ECE 2704
Signals and Systems
Spring 2006

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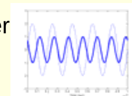
Lecture #10: Properties of The Fourier Series



Overview

- Today we continue to discuss the concept of the Continuous Time Fourier Series (CTFS) with an emphasis on properties including
  - Linearity
  - Time-shifting
  - Frequency-shifting
  - Time reversal
  - Time scaling
  - Change of period
  - Time differentiation
  - Time integration
  - Multiplication-Convolution
  - Parseval's Theorem
- What to read – Section 4.4 in the text

Preliminaries

- Let us consider periodic signals *x(t)* and *y(t)* with fundamental periods T_{Fx} and T_{Fy}
- We can represent the signals over all time through their Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t} \qquad y(t) = \sum_{k=-\infty}^{\infty} Y[k] e^{j2\pi k f_{Fy} t}$$

$$X[k] = \frac{1}{T_{Fx}} \int_{T_{Fx}} x(t) e^{-j2\pi k f_{Fx} t} dt \qquad Y[k] = \frac{1}{T_{Fy}} \int_{T_{Fy}} y(t) e^{-j2\pi k f_{Fy} t} dt$$

where integration occurs over one period T_{Fx} and T_{Fy}.

- If T_{Fx} and T_{Fy} are not the same, we must change them such that they are. We will discuss this later.

Linearity

- If $$z(t) = \alpha x(t) + \beta y(t)$$
- Then
$$Z[k] = \frac{1}{T_{Fz}} \int_{T_{Fz}} z(t) e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fz}} \int_{T_{Fz}} \{\alpha x(t) + \beta y(t)\} e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{\alpha}{T_{Fx}} \int_{T_{Fx}} x(t) e^{-j2\pi k f_{Fx} t} dt + \frac{\beta}{T_{Fy}} \int_{T_{Fy}} y(t) e^{-j2\pi k f_{Fy} t} dt$$
$$= \alpha X[k] + \beta Y[k]$$

Assuming that $T_{Fx}=T_{Fy}=T_{Fz}$

- In other words
$$\alpha x(t) + \beta y(t) \xleftrightarrow{FS} \alpha X[k] + \beta Y[k]$$

Time-shifting

- Let $$z(t) = x(t - t_o)$$
- Then
$$Z[k] = \frac{1}{T_{Fz}} \int_{T_{Fz}} z(t) e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fz}} \int_{T_{Fz}} x(t - t_o) e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fz}} \int_{T_{Fz}} x(\tau) e^{-j2\pi k f_{Fz} (\tau + t_o)} d\tau$$
$$= e^{-j2\pi k f_{Fz} t_o} \frac{1}{T_{Fx}} \int_{T_{Fx}} x(\tau) e^{-j2\pi k f_{Fx} \tau} d\tau$$
$$= e^{-j2\pi k f_{Fz} t_o} X[k]$$

Assumes that $T_{Fx}=T_{Fz}$

$$x(t - t_o) \xleftrightarrow{FS} e^{-j2\pi k f_{Fx} t_o} X[k]$$

Time-shifting: Interpretation

$$x(t - t_o) \xleftrightarrow{FS} e^{-j2\pi k f_{Fx} t_o} X[k]$$

- Recall that time-delaying a sinusoid is equivalent to applying a phase shift.
- Thus, when we delay a function which is made up of sinusoids, we must phase shift each sinusoid by an appropriate amount
- Each sinusoid must be phase shifted by

$$2\pi k f_{Fx} t_o$$

- For higher frequencies, a fixed amount of time corresponds to a larger phase shift

Frequency-shifting

- Let $z(t) = e^{j2\pi k_o f_{Fx} t} x(t)$

k_o = integer

- Then

$$Z[k] = \frac{1}{T_{Fz}} \int_{T_{Fz}} z(t) e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fz}} \int_{T_{Fz}} e^{j2\pi k_o f_{Fx} t} x(t) e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fx}} \int_{T_{Fx}} x(t) e^{-j2\pi (k-k_o) f_{Fx} t} dt$$
$$= X[k - k_o]$$

