2704: Signals and Systems

Homework #4
Solutions

19. (a)

$y'(t) + 5y(t) = x(t)$

The standard solution for a First-Order Differential Equation is:
$y(t) = Ke^{-at}u(t)$, where $a = 5$.

To find the impulse response,

$$h'(t) + 5h(t) = \delta(t)$$

$$h(0^+) - h(0^-) + a\int_{0^-}^{0^+} h(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

$$Since \int_{0^-}^{0^+} h(t)dt = 0$$

$$h(0^+) - h(0^-) = 1$$

Since we are looking at the impulse response, $h(0^-)$ =0 since nothing exists before t = 0.

This gives us $h(0^+) = 1 = Ke^{-a(0)} = K$
This finally gives us:
$h(t) = e^{-5t}u(t)$

(b)

$$y''(t) + 6y'(t) + 4y(t) = x(t)$$
$$h''(t) + 6h'(t) + 4h(t) = \delta(t)$$
$$\therefore h(t) = \left(K_1e^{-at} + K_2e^{-bt}\right)\cdot u(t)$$
$$h'(0^+) - h'(0^-) + 6\left[h(0^+) - h(0^-)\right] + 4\int_{0^-}^{0^+} h(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

We know that the impulse response cannot contain an impulse because its second derivative would be a triplet and there is no triplet excitation. We also know that the response cannot be discontinuous at time $t = 0$ because if it were the second derivative would be a doublet and there is no doublet excitation. Therefore,

$$h'(0^+) - h'(0^-) = 1 \rightarrow h'(0^+) = 1$$

Therefore,

$$\lambda^2 + 6\lambda + 4 = 0 \rightarrow \lambda = [-0.764, -5.24] \therefore h'(t) = -0.764K_1e^{-0.764t} - 5.24K_2e^{-5.24}$$
$$h'(0^+) = 1 = -5.24K_1 - 0.76K_2 = 1$$

We can also show:

$$h(0^+) = 0 = K_1e^{-5.24(0)} + K_2e^{-0.76(0)} = K_1 + K_2 \rightarrow K_1 = -K_2 \therefore K_1 = -0.2236, K_2 = 0.2236$$

Thus, the total response is:

$$h(t) = 0.2236\left(e^{-0.76t} - e^{-5.24t}\right)u(t)$$

(c)

$$2y'(t)+3y(t)=x'(t)$$

Thus, we know $h_h(t)=K_h e^{-\frac{3}{2}t}$. The impulse response will be of the form

$$h(t)=K_h e^{-\frac{3}{2}t}u(t)+K_i\delta(t)$$

Integrating from 0^- to 0^+,

$$2\left[h(0^+)-h(0^-)\right]+3\int_{0^-}^{0^+}h(t)dt=x(0^+)-x(0^-)=0=2K_h+3K_i$$

Integrating a second time from 0^- to 0^+,

$$2\int_{0^-}^{0^+}h(t)dt=\int_{0^-}^{0^+}\delta(t)dt=1\rightarrow 2K_i=1\therefore K_i=\frac{1}{2}\therefore K_h=-\frac{3}{4}$$

Therefore,

$$h(t)=-\frac{3}{4}e^{-\frac{3}{2}t}u(t)+\frac{1}{2}\delta(t)$$

(d)

$$4y'(t)+9y(t)=2x(t)+x'(t)$$

Thus, we know $h_h(t)=K_h e^{-\frac{9}{4}t}$, and the impulse response is of the form

$$h(t)=K_h e^{-\frac{9}{4}t}u(t)+K_i\delta(t)$$

Integrating from 0^- to 0^+,

$$4\left[h(0^+)-h(0^-)\right]+9\int_{0^-}^{0^+}h(t)dt=2\int_{0^-}^{0^+}\delta(t)dt+x(0^+)-x(0^-)=2=4K_h+9K_i$$

Integrating a second time from 0^- to 0^+,

$$4\int_{0^-}^{0^+}h(t)dt=1\rightarrow 4K_i=1\therefore K_i=\frac{1}{4}\therefore K_h=-\frac{1}{16}$$

