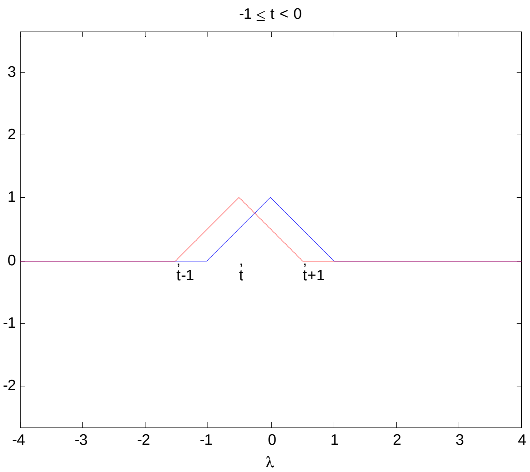
$$g_1(t) = \int_{-1}^{t+1} \overbrace{(\lambda+1)}^{blue-left}\overbrace{(1+t-\lambda)}^{red-right} d\lambda = \int_{-1}^{t+1} \left(1+t+t\lambda-\lambda^2\right) d\lambda = \left[\lambda + t\lambda + \frac{t\lambda^2}{2} - \frac{\lambda^3}{3}\right]_{-1}^{t+1}$$

$$= -\frac{(t+1)^3}{3} + \frac{t\left[(t+1)^2-1\right]}{2} + t^2 + 3t + \frac{5}{3} = \left\{\frac{t^3}{6} + t^2 + 2t + \frac{4}{3}\right\} u(t+2)u(-1-t)$$

NOTE: We use multiplied unit step functions above to bound the output of this function between t = -2 and t = -1.

-1 ≤ *t* < 0

In this interval, red-left intersects with blue-left, right-right intersects with blue-left, and red-right intersects with blue-right. This gives us:



$$g_2(t) = \int_{-1}^{t} (\lambda+1)(\lambda+1-t) d\lambda + \int_{t}^{0} (\lambda+1)(1+t-\lambda) d\lambda + \int_{0}^{t+1} (1-\lambda)(1+t-\lambda) d\lambda$$

$$= \left\{-\frac{t^3}{6} + \frac{t}{2} + \frac{1}{3}\right\} + \left\{-\frac{t^3}{6} - t^2 - t\right\} + \left\{-\frac{t^3}{6} + \frac{t}{2} + \frac{1}{3}\right\}$$

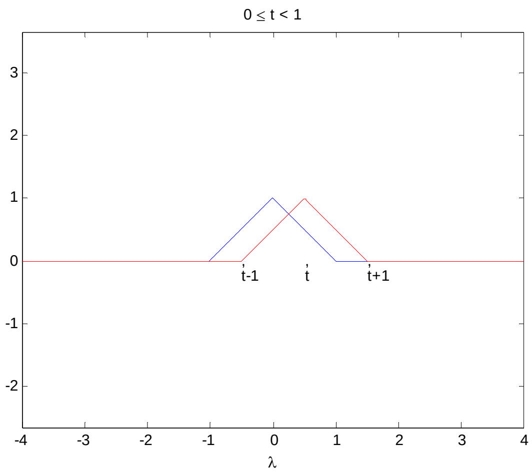
$$= \left\{-\frac{t^3}{2} - t^2 + \frac{2}{3}\right\} u(t+1)u(-t)$$

0 ≤ *t* < 1

In this interval, red-left intersects with blue-left and blue-right. Red-right intersects with blue-right only. This gives us:

$$g_3(t) = \int_{t-1}^{0} (\lambda+1)(\lambda+1-t) d\lambda + \int_{0}^{t} (1-\lambda)(\lambda+1-t) d\lambda + \int_{t}^{1} (1-\lambda)(1+t-\lambda) d\lambda$$