Assumes that $T_{Fx}=T_{Fz}$

$$e^{j2\pi k_o f_{Fx} t} x(t) \xleftrightarrow{FS} X[k - k_o]$$

Time-Reversal

- Let $z(t) = x(-t)$
- Then

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t}$$
$$x(-t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx}(-t)} = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi(-k) f_{Fx} t}$$

- Letting $q = -k$

$$x(-t) = \sum_{q=\infty}^{-\infty} X[-q] e^{j2\pi q f_{Fx} t} = \sum_{q=-\infty}^{\infty} X[-q] e^{j2\pi q f_{Fx} t}$$
$$z(t) = \sum_{q=-\infty}^{\infty} Z[q] e^{j2\pi q f_{Fz} t}$$

- *Thus,*

$$Z[k] = X[-k]$$

$$x(-t) \xleftrightarrow{FS} X[-k]$$

Assuming that $T_{Fx}=T_{Fz}$

Time scaling

- Let $z(t) = x(at)$
- Since *x(t)* is periodic with period T_{Fx}, then *z(t)* is periodic with period $T_{Fz}=T_{Fx}/a$.
- There are two cases we wish to consider:
  - Case 1 – The representation interval of *z(t)* is equal to the period of *z(t)*. In this case the representation interval is reduced by a factor of *a.*
  - Case 2 - The representation interval of *z(t)* is equal to the period of x*(t)*. In this case the representation interval remains the same.

Case 1 – $T_{Fz} = T_{Fx}/a$

- Examining *Z[k]*

$$\begin{aligned} Z[k] &= \frac{1}{T_{Fz}} \int_{t_o}^{t_o+T_{Fz}} z(t) e^{-j2\pi k f_{Fz} t} dt \\ &= \frac{a}{T_{Fx}} \int_{t_o}^{t_o+T_{Fx}/a} x(at) e^{-j2\pi k a f_{Fx} t} dt \\ &= \frac{1}{T_{Fz}} \int_{at_o}^{at_o+T_{Fx}} x(\lambda) e^{-j2\pi k f_{Fx} \lambda} d\lambda \\ &= X[k] \end{aligned}$$

Let λ=at

- Thus, the Fourier Series terms are identical, but the *representations* are not since they have different fundamental frequencies $f_{Fz} = a\, f_{Fx}$

Case 2 – $T_{Fz} = T_{Fx}$

- Examining *Z[k]*

$$\begin{aligned} Z[k] &= \frac{1}{T_{Fz}} \int_{t_o}^{t_o+T_{Fz}} z(t) e^{-j2\pi k f_{Fz} t} dt \\ &= \frac{1}{T_{Fx}} \int_{t_o}^{t_o+T_{Fx}} x(at) e^{-j2\pi k f_{Fx} t} dt \\ &= \frac{1}{aT_{Fx}} \int_{at_o}^{at_o+aT_{Fx}} x(\lambda) e^{-j2\pi k f_{Fx} \lambda / a} d\lambda \\ &= \frac{1}{aT_{Fx}} \int_{at_o}^{at_o+aT_{Fx}} x(\lambda) e^{-j2\pi k (f_{Fx}/a) \lambda} d\lambda \end{aligned}$$

- For non-integer values of *a,* we cannot simplify this further.
- For integer values of *a*

$$Z[k] = \begin{cases} X\left[\frac{k}{a}\right] & \frac{k}{a} = \text{integer} \\ 0 & \text{else} \end{cases}$$

Change of representation interval

- If the CTFS of a periodic function *x(t)* is defined over one period T_{Fx} as X[k], then we can find the CTFS representation of *x(t)* over a period mT_{Fx} where *m* is a positive integer

$$\begin{aligned} X_m[k] &= \frac{1}{mT_{Fx}} \int_{t_o}^{t_o+mT_{Fx}} x(t) e^{-j2\pi k (f_{Fx}/m) t} dt \\ &= \frac{1}{mT_{Fx}} \int_{t_o}^{t_o+mT_{Fx}} x(t) e^{-j2\pi (k/m) f_{Fx} t} dt \\ &= \begin{cases} X\left[\frac{k}{m}\right] & \frac{k}{m} = \text{integer} \\ 0 & \text{else} \end{cases} \end{aligned}$$