Therefore,

$$h(t)=-\frac{1}{16}e^{-\frac{9}{4}t}u(t)+\frac{1}{4}\delta(t)$$

23)

$$h(t) = 4e^{-4t}u(t)$$

$$x(t) = rect\left(2\left(t - \frac{1}{4}\right)\right) = rect\left(\frac{4t-1}{2}\right)$$

If you are unsure how this *rect* function will look, remember that *rect*(t) is on at –½ to +½, so if we just check the arguments for those values, we see this *rect* is one high for t = [0, 0.5].

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

Since we are dealing with a rect and an infinite length function, we will have three ranges:

1) $(t+0.5) < 0$: No intersection
2) $0 \le t \le 0.5$: Entering
3) $t > 0.5$: Fully intersecting

$0 \le t \le 0.5$

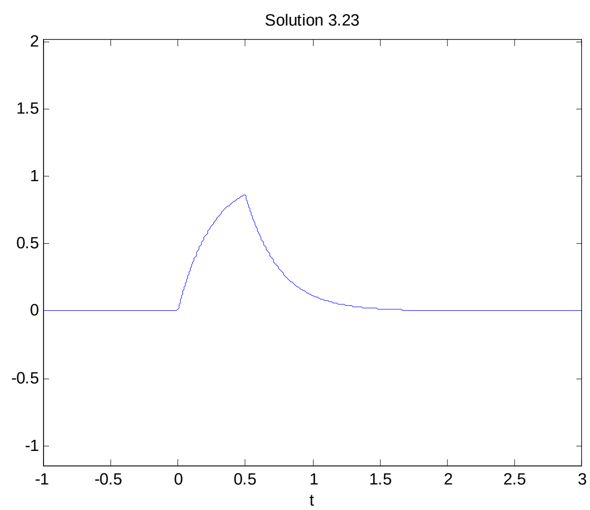
$$y_1(t) = \int_0^t 4e^{-4\lambda}d\lambda = -e^{-4\lambda}\Big|_0^t = 1 - e^{-4t}$$

t > 0.5

$$y_2(t) = \int_{t-0.5}^{t} 4e^{-4\lambda}d\lambda = -e^{-4\lambda}\Big|_{t-0.5}^{t} = e^{-4t+2} - e^{-4t}$$

Therefore

$$y(t) = \begin{cases} 0 & o.w. \\ 1 - e^{-4t} & 0 \le t \le 0.5 \\ e^{-4t+2} - e^{-4t} & t > 0.5 \end{cases}$$



24)

$$h(t) = \delta(t) - 4e^{-4t}u(t)$$

$$x(t) = rect\left(2\left(t - \frac{1}{4}\right)\right) = rect\left(\frac{4t-1}{2}\right)$$

We can quickly solve this problem by using the solution above. We know that the only difference is that we will be retaining a copy of the original input (the *rect* function) and adding that to a flipped version of the result.

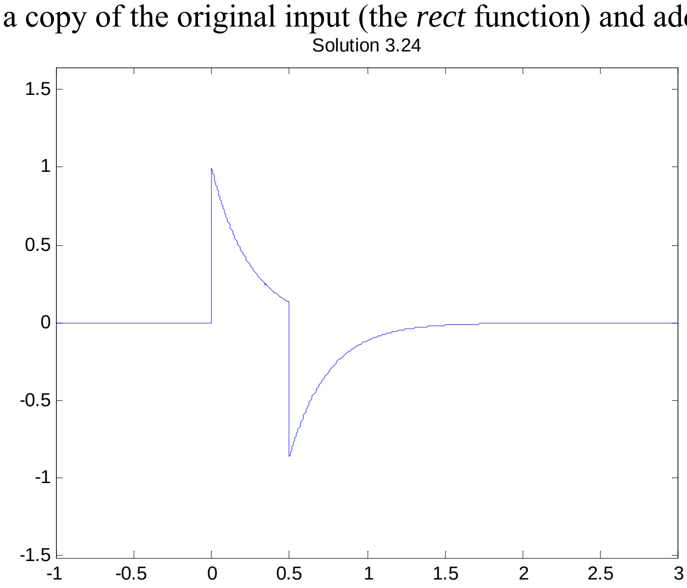
Recall:

$$y_{3.23}(t) = \begin{cases} 0 & o.w. \\ 1 - e^{-4t} & 0 \le t \le 0.5 \\ e^{-4t+2} - e^{-4t} & t > 0.5 \end{cases}$$

$$y(t) = x(t) - y_{3.23}(t)$$

Therefore,

$$y(t) = \begin{cases} 0 & o.w. \\ e^{-4t} & 0 \le t \le 0.5 \\ e^{-4t} - e^{-4t+2} & t > 0.5 \end{cases}$$

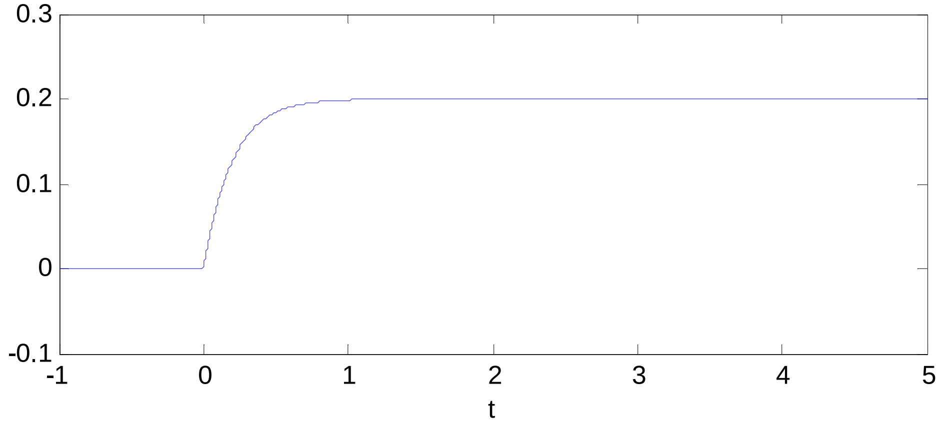


31)

(a) $h(t)=e^{-5t}u(t)$

$$h_{-1}=\int_{-\infty}^{\infty}e^{-5\lambda}u(\lambda)u(t-\lambda)d\lambda=\int_{0}^{t}e^{-5\lambda}d\lambda=\frac{1-e^{-5t}}{5}u(t)$$

Solution 3.31(a)

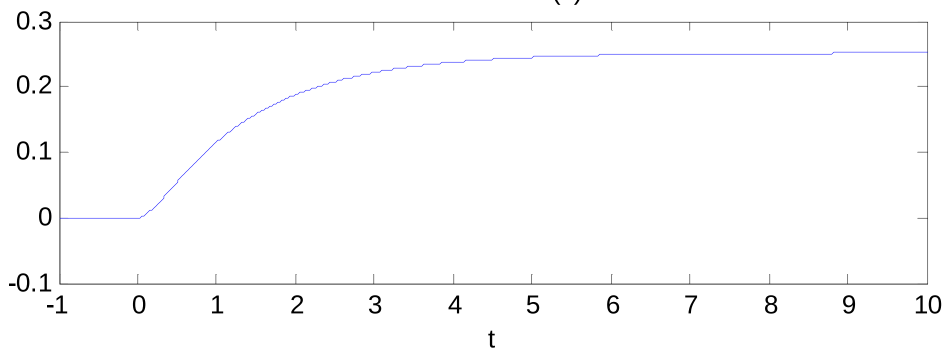


(b) $h(t)=0.2236\left(e^{-0.76t}-e^{-5.24t}\right)u(t)$

$$h_{-1}=\int_{-\infty}^{\infty}0.2236\left(e^{-0.76\lambda}-e^{-5.24\lambda}\right)u(\lambda)u(t-\lambda)d\lambda=\int_{0}^{t}0.2236\left(e^{-0.76\lambda}-e^{-5.24\lambda}\right)d\lambda$$

$$=0.2236\left(1.316\left(1-e^{-0.76t}\right)-0.1912\left(1-e^{-5.23t}\right)\right)\cdot u(t)$$

