2704: Signals and Systems

Homework #5
Solutions

3) (a) x(t) = tri(t)

$$X(f) = \int_{-\infty}^{\infty} tri(t)e^{-j2\pi ft}dt = \int_{-1}^{1}\left(1-|t|\right)e^{-j2\pi ft}dt = \int_{-1}^{1}\underbrace{\left(1-|t|\right)}_{even}\left(\underbrace{\cos(2\pi ft)}_{even} - \underbrace{j\sin(2\pi ft)}_{odd}\right)dt$$

$$X(f) = 2\int_{0}^{1}\underbrace{\left(1-|t|\right)}_{even}\underbrace{\cos(2\pi ft)}_{even}dt = 2\int_{0}^{1}\cos(2\pi ft)dt - 2\int_{0}^{1}t\cos(2\pi ft)dt$$

$$X(f) = 2\int_{0}^{1}\cos(2\pi ft)dt - 2\int_{0}^{1}t\cos(2\pi ft)dt = 2\left\{\left.\frac{\sin(2\pi ft)}{2\pi f}\right|_0^1 - \left[\left.t\frac{\sin(2\pi ft)}{2\pi f}\right|_0^1 + \left.\frac{\cos(2\pi ft)}{(2\pi f)}\right|_0^1\right]\right\}$$

$$X(f) = 2\left\{\underbrace{\left.\frac{\sin(2\pi ft)}{2\pi f}\right|_0^1 - \left.t\frac{\sin(2\pi ft)}{2\pi f}\right|_0^1}_{=0} - \left.\frac{\cos(2\pi ft)}{(2\pi f)^2}\right|_0^1\right\} = -2\left.\frac{\cos(2\pi ft)}{(2\pi f)^2}\right|_0^1 = \frac{1-\cos(2\pi f)}{2(\pi f)^2}$$

Since $\sin^2(x) = \frac{1-\cos(2x)}{2}$, we know $X(f) = \frac{1-\cos(2\pi ft)}{2(\pi f)^2} = \frac{\sin^2(\pi f)}{(\pi f)^2} = \text{sinc}^2(f)$

(b) $x(t) = \delta\left(t+\frac{1}{2}\right) - \delta\left(t-\frac{1}{2}\right)$

$$X(f) = \int_{-\infty}^{\infty}\left[\delta\left(t+\frac{1}{2}\right) - \delta\left(t-\frac{1}{2}\right)\right]e^{-j2\pi ft}dt = e^{j\pi f} - e^{-j\pi f} = j2\sin(\pi f)$$

6) $x(t) = A\sin(\omega_0 t) \xleftrightarrow{CTFS} X(j\omega) = \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} Ae^{-\sigma|t|}\sin(\omega_0 t)e^{-j\omega t}dt$

$$X(j\omega) = A\lim_{\sigma\to 0^+}\left[\overbrace{\int_{-\infty}^{\infty} e^{-\sigma|t|}\sin(\omega_0 t)\cos(\omega t)dt}^{=0} - j\int_{-\infty}^{\infty} e^{-\sigma|t|}\sin(\omega_0 t)\sin(\omega t)dt\right]$$

$$X(j\omega) = -j2A\lim_{\sigma\to 0^+}\int_0^{\infty} e^{-\sigma|t|}\sin(\omega_0 t)\sin(\omega t)dt = -j2A\lim_{\sigma\to 0^+}\int_0^{\infty} e^{-\sigma|t|}\left\{\frac{\cos[(\omega_0-\omega)t]-\cos[(\omega_0+\omega)t]}{2}\right\}dt$$

$$X(j\omega) = -jA\lim_{\sigma\to 0^+}\int_0^{\infty} e^{-\sigma|t|}\{\cos[(\omega_0-\omega)t]-\cos[(\omega_0+\omega)t]\}dt$$

$$X(j\omega) = -jA\lim_{\sigma\to 0^+}\left[\pm\frac{e^{-\sigma t}}{\sigma^2+(\omega_0\mp\omega)^2}(\omega\sin[(\omega_0\mp\omega)t]-\sigma\cos[(\omega_0\mp\omega)t])\right]$$

$$X(j\omega) = -jA\left[\lim_{\sigma\to 0^+}\frac{\sigma}{\sigma^2+(\omega_0-\omega)^2} - \lim_{\sigma\to 0^+}\frac{\sigma}{\sigma^2+(\omega_0+\omega)^2}\right]$$