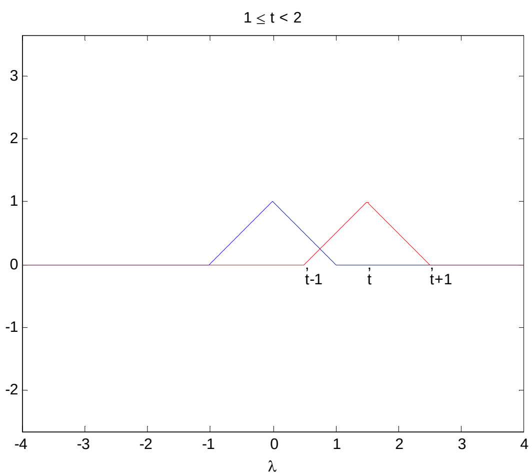
$$= \left\{\frac{t^3}{2} - t^2 + \frac{2}{3}\right\} u(t)u(1-t)$$



1 ≤ *t* < 2

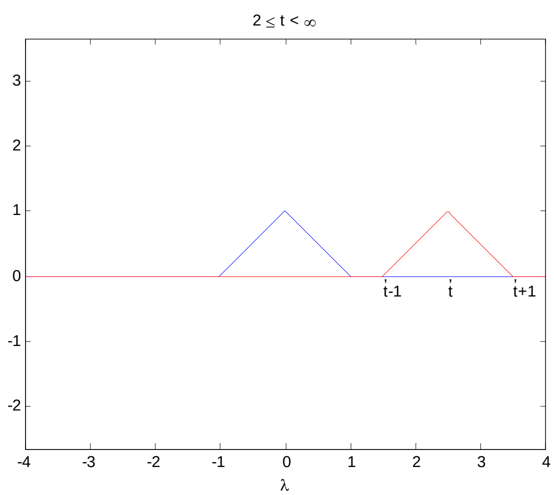
Here we have blue-right intersecting with red-left. This gives us:

$$g_4(t) = \int_{t-1}^{1} (1-\lambda)(\lambda + 1 - t)d\lambda = \left\{ -\frac{t^3}{6} + t^2 - 2t + \frac{4}{3} \right\} u(t-1)u(2-t)$$



***t* ≥ 2**

Finally, we have the case where the red-triangle has completely passed the left-triangle, so there is no intersection.

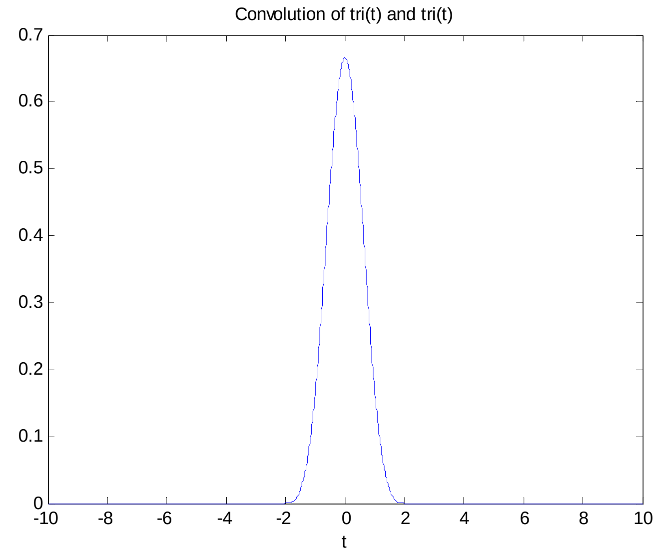


Thus, our convolution solution can be written as:

$$g(t) = g_1(t) + g_2(t) + g_3(t) + g_4(t)$$

$$g(t) = \left\{\frac{t^3}{6} + t^2 + 2t + \frac{4}{3}\right\}u(t+2)u(-1-t) + \left\{-\frac{t^3}{2} - t^2 + \frac{2}{3}\right\}u(t+1)u(-t) + \left\{\frac{t^3}{2} - t^2 + \frac{2}{3}\right\}u(t)u(1-t)$$

$$+ \left\{-\frac{t^3}{6} + t^2 - 2t + \frac{4}{3}\right\}u(t-1)u(2-t)$$



(b) What is the duration? Does this match theory?

We can see from the plot to the left that the duration is 4. Theory tells us that the total duration of the convolution will be the sum of the durations of the functions being convolved. Since each triangle was of duration 2, this matches theory.