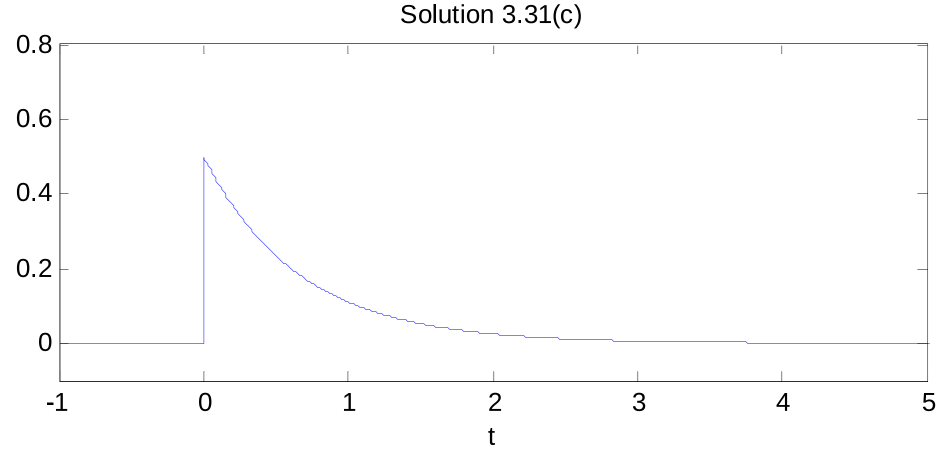
Solution 3.31(b)



(c) $h(t)=\frac{1}{2}\delta(t)-\frac{3}{4}e^{-\frac{3}{2}t}u(t)$

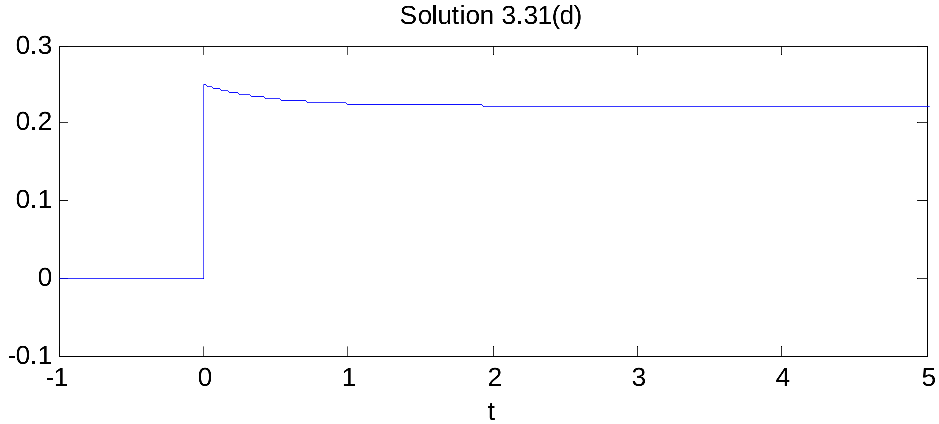
$$h_{-1}=\int_{-\infty}^{\infty}\left[\frac{1}{2}\delta(\lambda)-\frac{3}{4}e^{-\frac{3}{2}\lambda}u(\lambda)\right]u(t-\lambda)d\lambda=\frac{1}{2}-\frac{3}{4}\int_{0}^{t}e^{-\frac{3}{2}\lambda}d\lambda=\frac{1}{2}u(t)+\frac{e^{-\frac{3}{2}t}-1}{2}u(t)=\frac{1}{2}e^{-\frac{3}{2}t}u(t)$$

Solution 3.31(c)