When $\omega_0 \neq \omega$, $\lim_{\sigma\to 0^+}\frac{\sigma}{\sigma^2+(\omega_0-\omega)^2} = 0$, and when $\omega_0 \neq -\omega$, $\lim_{\sigma\to 0^+}\frac{\sigma}{\sigma^2+(\omega_0+\omega)^2} = 0$

The area under these functions can be found as:

$$Area = \lim_{\sigma\to 0^+}\int_{-\infty}^{\infty}\frac{\sigma}{\sigma^2+(\omega_0\pm\omega)^2}d\omega \underset{\omega_0\pm\omega=\lambda}{=} \lim_{\sigma\to 0^+}\int_{-\infty}^{\infty}\frac{\sigma}{\sigma^2+(\lambda)^2}d\lambda = \pi$$

Therefore,

$$X(j\omega) = -jA[\pi\delta(\omega-\omega_0)-\pi\delta(\omega+\omega_0)] = j\pi A[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

7) (a) From Appendix E, we see that the CTFS is

$$A\cos(2\pi f_0 t) \xleftrightarrow{CTFS} \frac{A}{2}(\delta[k-1]+\delta[k+1])$$

Similarly, the CTFT is

$$A\cos(2\pi f_0 t) \xleftrightarrow{CTFT} \frac{A}{2}(\delta(f-f_0)+\delta(f+f_0)) = X[1]\delta(f-f_0)+X[-1]\delta(f+f_0)$$

(b) $x(t) = comb(t)$

The CTFS is found from

$$X[k] = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} comb(t)e^{-j2\pi k f_0 t}dt = \frac{1}{1}\int_{\frac{-1}{2}}^{\frac{1}{2}}\delta(t)e^{-j2\pi k f_0 t}dt = 1$$

The CTFT is found from

$$X(f) = comb(f) = \sum_{k=-\infty}^{\infty}\delta(f-k) = \sum_{k=-\infty}^{\infty}X[k]\delta(f-kf_0)$$

Finally observation and deduction reveals a general formula is,

$$X(f) = \sum_{k=-\infty}^{\infty}X[k]\delta(f-kf_0)$$

8)

$$x\left(t-\frac{1}{40}\right)=2\cos\left(4\pi\left(t-\frac{1}{40}\right)\right)+5\cos\left(15\pi\left(t-\frac{1}{40}\right)\right)$$

$$x\left(t-\frac{1}{40}\right)=2\cos\left(4\pi t-\underbrace{\frac{\pi}{10}}_{\text{phase shift}}\right)+5\cos\left(15\pi t-\underbrace{\frac{3\pi}{8}}_{\text{phase shift}}\right)$$

$$x\left(t-\frac{1}{40}\right)\xleftrightarrow{\mathcal{F}}\left[\delta(f-2)+\delta(f+2)\right]e^{-j\frac{\pi f}{20}}+\frac{5}{2}\left[\delta\left(f-\frac{15}{2}\right)+\delta\left(f+\frac{15}{2}\right)\right]e^{-j\frac{3\pi f}{8}}$$

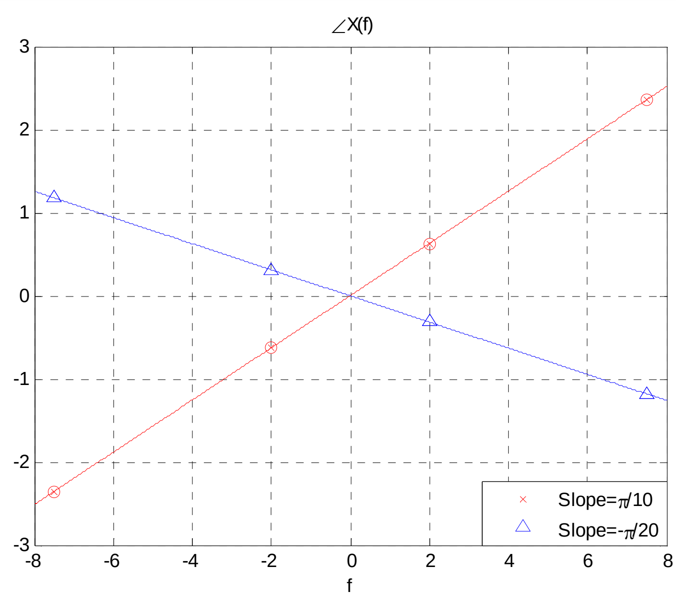
$$x\left(t-\frac{1}{40}\right)\xleftrightarrow{\mathcal{F}}\delta(f-2)e^{-j\frac{\pi}{10}}+\delta(f+2)e^{+j\frac{\pi}{10}}+\frac{5}{2}\delta\left(f-\frac{15}{2}\right)e^{-j\frac{45\pi}{16}}+\frac{5}{2}\delta\left(f+\frac{15}{2}\right)e^{+j\frac{45\pi}{16}}$$