2. Determine $x_3(t) = x_1(t) * x_2(t)$ where $x_1(t) =$ sinc(tT), $x_2(t)$=δ(t-100T).

Given that:

$x_1(t) * \delta(t-t_0) = x_1(t-t_0)$

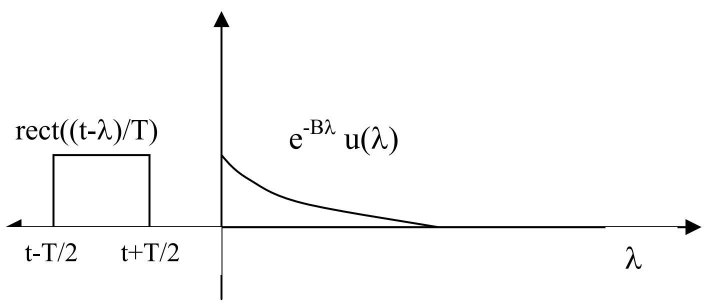
We see that:

$x_3(t) = x_1(t-t_0)$ = sinc((t-100T)T) = <u>sinc(Tt – 100T^2)</u>

3. Determine the convolution of $x_1(t) = rect\left(\frac{t}{T}\right)$ with $x_2(t) = e^{-Bt}u(t)$.

$$g(t) = \int_{-\infty}^{\infty} e^{-B\lambda} u(\lambda) rect\left(\frac{t-\lambda}{T}\right) d\lambda = \int_{-\infty}^{\infty} e^{-B\lambda} u(\lambda) u\left(t-\lambda+\frac{T}{2}\right) u\left(\lambda - t + \frac{T}{2}\right) d\lambda$$

$$= \int_{0}^{\infty} e^{-B\lambda} u\left(t-\lambda+\frac{T}{2}\right) u\left(\lambda - t + \frac{T}{2}\right) d\lambda$$



We can see that the function *g(t)* will be 0 for t < -T/2, so if we break up the function into two non-zero intervals:

-T/2 ≤ t < T/2

$$g(-T/2 \le t < T/2) = \int_{0}^{t+T/2} e^{-B\lambda} d\lambda = -\frac{1}{B}\left[e^{-B\lambda}\right]_0^{t+T/2} = \frac{1-e^{-\frac{BT}{2}-Bt}}{B}$$

t ≥ T/2

$$g(t \ge T/2) = \int_{t-T/2}^{t+T/2} e^{-B\lambda} d\lambda = -\frac{1}{B}\left[e^{-B\lambda}\right]_{t-T/2}^{t+T/2} = \frac{e^{-Bt+\frac{BT}{2}} - e^{-\frac{BT}{2}-Bt}}{B}$$

Therefore, our final solution is:

$$\underline{\underline{g(t) = \frac{1-e^{-\frac{BT}{2}-Bt}}{B} u\left(t+\frac{T}{2}\right) u\left(\frac{T}{2}-t\right) + \frac{e^{-Bt+\frac{BT}{2}} - e^{-\frac{BT}{2}-Bt}}{B} u\left(t-\frac{T}{2}\right)}}$$

To check this solution, we can look at the boundary of t = T/2

$$g(T/2) = \frac{1-e^{-\frac{BT}{2}-BT/2}}{B} u\left(T/2+\frac{T}{2}\right) u\left(\frac{T}{2}-T/2\right) + \frac{e^{-BT/2+\frac{BT}{2}} - e^{-\frac{BT}{2}-BT/2}}{B} u\left(T/2-\frac{T}{2}\right)$$

$$= g(T/2) = \frac{1-e^{-\frac{2BT}{2}}}{B} u(T)u(0) + \frac{e^0 - e^{-\frac{2BT}{2}}}{B} u(0) = \frac{1-e^{-\frac{2BT}{2}}}{B}(1)\left(\frac{1}{2}\right) + \frac{1-e^{-\frac{2BT}{2}}}{B}\left(\frac{1}{2}\right) = \frac{1-e^{-\frac{2BT}{2}}}{B}$$

which is the same value both pieces result to at t = T/2

4. Determine the convolution of $x_1(t) = 2\text{rect}\left(\frac{\text{t-5}}{2}\right)$ with $x_2(t) = rect\left(\frac{t+2}{4}\right)$.