(d) $h(t)=\frac{1}{4}\delta(t)-\frac{1}{16}e^{-\frac{9}{4}t}u(t)$

$$h_{-1}=\int_{-\infty}^{\infty}\left[\frac{1}{4}\delta(\lambda)-\frac{1}{16}e^{-\frac{9}{4}\lambda}u(\lambda)\right]u(t-\lambda)d\lambda=\frac{1}{4}-\frac{1}{16}\int_{0}^{t}e^{-\frac{9}{4}\lambda}d\lambda=\frac{1}{4}u(t)+\frac{e^{-\frac{9}{4}t}-1}{36}u(t)=\frac{2}{9}u(t)+\frac{1}{36}e^{-\frac{9}{4}t}u(t)$$

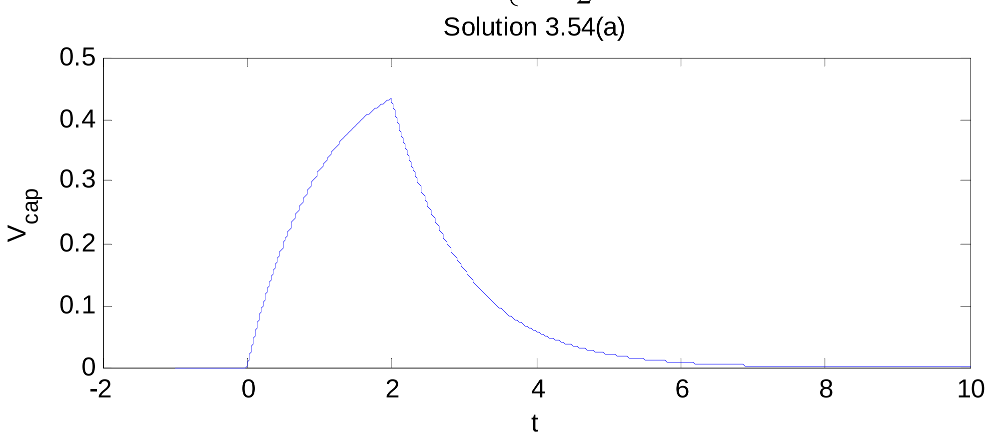


54) From the example in the lecture, we know that for an RC circuit with the voltage measured across C the impulse response is:

$$h(t)=\frac{1}{RC}e^{-\frac{1}{RC}t}u(t)=\frac{1}{(10k\Omega)(100\mu F)}e^{-\frac{1}{(10k\Omega)(100\mu F)}t}u(t)=e^{-t}u(t)$$

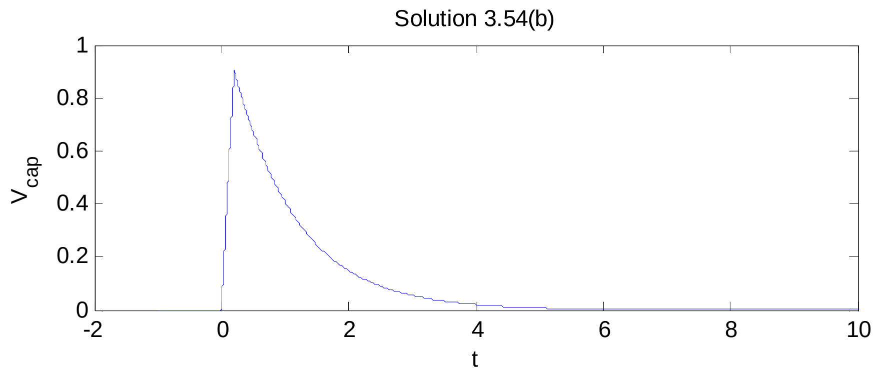
(a) The pulse can be written as $V_{in}(t)=\frac{1}{2}rect\left(\frac{t-1}{2}\right)$

$$V_{cap}(t)=\frac{1}{2}\int_{-\infty}^{\infty}e^{-\lambda}u(\lambda)rect\left(\frac{(t-\lambda)-1}{2}\right)d\lambda=\begin{cases}0 & t<0\\ \frac{1-e^{-t}}{2} & 0\le t\le 2\\ \frac{e^{-t+2}-e^{-t}}{2} & t>2\end{cases}$$