$$x\left(t+\frac{1}{20}\right)=2\cos\left(4\pi\left(t+\frac{1}{20}\right)\right)+5\cos\left(15\pi\left(t+\frac{1}{20}\right)\right)$$

$$x\left(t+\frac{1}{20}\right)=2\cos\left(4\pi t+\underbrace{\frac{\pi}{5}}_{\text{phase shift}}\right)+5\cos\left(15\pi t+\underbrace{\frac{3\pi}{4}}_{\text{phase shift}}\right)$$

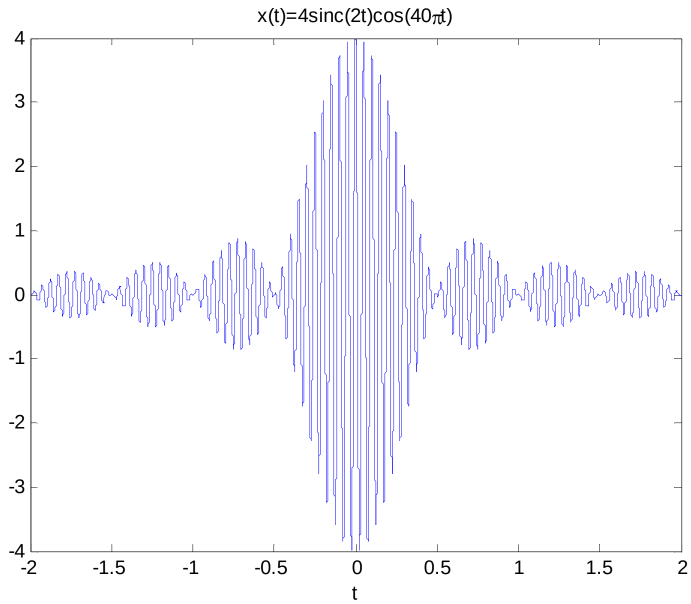
$$x\left(t+\frac{1}{20}\right)\xleftrightarrow{\mathcal{F}}\left[\delta(f-2)+\delta(f+2)\right]e^{+j\frac{\pi f}{10}}+\frac{5}{2}\left[\delta\left(f-\frac{15}{2}\right)+\delta\left(f+\frac{15}{2}\right)\right]e^{+j\frac{3\pi f}{4}}$$

$$x\left(t+\frac{1}{20}\right)\xleftrightarrow{\mathcal{F}}\delta(f-2)e^{+j\frac{\pi}{5}}+\delta(f+2)e^{-j\frac{\pi}{5}}+\frac{5}{2}\delta\left(f-\frac{15}{2}\right)e^{+j\frac{45\pi}{8}}+\frac{5}{2}\delta\left(f+\frac{15}{2}\right)e^{-j\frac{45\pi}{8}}$$



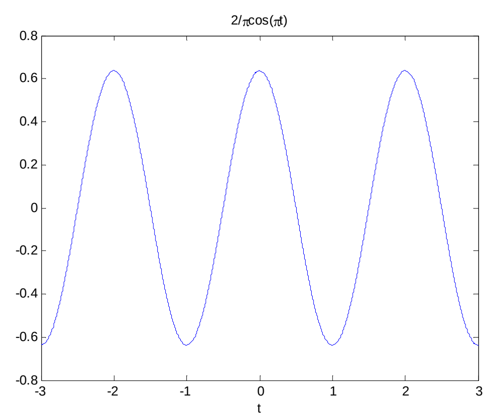
In general, the slope of the line is -2π*f* times the delay

9) $x(t) = 2\mathrm{sinc}(2t)e^{j2(20)\pi t} + 2\mathrm{sinc}(2t)e^{-j2(20)\pi t} = 4\mathrm{sinc}(2t)\cos(40\pi t)$