This problem affords us very simple graphical and analytical solutions. To simplify the problem even move, we rely on the fact that $x_3(t) = x_1(t) * x_2(t) \to x_1(t-a)x_2(t-b) = x_3(t-a-b)$, we can remove the shift during the convolution, then add an overall shift of -5+2=-3 (3 to the right) at the end.

We will denote the unshifted function as $x_3'(t)$:

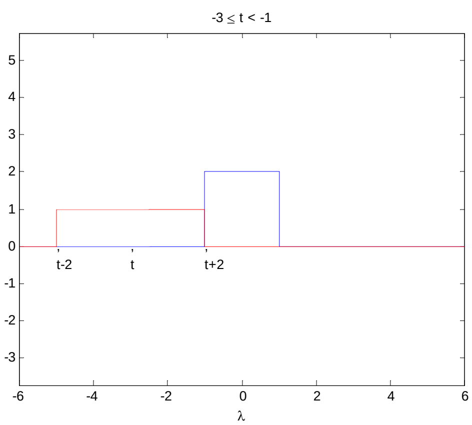
$$x_3'(t) = 2rect\left(\frac{t}{2}\right) * rect\left(\frac{t}{4}\right) = \int_{-\infty}^{\infty} 2rect\left(\frac{\lambda}{2}\right) rect\left(\frac{t-\lambda}{4}\right) d\lambda$$

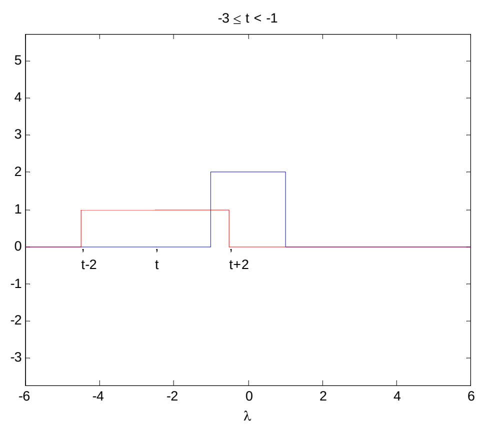
Solving the convolution, we break it up into several intervals:

$t < -3$	: There is no intersection, so $x_3'(t) = 0$
$-3 \le t < -1$	: The two blocks begin to intersect, so $x_3'(t)$ increases linearly
$-1 \le t < 1$	: The two blocks completely intersect, $x_3'(t)$ remains constant
$1 \le t < 3$	: The convolving block begins to leave, $x_3'(t)$ decreases linearly
$t \ge 3$	: The two blocks no longer intersect, $x_3'(t) = 0$

Case 1: $-3 \le t < -1$

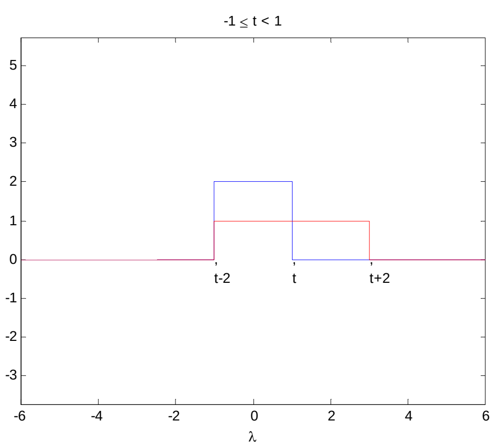
$$x_3'(t) = \int_{-1}^{t+2} 2d\lambda = 2(t+2) + 2 = 2t + 6$$