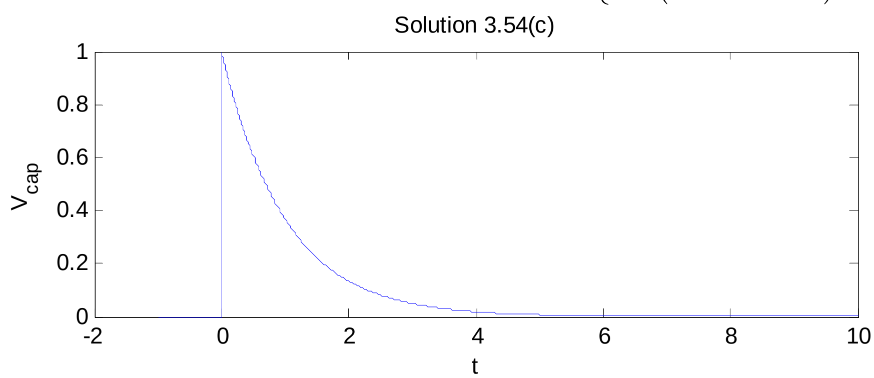
(b) The pulse can be written as $V_{in}(t) = 5rect\left(\frac{t-0.1}{0.2}\right)$

$$V_{cap}(t) = 5\int_{-\infty}^{\infty} e^{-\lambda}u(\lambda)rect\left(\frac{(t-\lambda)-0.1}{0.2}\right)d\lambda = \begin{cases} 0 & t<0 \\ 5-5e^{-t} & 0 \le t \le 0.2 \\ 5\left(e^{-t+0.2}-e^{-t}\right) & t>0.2 \end{cases}$$



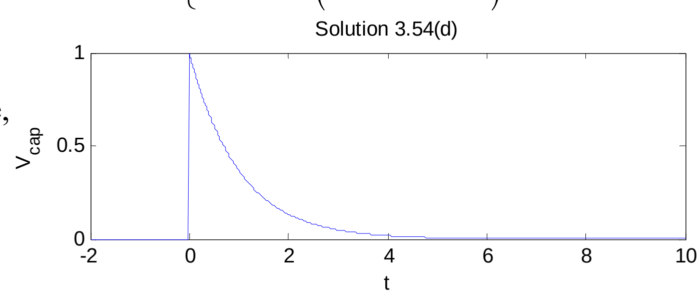
(c) The pulse can be written as $V_{in}(t) = 50rect\left(\frac{t-0.001}{0.002}\right)$

$$V_{cap}(t) = 500\int_{-\infty}^{\infty} e^{-\lambda}u(\lambda)rect\left(\frac{(t-\lambda)-0.001}{0.002}\right)d\lambda = \begin{cases} 0 & t<0 \\ 500-500e^{-t} & 0 \le t \le 0.002 \\ 500\left(e^{-t+0.002}-e^{-t}\right) & t>0.002 \end{cases}$$



(d) 2×10^{-6} The pulse can be written as $V_{in}(t) = 500\times10^{3}rect\left(\frac{t-10^{-6}}{2\times10^{-6}}\right)$

$$V_{cap}(t) = 500\times10^{3}\int_{-\infty}^{\infty} e^{-\lambda}u(\lambda)rect\left(\frac{(t-\lambda)-10^{-6}}{2\times10^{-6}}\right)d\lambda = \begin{cases} 0 & t<0 \\ 500\times10^{3}-500\times10^{3}e^{-t} & 0 \le t \le 2\times10^{-6} \\ 500\times10^{3}\left(e^{-t+2\times10^{-6}}-e^{-t}\right) & t>2\times10^{-6} \end{cases}$$



From these results, we see that as the voltage pulse tends towards an impulse, the output tends towards the impulse response (as expected).