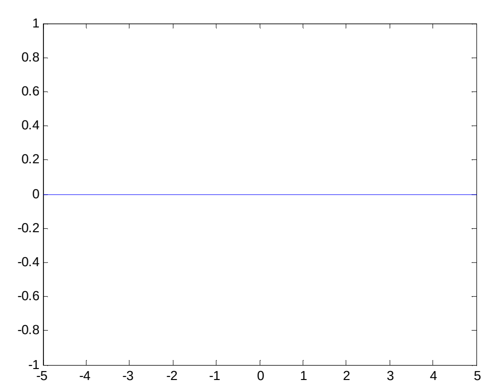


11) (a) $y(t) = rect(t) * \cos(\pi t) \xleftrightarrow{CTFT} \mathrm{sinc}(f)\dfrac{\delta(f + 1/2) + \delta(f - 1/2)}{2}$

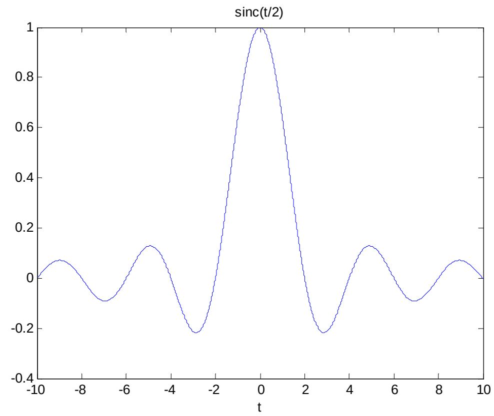
$$\mathrm{Y}(f) = \frac{1}{2}\mathrm{sinc}(-\frac{1}{2})\,\delta(f + 1/2) + \frac{1}{2}\mathrm{sinc}(\frac{1}{2})\,\delta(f - 1/2) = \frac{1}{\pi}\left[\delta\left(f - \frac{1}{2}\right) + \delta\left(f + \frac{1}{2}\right)\right] \xleftrightarrow{ICTFT} \frac{2}{\pi}\cos(\pi t)$$



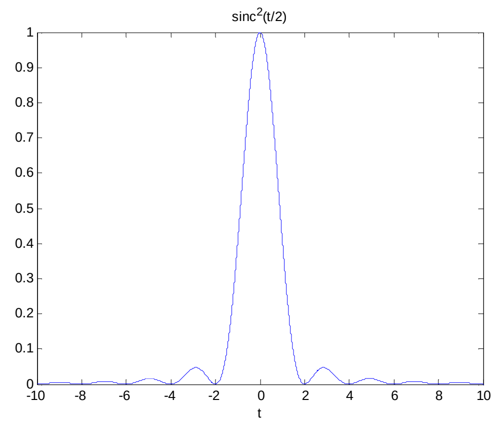
(b) $y(t) = rect(t) * \cos(\pi t) \xleftrightarrow{CTFT} \mathrm{sinc}(f)\dfrac{\delta(f + 1) + \delta(f - 1)}{2} = 0$



(c) $y(t)=\mathrm{sinc}(t)*\mathrm{sinc}\left(\frac{t}{2}\right)\xleftrightarrow{CTFT} rect(f)2rect(2f)=2rect(2f)\xleftrightarrow{ICTFT}\mathrm{sinc}\left(\frac{t}{2}\right)$



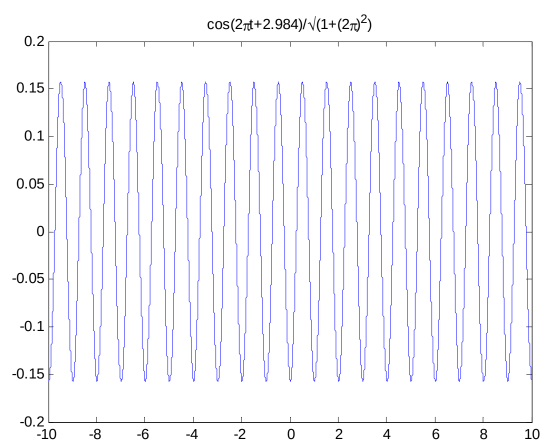
(d) $y(t)=\mathrm{sinc}(t)*\mathrm{sinc}^2\left(\frac{t}{2}\right)\xleftrightarrow{CTFT} rect(f)2tri(2f)=2tri(2f)\xleftrightarrow{ICTFT}\mathrm{sinc}^2\left(\frac{t}{2}\right)$



