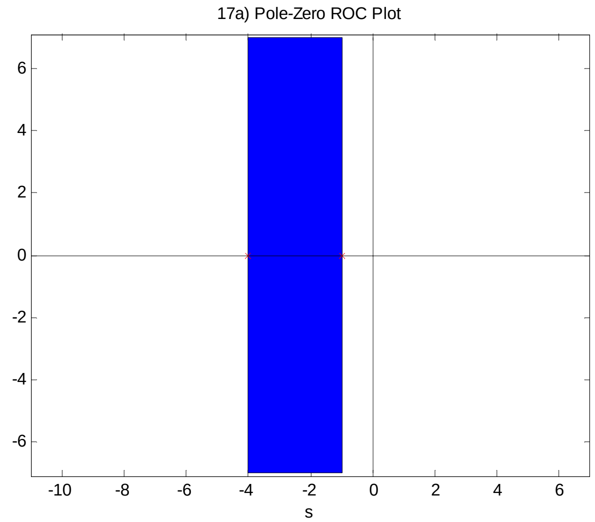
2704: Signals and Systems

Homework #8
Solutions

17) (a) $\Re\{s\} > -4 \cap \Re\{s\} < -1 = -4 \le \Re\{s\} \le -1$



(b) $\Re\{s\} > 1 \cap \Re\{s\} < -2 \therefore$ no ROC.

18) (a) $g(t) = e^{-at}u(t)$

$$G(s) = \int_{0^-}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0^-}^{\infty} e^{-(s+a)t}dt = -\frac{e^{-(s+a)t}}{s+a}\Bigg]_{0^-}^{\infty} = \frac{1}{s+a}, \Re\{s\} > -a$$

(b) $g(t) = e^{-a(t-\tau)}u(t-\tau), \tau > 0$

$$G(s) = \int_{0^-}^{\infty} e^{-a(t-\tau)}u(t-\tau)e^{-st}dt = e^{a\tau}\int_{\tau^-}^{\infty} e^{-(s+a)t}dt = -e^{a\tau}\frac{e^{-(s+a)t}}{s+a}\Bigg]_{\tau^-}^{\infty} = \frac{e^{-s\tau}}{s+a}, \tau > 0$$

(c) $g(t) = e^{-a(t+\tau)}u(t+\tau), \tau > 0$

$$G(s) = \int_{0^-}^{\infty} e^{-a(t+\tau)}u(t+\tau)e^{-st}dt = e^{-a\tau}\int_{0^-}^{\infty} e^{-(s+a)t}dt = -e^{a\tau}\frac{e^{-(s+a)t}}{s+a}\Bigg]_{0^-}^{\infty} = \frac{e^{-a\tau}}{s+a}, \tau > 0$$

(d) $g(t) = \sin(\omega_0 t)u(t)$

$$G(s) = \int_{0^-}^{\infty} \sin(\omega_0 t)u(t)e^{-st}dt = \int_{0^-}^{\infty} \frac{e^{j\omega_0} - e^{-j\omega_0}}{j2}e^{-st}dt = \frac{1}{j2}\int_{0^-}^{\infty}\left[e^{-(s-j\omega_0)t} - e^{-(s+j\omega_0)t}\right]dt$$

$$G(s) = \frac{1}{j2}\left[-\frac{e^{-(s-j\omega_0)t}}{s-j\omega_0} + \frac{e^{-(s+j\omega_0)t}}{s+j\omega_0}\right]_{0^-}^{\infty} = \frac{1}{j2}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right] = \frac{\omega_0}{s^2+\omega_0{}^2}, \Re\{s\} > 0$$

(e) $g(t) = rect(t)$

$$G(s) = \int_{0^-}^{\infty} rect(t)e^{-st}dt = \int_{0^-}^{1/2} e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_{0^-}^{1/2} = \frac{1-e^{-s/2}}{s}, \Re\{s\} > 0$$