Time Differentiation

- Let $$z(t) = \frac{d}{dt} x(t)$$
- Then
$$z(t) = \frac{d}{dt}\left\{ \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t} \right\}$$
$$= \sum_{k=-\infty}^{\infty} j2\pi k f_{Fx} X[k] e^{j2\pi k f_{Fx} t}$$
$$= \sum_{k=-\infty}^{\infty} Z[k] e^{j2\pi k f_{Fx} t}$$
- Thus, $$Z[k] = \left(j2\pi k f_{Fx} \right) X[k]$$

Assuming that T_{Fx}=T_{Fz}

$$\frac{d}{dt}\{x(t)\} \xleftrightarrow{FS} \left(j2\pi k f_{Fx} \right) X[k]$$

Time integration

- Let $$z(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$
- Then
$$z(t) = \int_{-\infty}^{t} \left\{ \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} \tau} \right\} d\tau$$
$$= \sum_{k=-\infty}^{\infty} X[k] \int_{-\infty}^{t} e^{j2\pi k f_{Fx} \tau} d\tau$$
$$= \sum_{k=-\infty}^{\infty} \left\{ X[k] \frac{1}{j2\pi k f_{Fx}} \right\} e^{j2\pi k f_{Fx} t}$$
$$Z[k] = \frac{1}{j2\pi k f_{Fx}} X[k]$$
- Thus,

Assuming that T_{Fx}=T_{Fz}

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FS} \frac{1}{j2\pi k f_{Fx}} X[k]$$

Multiplication-Convolution

- Let $$z(t) = x(t) y(t)$$
- Then
$$Z[k] = \frac{1}{T_{Fz}} \int_{t_o}^{t_o + T_{Fz}} z(t) e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fz}} \int_{t_o}^{t_o + T_{Fz}} x(t) y(t) e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fz}} \int_{t_o}^{t_o + T_{Fz}} x(t) \left\{ \sum_{q=-\infty}^{\infty} Y[q] e^{j2\pi q f_{Fy} t} \right\} e^{-j2\pi k f_{Fz} t} dt$$
$$= \frac{1}{T_{Fz}} \sum_{q=-\infty}^{\infty} Y[q] \int_{t_o}^{t_o + T_{Fx}} x(t) e^{j2\pi q f_{Fy} t} e^{-j2\pi k f_{Fx} t} dt$$

Multiplication-Convolution (cont.)

- Continuing

$$Z[k] = \frac{1}{T_{Fz}} \sum_{q=-\infty}^{\infty} Y[q] \int_{t_o}^{t_o+T_{Fx}} x(t) e^{j2\pi q f_{Fy} t} e^{-j2\pi k f_{Fx} t} dt$$

$$= \sum_{q=-\infty}^{\infty} Y[q] \underbrace{\frac{1}{T_{Fx}} \int_{t_o}^{t_o+T_{Fx}} x(t) e^{-j2\pi (k-q) f_{Fx} t} dt}_{X[k-q]}$$

Assuming that $T_{Fx}=T_{Fy}=T_{Fz}$

$$= \sum_{q=-\infty}^{\infty} Y[q] X[k-q]$$

$$\boxed{x(t)\,y(t) \overset{FS}{\longleftrightarrow} \sum_{q=-\infty}^{\infty} Y[q] X[k-q]}$$

Convolution sum

Multiplication-Convolution II

- Let $Z[k] = X[k]Y[k]$
- Then $z(t) = \sum_{k=-\infty}^{\infty} X[k]Y[k] e^{j2\pi k f_{Fz} t}$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T_{Fx}} \left(\int_{T_{Fx}} x(\tau) e^{-j2\pi k f_{Fx} \tau} d\tau \right) Y[k] e^{j2\pi k f_{Fz} t}$$

Assuming that $T_{Fx}=T_{Fy}=T_{Fz}$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T_{Fx}} \left(\int_{T_{Fx}} x(\tau) Y[k] e^{j2\pi k f_{Fx}(t-\tau)} d\tau \right)$$

$$= \frac{1}{T_{Fx}} \int_{T_{Fx}} x(\tau) \underbrace{\sum_{k=-\infty}^{\infty} Y[k] e^{j2\pi k f_{Fy}(t-\tau)}}_{y(t-\tau)} d\tau$$

Multiplication-Convolution (cont.)