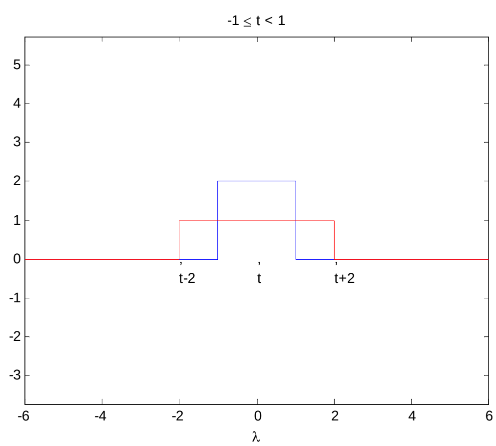




Case 2: $-1 \le t < 1$

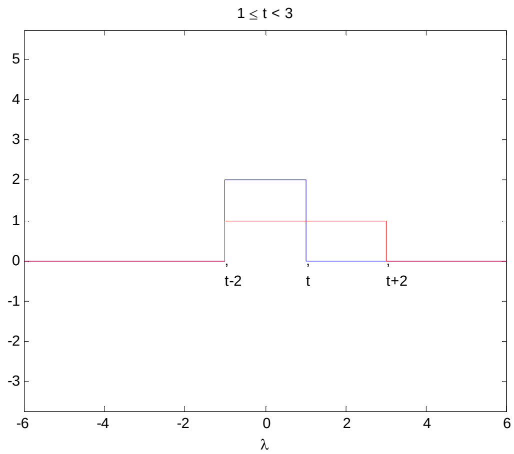
$$x_3'(t) = \int_{-1}^{1} 2d\lambda = 2 + 2 = 4$$

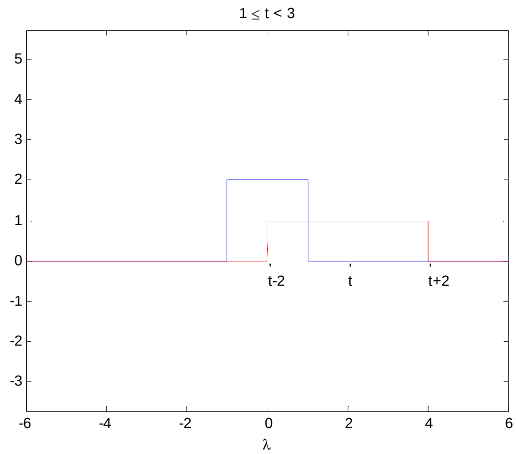




Case 3: $1 \le t < 3$

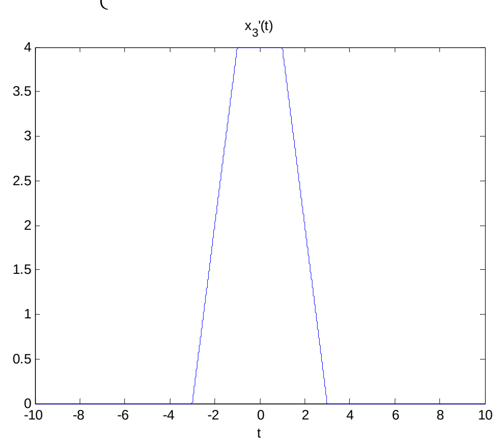
$$x_3'(t) = \int_{t-2}^{1} 2\,d\lambda = 2 - 2(t-2) = 6 - 2t$$



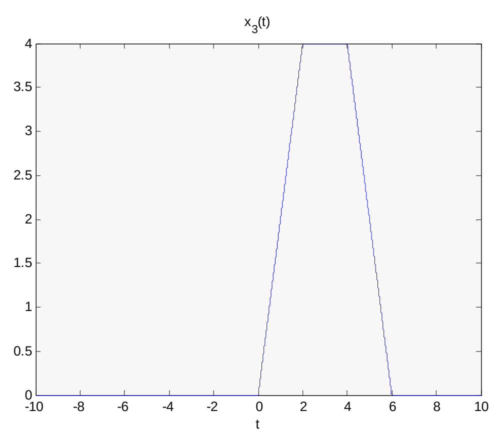


Final result:

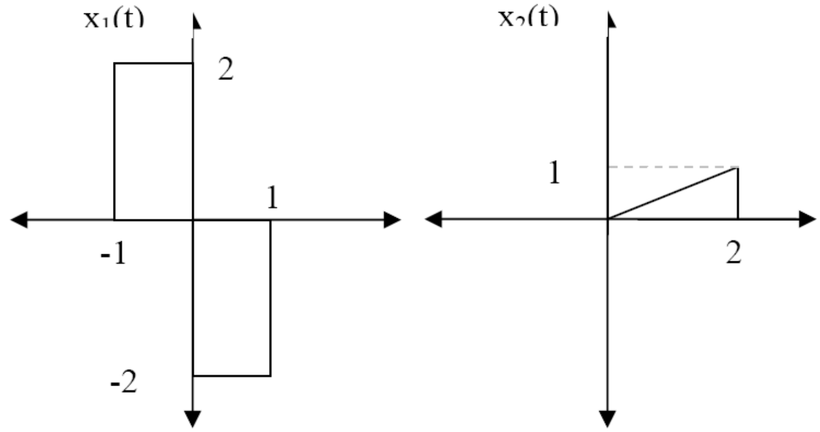
$$x_3'(t) = \begin{cases} 0 & -\infty < t < -3 \\ 2t+6 & -3 \le t < -1 \\ 4 & -1 \le t < 1 \\ 6-2t & 1 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} \Rightarrow x_3(t) = x_3'(t-3) = \begin{cases} 0 & -\infty < t < 0 \\ 2t & 0 \le t < 2 \\ 4 & 2 \le t < 4 \\ 12-2t & 4 \le t < 6 \\ 0 & 6 \le t < \infty \end{cases}$$

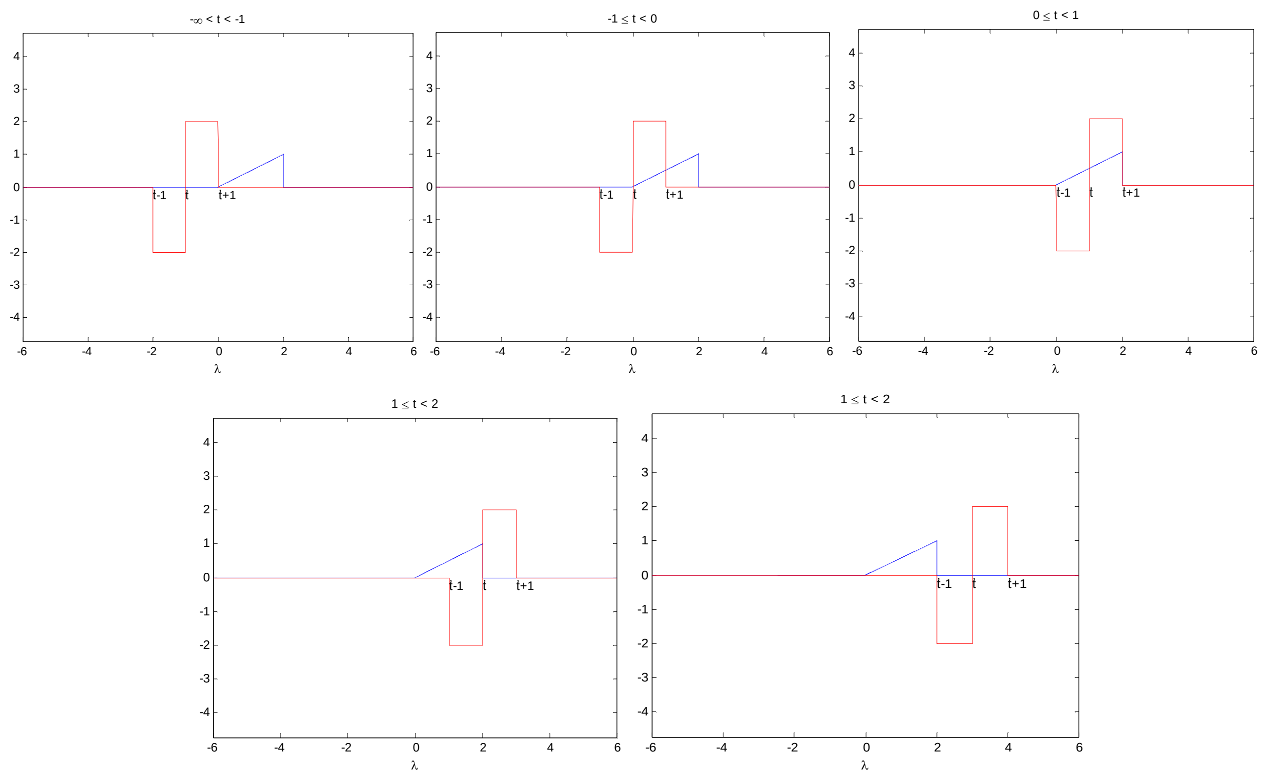


$\Rightarrow$



5. Determine the convolution of $x_1(t)$ and $x_2(t)$.





Case 1: $-1 \le t < 0$

$$g(-1 \le t < 0) = \int_0^{t+1} \frac{\lambda}{2} 2 d\lambda = \frac{(t+1)^2}{2}$$