(f) $g(t) = rect\left(t - \frac{1}{2}\right)$

$$G(s)=\int_{0^-}^{\infty} rect\left(t-\frac{1}{2}\right)e^{-st}dt=\int_{0^-}^{1} e^{-st}dt=\left[-\frac{e^{-st}}{s}\right]_{0^-}^{1}=\frac{1-e^{-s}}{s},\Re\{s\}>0$$

20)

(a) $g(t)=5\sin(2\pi(t-1))u(t-1)$

$$\sin(\beta t)u(t)\xleftrightarrow{\mathcal{L}}\frac{\beta}{s^2+\beta^2}$$

$$\sin(2\pi t)u(t)\xleftrightarrow{\mathcal{L}}\frac{2\pi}{s^2+(2\pi)^2}$$

Linearity $$5\sin(2\pi t)u(t)\xleftrightarrow{\mathcal{L}}\frac{10\pi}{s^2+(2\pi)^2}$$

Time Shifting $$5\sin(2\pi(t-1))u(t-1)\xleftrightarrow{\mathcal{L}}\frac{10\pi e^{-s}}{s^2+(2\pi)^2}$$

(b) $g(t)=5\sin(2\pi t)u(t-1)$

$$\sin(2\pi t)=\sin(2\pi(t-1))$$

Therefore

$$5\sin(2\pi t)u(t-1)\xleftrightarrow{\mathcal{L}}\frac{10\pi e^{-s}}{s^2+(2\pi)^2}$$

(c) $g(t)=2\cos(10\pi t)\cos(100\pi t)u(t)$

$$\cos(10\pi t)\cos(100\pi t)=\frac{1}{2}[\cos(10\pi t-100\pi t)+\cos(10\pi t+100\pi t)]$$

$$\cos(10\pi t)\cos(100\pi t)=\frac{1}{2}[\cos(-90\pi t)+\cos(110\pi t)]$$

$$2\cos(10\pi t)\cos(100\pi t)u(t)=[\cos(90\pi t)+\cos(110\pi t)]u(t)$$

$$\cos(\beta t)u(t)\xleftrightarrow{\mathcal{L}}\frac{s}{s^2+\beta^2}$$

$$[\cos(90\pi t)+\cos(110\pi t)]u(t)\xleftrightarrow{\mathcal{L}}\frac{s}{s^2+(-90\pi)^2}+\frac{s}{s^2+(110\pi)^2}$$

$$2\cos(10\pi t)\cos(100\pi t)u(t)\xleftrightarrow{\mathcal{L}}\frac{s}{s^2+(90\pi)^2}+\frac{s}{s^2+(110\pi)^2}$$

(d) $g(t)=\frac{d}{dt}(u(t-2))$

$$u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s}$$

Time Shifting $$u(t-2)\xleftrightarrow{\mathcal{L}}\frac{e^{-2s}}{s}$$

Time Differentiation Once $$\frac{d}{dt}(u(t-2))\xleftrightarrow{\mathcal{L}}s\frac{e^{-2s}}{s}-u(-2^-)=e^{-2s}$$

(e) $\mathrm{g}(t) = \int_{0^-}^{t} \mathrm{u}(\tau)\,d\tau$

Integration $\int_{0^-}^{t} \mathrm{u}(\tau)\,d\tau \xleftrightarrow{L} \frac{1}{s^2}$

(f) $\mathrm{g}(t) = \frac{d}{dt}\left(5e^{-\frac{t-\tau}{2}}\mathrm{u}(t-\tau)\right)\ ,\ \ \tau > 0$

$$e^{-\alpha t}\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha}$$

Frequency Shifting $e^{-\frac{t}{2}}\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\frac{1}{2}}$

Time Shifting $e^{-\frac{t-\tau}{2}}\mathrm{u}(t-\tau) \xleftrightarrow{\mathcal{L}} \frac{e^{-\tau s}}{s+\frac{1}{2}}$