(e) $y(t)=e^{-t}u(t)*\sin(2\pi t)\xleftrightarrow{CTFT}\frac{1}{1+j2\pi f}\frac{j}{2}\left[\delta(f+1)-\delta(f-1)\right]=\frac{j}{2}\left[\frac{\delta(f+1)(1+j2\pi)-\delta(f-1)(1-j2\pi)}{(1-j2\pi)(1+j2\pi)}\right]$

$$=\frac{j}{2}\left[\frac{\delta(f+1)-\delta(f-1)+j2\pi\left[\delta(f+1)+\delta(f-1)\right]}{1+(2\pi)^2}\right]=\frac{j}{2}\frac{\delta(f+1)-\delta(f-1)}{1+(2\pi)^2}-\pi\frac{\delta(f+1)+\delta(f-1)}{1+(2\pi)^2}$$

$$=\frac{\sin(2\pi t)-2\pi\cos(2\pi t)}{1+(2\pi)^2}=\frac{\sqrt{1^2+(-2\pi)^2}\cos\left(2\pi t+\tan^{-1}\left(\frac{1}{-2\pi}\right)\right)}{1+(2\pi)^2}=\frac{\cos(2\pi t+2.984)}{\sqrt{1+(2\pi)^2}}$$



14)

(a) $\mathrm{x}(t) = 4\,\mathrm{sinc}\left(\frac{t}{5}\right)$

$$E_x = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2\,dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2\,df = \int_{-\infty}^{\infty} |20\,\mathrm{rect}(5f)|^2\,df = 400\int_{-\infty}^{\infty} \mathrm{rect}(5f)\,df$$

$$E_x = 400\int_{-\frac{1}{10}}^{\frac{1}{10}} df = 80$$

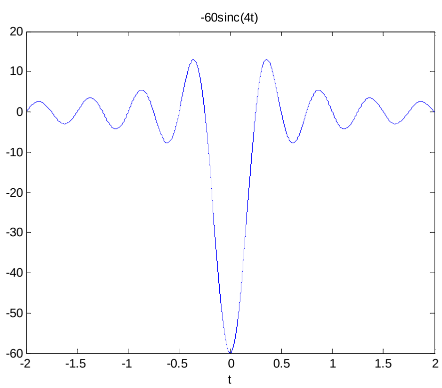
(b) $\mathrm{x}(t) = 2\,\mathrm{sinc}^2(3t)$

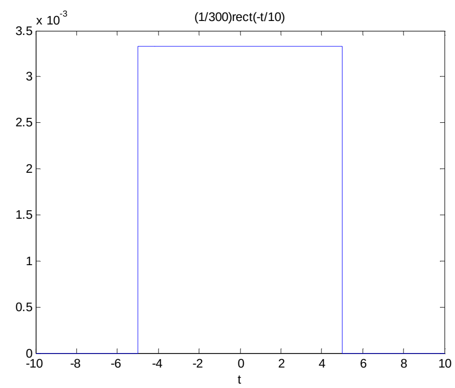
$$E_x = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2\,dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2\,df = \int_{-\infty}^{\infty} \left|\frac{2}{3}\mathrm{tri}\left(\frac{f}{3}\right)\right|^2 df = \frac{4}{9}\int_{-\infty}^{\infty} \mathrm{tri}^2\left(\frac{f}{3}\right)df$$

$$E_x = \frac{8}{9}\int_0^3 \mathrm{tri}^2\left(\frac{f}{3}\right)df = \frac{8}{9}\int_0^3 \left(1-\frac{f}{3}\right)^2 df = \frac{8}{9}\int_0^3 \left(1-\frac{2f}{3}+\frac{f^2}{9}\right)df$$

$$E_x = \frac{8}{9}\left[f-\frac{f^2}{3}+\frac{f^3}{27}\right]_0^3 = \frac{8}{9}\left[3-\frac{9}{3}+\frac{27}{27}\right] = \frac{8}{9}$$

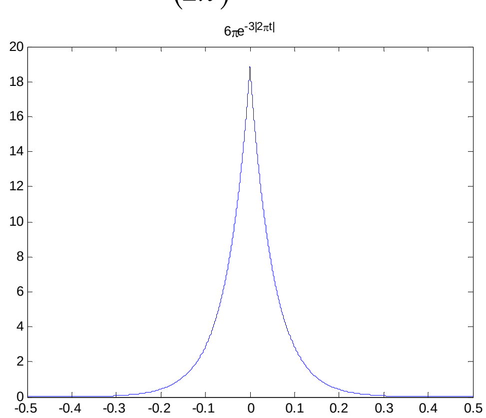
19) (a) $X(f) = -15\,rect\left(\frac{f}{4}\right) \xleftrightarrow{ICTFT} -60\,\mathrm{sinc}(4t)$