Case 2: $0 \le t < 1$

$$g(0 \le t < 1) = \int_t^{t+1} \frac{\lambda}{2} 2 d\lambda - \int_0^{t} \frac{\lambda}{2} 2 d\lambda = \frac{t^2+2t+1}{2} - \frac{t^2}{2} - \frac{t^2}{2} = \frac{-t^2+2t+1}{2}$$

Case 3: $1 \le t < 2$

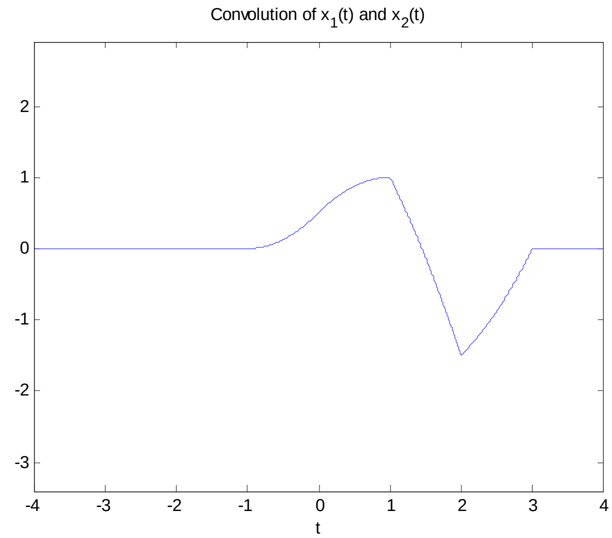
$$g(1 \le t < 2) = \int_t^{2} \frac{\lambda}{2} 2 d\lambda - \int_{t-1}^{t} \frac{\lambda}{2} 2 d\lambda = \frac{4-t^2}{2} - \frac{t^2-(t-1)^2}{2} = \frac{4-t^2}{2} - \frac{t^2-(t^2-2t+1)}{2} = \frac{5-2t-t^2}{2}$$

Case 4: $2 \le t < 3$

$$g(2 \le t < 3) = -\int_{t-1}^{2} \frac{\lambda}{2} 2 d\lambda = -\left[\frac{4}{2} - \frac{(t-1)^2}{2}\right] = \frac{t^2-2t-3}{2}$$

Final solution:

$$g(t)=\begin{cases}0 & t<-1\\ \dfrac{(t+1)^2}{2} & -1\le t<0\\ \dfrac{-t^2+2t+1}{2} & 0\le t<1\\ \dfrac{5-2t-t^2}{2} & 1\le t<2\\ \dfrac{t^2-2t-3}{2} & 2\le t<3\\ 0 & t\ge 3\end{cases}$$



6. Determine the convolution of the unit ramp with $x_2(t) = e^{-Bt}u(t)$

$$g(t)=\int_{-\infty}^{\infty} e^{-B\lambda}u(\lambda)ramp(t-\lambda)d\lambda=\int_{-\infty}^{\infty} e^{-B\lambda}u(\lambda)(t-\lambda)u(t-\lambda)d\lambda=t\int_0^t e^{-B\lambda}d\lambda-\int_0^t \lambda e^{-B\lambda}d\lambda$$

$$t\left[-\frac{e^{-B\lambda}}{B}\right]_0^t u(t)-\left[\frac{1}{B^2}(-B\lambda-1)e^{-B\lambda}\right]_0^t u(t)=\left[\frac{t-te^{-Bt}}{B}+\left(\frac{Bte^{-Bt}+e^{-Bt}-1}{B^2}\right)\right]u(t)$$

$$=\left(\frac{Bt+e^{-Bt}-1}{B^2}\right)u(t)$$