Linearity $5e^{-\frac{t-\tau}{2}}\mathrm{u}(t-\tau) \xleftrightarrow{\mathcal{L}} \frac{5e^{-\tau s}}{s+\frac{1}{2}}$

Time Differentiation Once $\frac{d}{dt}\left(5e^{-\frac{t-\tau}{2}}\mathrm{u}(t-\tau)\right) \xleftrightarrow{\mathcal{L}} s\frac{5e^{-\tau s}}{s+\frac{1}{2}} - 5e^{-\frac{-\tau^-}{2}}\underbrace{\mathrm{u}(-\tau^-)}_{=0}$

$$\frac{d}{dt}\left(5e^{-\frac{t-\tau}{2}}\mathrm{u}(t-\tau)\right) \xleftrightarrow{\mathcal{L}} \frac{5se^{-\tau s}}{s+\frac{1}{2}}$$

(g) $\mathrm{g}(t) = 2e^{-5t}\cos(10\pi t)\mathrm{u}(t)$

$$e^{-\alpha t}\cos(\beta t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

$$e^{-5t}\cos(10\pi t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s+5}{(s+5)^2+(10\pi)^2}$$

Linearity $2e^{-5t}\cos(10\pi t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 2\frac{s+5}{(s+5)^2+(10\pi)^2}$

(h) $\mathrm{x}(t) = 5\sin\left(\pi t - \frac{\pi}{8}\right)\mathrm{u}(t)$

$$\mathrm{x}(t) = 5\sin\left(\pi\left(t - \frac{1}{8}\right)\right)\mathrm{u}(t) = \frac{5}{j2}\left[e^{j\pi\left(t-\frac{1}{8}\right)} - e^{-j\pi\left(t-\frac{1}{8}\right)}\right]\mathrm{u}(t)$$

$$\mathrm{x}(t) = \frac{5}{j2}\left[e^{j\pi t}e^{-\frac{j\pi}{8}} - e^{-j\pi t}e^{\frac{j\pi}{8}}\right]\mathrm{u}(t)$$

$$\mathrm{X}(s) = \frac{5}{j2}\left[\frac{e^{-\frac{j\pi}{8}}}{s - j\pi} - \frac{e^{\frac{j\pi}{8}}}{s + j\pi}\right] = \frac{5}{j2}\frac{(s + j\pi)e^{-\frac{j\pi}{8}} - (s - j\pi)e^{\frac{j\pi}{8}}}{s^2 + \pi^2}$$

$$\mathrm{X}(s) = \frac{5}{j2}\frac{s\left(e^{-\frac{j\pi}{8}} - e^{\frac{j\pi}{8}}\right) + j\pi\left(e^{-\frac{j\pi}{8}} + e^{\frac{j\pi}{8}}\right)}{s^2 + \pi^2} = \frac{5}{j2}\frac{-j2\sin\left(\frac{\pi}{8}\right)s + j2\pi\cos\left(\frac{\pi}{8}\right)}{s^2 + \pi^2}$$

$$\mathrm{X}(s) = 5\frac{\pi\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)s}{s^2 + \pi^2}$$

21)

(a) $\mathrm{g}(2t)$

Time Scaling $\mathrm{g}(2t) \xleftrightarrow{\mathcal{L}} \frac{1}{2}\frac{\frac{s}{2} + 1}{\frac{s}{2}\left(\frac{s}{2} + 4\right)}$

$$\mathrm{g}(2t) \xleftrightarrow{\mathcal{L}} \frac{s + 2}{s(s + 8)}$$

(b) $\frac{d}{dt}(\mathrm{g}(t))$

Time Differentiation Once $\frac{d}{dt}(\mathrm{g}(t)) \xleftrightarrow{\mathcal{L}} s\frac{s + 1}{s(s + 4)} - \mathrm{g}(0^-)$

$$\frac{d}{dt}(\mathrm{g}(t)) \xleftrightarrow{\mathcal{L}} \frac{s + 1}{s + 4} - \mathrm{g}(0^-)$$

Initial Value Theorem $\mathrm{g}(0^+) = \lim_{s\to\infty} s\mathrm{G}(s) = 1$

$$\frac{d}{dt}(\mathrm{g}(t)) \xleftrightarrow{\mathcal{L}} \frac{s + 1}{s + 4} - 1$$

$$\frac{d}{dt}(\mathrm{g}(t)) \xleftrightarrow{\mathcal{L}} -\frac{3}{s + 4}$$

(This is correct if $\mathrm{g}(0^-) = \mathrm{g}(0^+)$. That is, if g is continuous at time, $t = 0$.)