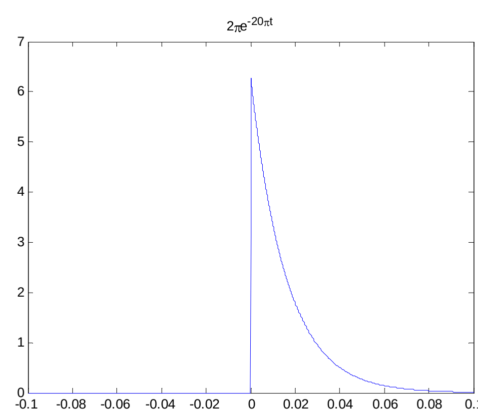




(b) $X(f) = \frac{1}{30}\,\mathrm{sinc(-10}f) \xleftrightarrow{ICTFT} \frac{1}{30}\left[\frac{1}{10}\,rect\left(\frac{t}{-10}\right)\right] = \frac{1}{300}\,rect\left(-\frac{t}{10}\right)$

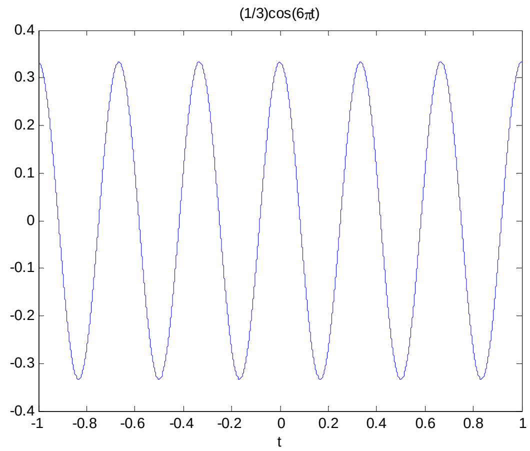
(c) $X(f) = \frac{18}{9+f^2} = \frac{2(3)^2}{(3)^2+\frac{(2\pi f)^2}{(2\pi)^2}} \xrightarrow{\lambda=\frac{f}{2\pi}} 3\frac{2(3)}{(3)^2+(2\pi\lambda)^2} \xleftrightarrow{ICTFT} 3(2\pi)e^{-3|2\pi t|} = 6\pi e^{-3|2\pi t|}$

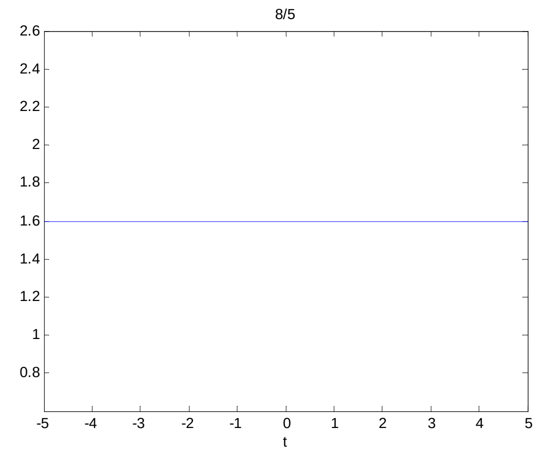




(d) $X(f)=\dfrac{1}{10+jf}\xrightarrow{\lambda=\frac{f}{2\pi}}\dfrac{1}{10+j2\pi\lambda}\xleftrightarrow{ICTFT}(2\pi)e^{-10(2\pi t)}u(t)=2\pi e^{-20\pi t}u(t)$

(e) $X(f)=\dfrac{\delta(f-3)+\delta(f+3)}{6}=\dfrac{1}{3}\dfrac{\delta(f-3)+\delta(f+3)}{2}\xleftrightarrow{ICTFT}\dfrac{1}{3}\cos\left(2\pi(3)t\right)=\dfrac{1}{3}\cos\left(6\pi t\right)$





(f) $X(f)=8\delta(5f)=\dfrac{8}{5}\delta(f)\xleftrightarrow{ICTFT}\dfrac{8}{5}$

(g) $X(f)=-\dfrac{3}{j\pi f}=-3\dfrac{1}{j\pi f}\xleftrightarrow{ICTFT}-3\,\mathrm{sgn}(t)$