56)

(a)

$$y(t) = \frac{d(t)}{R}$$

$$\frac{d}{dt}\left(\underbrace{Ad(t)}_{volume}\right) = x(t) - y(t)$$

$$Ad'(t) = x(t) - \frac{d(t)}{R}$$

$$Ad'(t) + \frac{d(t)}{R} = x(t)$$

(b)

Find when $d'(t) = 0$.

$$d(t) = Rx(t) = 10\frac{s}{m^2} \times 0.05\frac{m^3}{s} = 0.5m$$

(c)

Dumping 1 m^3 of water into an empty tank is exciting this system with a unit impulse of water inflow. The impulse response, $h(t)$, of the system is the solution of

$$Ah'(t) + \frac{h(t)}{R} = \delta(t) \rightarrow h(t) = Ke^{-\frac{t}{AR}}u(t)$$

We can find K by finding the initial water depth in response to 1 m^3 being suddenly dumped in. The surface area is 0.7854 m^2. Therefore the initial depth is 1.273 m and

$$h(t) = 1.273e^{-\frac{t}{AR}}u(t)$$

(d)

The response to a step of flow is the convolution of the impulse response with the step excitation

$$d(t) = h(t) * x(t) = 1.273e^{-\frac{t}{AR}}u(t) * 0.2u(t)$$

$$d(t) = 0.2546\int_{-\infty}^{\infty} e^{-\frac{\lambda}{AR}}u(\lambda)u(t-\lambda)d\lambda = 0.2546\int_{0}^{t} e^{-\frac{\lambda}{AR}}d\lambda = -0.2546AR\left(1 - e^{-\frac{t}{AR}}\right)$$

Therefore:

$$d(t) = 2\left(1 - e^{-\frac{t}{7.854}}\right)u(t) = 1.5m \rightarrow t = 10.886s$$

61)

(a) Summing forces:

$$f(t) - mg\sin(\theta) - k_f y'(t) = my''(t) \rightarrow f(t) = mg\sin(\theta) + k_f y'(t) + my''(t)$$

(b)

The zero-excitation response can be found by setting the force, *f*(*t*), to zero:

$$-mg\sin(\theta) = k_f y'(t) + my''(t)$$

Thus, $y_h(t) = K_{h1} + K_{h2}e^{-\frac{k_f}{m}t}$. The particular solution must be in the form of a linear function of t, to satisfy the differential equation. Choosing the form,

$$y_p(t) = K_p t$$

And solving, $K_p = -\frac{mg}{k_f}\sin(\theta)$. Thus, the total zero-excitation response is

$$y(t) = y_h(t) + y_p(t) = K_{h1} + K_{h2}e^{-\frac{k_f}{m}t} - \left(\frac{mg}{k_f}\sin(\theta)\right)t$$

Using the initial conditions,

$$y(0) = 0 = K_{h1} + K_{h2}$$

and

$$y'(0) = 10 = -\frac{k_f}{m}K_{h2} - \frac{mg}{k_f}\sin(\theta)$$

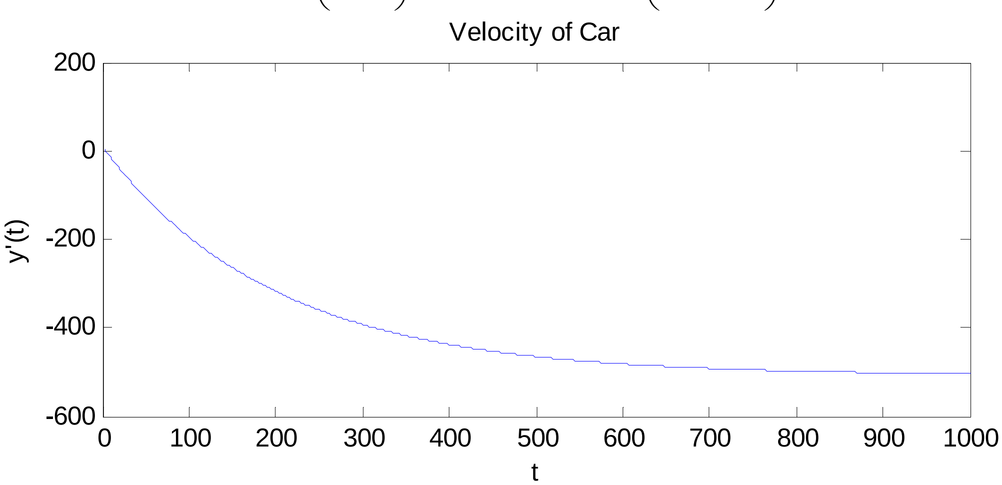
Solving yields:

$$K_{h2} = -\left(\frac{m}{k_f}\right)^2 g\sin(\theta) - \frac{m}{k_f}10 = -1.0146\times10^5 - 2000 = -1.0346\times10^5$$

$$K_{h1} = 1.0346\times10^5$$

$$y(t) = 1.0346\times10^5\left(1 - e^{-\frac{t}{200}}\right) - 507.28t$$

$$y'(t) = \frac{1.0346\times10^5}{200}\left(e^{-\frac{t}{200}}\right) - 507.28 = 517.28\left(e^{-\frac{t}{200}} - 1\right) + 10$$



(c)

We can write:

$$x(t) = f(t) - mg\sin(\theta) = 200 - mg\sin(\theta) = k_f y'(t) + my''(t)$$

Then the impulse response of the system is the solution of:

$$mh''(t) + k_f h'(t) = \delta(t)$$

which is of the form:

$$h(t) = \left(K_{h1} + K_{h2} e^{-\frac{k_f}{m}t} \right) u(t)$$

Integrating both sides of the equation from 0^- to 0^+,

$$mh'(0^+) + k^f h(0^+) = 1 = m\left(-\frac{k_f}{m} K_{h2} \right) + k_f \left(K_{h1} + K_{h2} \right)$$

$$= -k_f K_{h2} + k_f \left(K_{h1} + K_{h2} \right) = k_f K_{h1} \therefore K_{h1} = \frac{1}{k_f}$$

Integrating a second time gives us,

$$mh\left(0^+\right) = 0 = m\left(K_{h1} + K_{h2} \right) \rightarrow K_{h2} = -K_{h1} = -\frac{1}{k_f}$$

Therefore, the impulse response is,

$$h(t) = \frac{1 - e^{-\frac{k_f}{m}t}}{k_f} u(t)$$

Recall at the beginning we defined the excitation as,

$$x(t) = 200 - mg\sin(\theta)$$

For convenience sake, we will assume the car was held in place prior to the 200-newton push at time $t = 0$, making the excitation:

$$x(t) = \left(200 - mg\sin(\theta)\right) u(t)$$

Therefore,

$$y(t) = x(t) * h(t) = \left[200 - mg\sin(\theta)\right] u(t) * \frac{1 - e^{-\frac{k_f}{m}t}}{k_f} u(t) = \frac{200 - mg\sin(\theta)}{k_f} \int_0^t \left(1 - e^{-\frac{k_f}{m}\lambda} \right) d\lambda$$

Solving gives us,

$$y(t) = \frac{200 - mg\sin(\theta)}{k_f} \left[t + \frac{m}{k_f} e^{-\frac{k_f}{m}t} - \frac{m}{k_f} \right]$$

Terminal velocity is defined as $\lim_{t \to \infty} \frac{dy(t)}{dt}$. So to find the terminal velocity:

$$\lim_{t \to \infty} \frac{dy(t)}{dt} = \lim_{t \to \infty} \left\{ \left(\frac{200 - mg\sin(\theta)}{k_f} \right) \left[1 - e^{-\frac{k_f}{m}t} \right] \right\} = \frac{200 - mg\sin(\theta)}{k_f} = \frac{200 - 2536.43}{5} = -467.3 \, m/s$$