(c) $g(t-4)$

Time Shifting $$g(t-4) \xleftrightarrow{\mathcal{L}} \frac{s+1}{s(s+4)} e^{-4s}$$

(d) $g(t) * g(t)$

Time-domain Convolution $$g(t) * g(t) \xleftrightarrow{\mathcal{L}} \frac{s+1}{s(s+4)} \frac{s+1}{s(s+4)}$$

$$g(t) * g(t) \xleftrightarrow{\mathcal{L}} \frac{(s+1)^2}{s^2(s+4)^2}$$

22)

(a) $G(s) = \dfrac{4s}{(s+3)(s+8)}$

$$G(s) = \frac{-\frac{12}{5}}{s+3} + \frac{\frac{32}{5}}{s+8}$$

$$g(t) = \left(-\frac{12}{5} e^{-3t} + \frac{32}{5} e^{-8t}\right) u(t)$$

$$\lim_{t \to \infty} g(t) = \lim_{t \to \infty} \left[\left(-\frac{12}{5} e^{-3t} + \frac{32}{5} e^{-8t}\right) u(t)\right] = 0$$

$$\lim_{s \to 0^+} sG(s) = \lim_{s \to 0^+} \frac{4s^2}{(s+3)(s+8)} = 0 \quad , \quad \text{Check.}$$

$$\lim_{t \to 0^+} g(t) = \lim_{t \to 0^+} \left[\left(-\frac{12}{5} e^{-3t} + \frac{32}{5} e^{-8t}\right) u(t)\right] = 4$$

$$\lim_{s \to \infty} sG(s) = \lim_{s \to \infty} \frac{4s^2}{(s+3)(s+8)} = 4 \quad , \quad \text{Check.}$$

(b) $G(s) = \dfrac{4}{(s+3)(s+8)}$

$$G(s) = \frac{\frac{4}{5}}{s+3} - \frac{\frac{4}{5}}{s+8}$$

$$g(t) = \left(\frac{4}{5} e^{-3t} - \frac{4}{5} e^{-8t}\right) u(t)$$

$$\lim_{t \to \infty} g(t) = \lim_{t \to \infty} \left[\left(\frac{4}{5} e^{-3t} - \frac{4}{5} e^{-8t}\right) u(t)\right] = 0$$

$$\lim_{s\to 0^+} s\,\mathrm{G}(s) = \lim_{s\to 0^+} \frac{4s}{(s+3)(s+8)} = 0 \quad , \ \text{Check.}$$

$$\lim_{t\to 0^+} \mathrm{g}(t) = \lim_{t\to 0^+} \left[\left(\frac{4}{5} e^{-3t} - \frac{4}{5} e^{-8t} \right) \mathrm{u}(t) \right] = 0$$

$$\lim_{s\to \infty} s\,\mathrm{G}(s) = \lim_{s\to \infty} \frac{4s}{(s+3)(s+8)} = 0 \quad , \ \text{Check.}$$

(c) $\mathrm{G}(s) = \dfrac{s}{s^2+2s+2}$

$$\mathrm{G}(s) = \frac{s}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathrm{g}(t) = e^{-t}\left[\cos(t) - \sin(t)\right]\mathrm{u}(t)$$