- Continuing

$$z(t) = \frac{1}{T_{Fx}} \int_{T_{Fx}} x(\tau) \underbrace{\sum_{k=-\infty}^{\infty} Y[k] e^{j2\pi k f_{Fy}(t-\tau)}}_{y(t-\tau)} d\tau$$

$$= \frac{1}{T_{Fx}} \int_{T_{Fx}} x(\tau) y(t-\tau) d\tau$$

- What is this operation?
- It is similar to convolution except that the time interval is different (i.e., it only covers $t_o < \tau < t_o + T_{Fx}$)
- Further, we divide by T_{Fx} unlike in regular convolution.
- We term this *periodic convolution* $x(t) \otimes y(t) = \int_{T_{Fx}} x(\tau) y(t-\tau) d\tau$

$$\boxed{x(t) \otimes y(t) \overset{FS}{\longleftrightarrow} T_F X[k] Y[k]}$$

Conjugation

- Let $z(t) = x^*(t)$
- Then $\sum_{k=-\infty}^{\infty} Z[k] e^{j2\pi k f_{Fz} t} = \left(\sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t} \right)^*$

$$= \sum_{k=-\infty}^{\infty} X^*[k] e^{-j2\pi k f_{Fx} t}$$

$$= \sum_{q=\infty}^{-\infty} X^*[-q] e^{j2\pi q f_{Fx} t}$$

Assuming that $T_{Fx}=T_{Fz}$

$$= \sum_{q=-\infty}^{\infty} X^*[-q] e^{j2\pi q f_{Fx} t}$$

$$Z[k] = X^*[-k]$$

$$\boxed{x^*(t) \xleftrightarrow{FS} X^*[-k]}$$

Parseval's Theorem

- The signal energy in one fundamental period T_{Fx} of the periodic signal *x(t)* can be written as

$$E_{x,T_{Fx}} = \int_{T_{Fx}} |x(t)|^2 \, dt$$

$$= \int_{T_{Fx}} \left| \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t} \right|^2 dt$$

$$= \int_{T_{Fx}} \left(\sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t} \right) \left(\sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t} \right)^* dt$$

$$= \int_{T_{Fx}} \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} X[k] e^{j2\pi k f_{Fx} t} X^*[q] e^{-j2\pi q f_{Fx} t} dt$$

$$= \int_{T_{Fx}} \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} X[k] X^*[q] e^{j2\pi (k-q) f_{Fx} t} dt$$

Parseval's Theorem (cont.)

- Continuing…

$$E_{x,T_{Fx}} = \int_{T_{Fx}} \sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} X[k] X^*[q] e^{j2\pi (k-q) f_{Fx} t} dt$$

$$= \int_{T_{Fx}} \left(\sum_{k=-\infty}^{\infty} X[k] X^*[k] + \underbrace{\sum_{\substack{k=-\infty \\ k \neq q}}^{\infty} \sum_{q=-\infty}^{\infty} X[k] X^*[q] e^{j2\pi (k-q) f_{Fx} t}}_{=0,\, k \neq q} \right) dt$$

$$= \int_{T_{Fx}} \sum_{k=-\infty}^{\infty} |X[k]|^2 dt$$

$$= T_{Fx} \sum_{k=-\infty}^{\infty} |X[k]|^2$$

Parseval's Theorem Interpreted

- For any periodic signal we can write

$$\underbrace{\frac{1}{T_{Fx}}\int_{T_{Fx}} \left|x(t)\right|^2 dt}_{\text{Average power of } x(t)} = \sum_{k=-\infty}^{\infty} \left|X[k]\right|^2$$

- Thus, if we sum the magnitude squared of each Fourier Series term we arrive at the average power.
- Note that the magnitude squared of each Fourier Series term is simply the average power in that particular sinusoid. Thus, the average power in the time domain signal is equal to the power of the Fourier Series.

Summary

- In this lecture we have examined several properties of the Continuous Time Fourier Series.
- Knowing these properties will allow us to
  - Determine the CTFS of new signals from those which we already know
  - Determine the impact that a system will have on a signal
- We will examine the use of these properties in the next lecture.