Alternate Inverse:

$$\mathrm{G}(s) = \frac{\frac{1-j}{2}}{s+(1+j)} + \frac{\frac{1+j}{2}}{s+(1-j)}$$

$$\mathrm{g}(t) = \left(\frac{1-j}{2} e^{-(1+j)t} + \frac{1+j}{2} e^{-(1-j)t} \right) \mathrm{u}(t)$$

$$\mathrm{g}(t) = e^{-t}\left(\cos(t) - \sin(t)\right)\mathrm{u}(t)$$

$$\lim_{t\to \infty} \mathrm{g}(t) = \lim_{t\to \infty} \left[e^{-t}\left(\cos(t) - \sin(t)\right)\mathrm{u}(t) \right] = 0$$

$$\lim_{s\to 0^+} s\,\mathrm{G}(s) = \lim_{s\to 0^+} \frac{s^2}{s^2+2s+2} = 0 \quad , \ \text{Check.}$$

$$\lim_{t\to 0^+} \mathrm{g}(t) = \lim_{t\to 0^+} \left[e^{-t}\left(\cos(t) - \sin(t)\right)\mathrm{u}(t) \right] = 1$$

$$\lim_{s\to \infty} s\,\mathrm{G}(s) = \lim_{s\to \infty} \frac{s^2}{s^2+2s+2} = 1 \quad , \ \text{Check.}$$

(d) $\mathrm{G}(s) = \dfrac{e^{-2s}}{s^2+2s+2}$

$$\mathrm{G}(s) = \frac{e^{-2s}}{(s+1)^2+1}$$

$$\mathrm{g}(t) = \left[e^{-t}\sin(t)\mathrm{u}(t) \right]_{t\to t-2} = e^{-(t-2)}\sin(t-2)\mathrm{u}(t-2)$$

Alternate Inverse:

$$\mathrm{G}(s) = e^{-2s} \left[\frac{\frac{1}{-j2}}{s+(1+j)} + \frac{\frac{1}{j2}}{s+(1-j)} \right]$$

$$\mathrm{g}(t) = e^{-(t-2)}\sin(t-2)\mathrm{u}(t-2)$$

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty}\left[e^{-(t-2)}\sin(t-2)u(t-2)\right] = 0$$

$$\lim_{s\to 0^+} sG(s) = \lim_{s\to 0^+}\frac{e^{-2s}s}{s^2+2s+2} = 0 \quad , \text{ Check.}$$

$$\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+}\left[e^{-(t-2)}\sin(t-2)u(t-2)\right] = 0$$

$$\lim_{s\to\infty} sG(s) = \lim_{s\to\infty}\frac{e^{-2s}s}{s^2+2s+2} = 0 \quad , \text{ Check.}$$

23)

(a) $G\left(\frac{s}{3}\right)$

Frequency Scaling $\quad 3e^{-12t}u(3t) \xleftrightarrow{\mathcal{L}} G\left(\frac{s}{3}\right)$

$$3e^{-12t}u(3t) = 3e^{-12t}u(t)$$

(b) $G(s-2)+G(s+2)$

Frequency Shifting $\quad e^{2t}e^{-4t}u(t) \xleftrightarrow{\mathcal{L}} G(s-2)$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} G(s-2)$$

Frequency Shifting $\quad e^{-6t}u(t) \xleftrightarrow{\mathcal{L}} G(s+2)$

Linearity $\quad \left(e^{-2t}+e^{-6t}\right)u(t) \xleftrightarrow{\mathcal{L}} G(s-2)+G(s+2)$

(c) $\frac{G(s)}{s}$

Integration $\quad \int_{0^-}^{t} e^{-4\lambda}u(\lambda) \xleftrightarrow{\mathcal{L}} \frac{G(s)}{s}$

$$\left[-\frac{e^{-4\lambda}}{4}\right]_{0^-}^{t} u(t) \xleftrightarrow{\mathcal{L}} \frac{G(s)}{s}$$

$$\frac{1-e^{-4t}}{4}u(t) \xleftrightarrow{\mathcal{L}} \frac{G(s)}{s}$$