ECE 2704
Signals and Systems
Spring 2006

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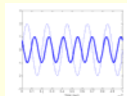
Lecture #11: The Frequency Domain
CTFS of Common Functions and
Using CTFS Tables



Overview

- Today we continue to discuss the concept of the Continuous Time Fourier Series (CTFS)
- Specific Topics include
  - The "Frequency Domain"
  - CTFS of common functions
  - Using tables and properties
- What to read – Sections 4.5-4.7 in the text

The Frequency Domain

- Let us consider periodic functions where the CTFS is defined over one period T_o.
- The CTFS coefficients can be written in function form by using the discrete time unit impulse

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & else \end{cases}$$

- When we relate two representations of a function through the CTFS, we say that one representation is in the *time domain* and one is in the *frequency domain* since it represents the amount of the signal at each frequency

$$\underbrace{x(t)}_{\substack{time \\ domain}} \overset{FS}{\longleftrightarrow} \underbrace{X[k]}_{\substack{frequency \\ domain}}$$

Example

$$x(t) = e^{j2\pi f_o t} \qquad T_F = T_o$$

- The Fourier Series Coefficients are calculated as

$$\begin{aligned} X[k] &= \frac{1}{T_F}\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi k f_F t}dt \\ &= \frac{1}{T_o}\int_0^{T_o} x(t)e^{-j2\pi k f_o t}dt \\ &= \frac{1}{T_o}\int_0^{T_o} e^{j2\pi f_o t}e^{-j2\pi k f_o t}dt \\ &= \frac{1}{T_o}\int_0^{T_o} e^{j2\pi f_o (1-k)t}dt \end{aligned}$$

Example (cont.)

- Evaluating the integral:

$$\begin{aligned} X[k] &= \frac{1}{j2\pi f_o(1-k)T_o} e^{j2\pi f_o(1-k)t}\Big|_0^{T_o} \\ &= \frac{e^{j2\pi f_o(1-k)T_o} - 1}{j2\pi f_o(1-k)T_o} \\ &= \begin{cases} \frac{0}{0} & k=1 \\ 0 & k \neq 1 \end{cases} \end{aligned}$$

- Using L'Hopital's Rule

$$\left.\frac{\frac{d}{dk}\left\{e^{j2\pi f_o(1-k)T_o} - 1\right\}}{\frac{d}{dk}\left\{j2\pi f_o(1-k)T_o\right\}}\right|_{k=1} = \left.\frac{-j2\pi f_o T_o e^{j2\pi f_o(1-k)T_o}}{-j2\pi f_o T_o}\right|_{k=1}$$

$$= e^{j2\pi f_o(1-k)T_o}\Big|_{k=1}$$

$$= 1$$

Example (final)

- Putting these together we have

$$X[k] = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$$

- This can be written as

$$X[k] = \delta[k-1]$$

- Thus, we have the CTFS pair

$$e^{j2\pi f_o t} \overset{FS}{\leftrightarrow} \delta[k-1] \qquad T_F = T_o$$

- If $T_F = mT_o$

$$e^{j2\pi f_o t} \overset{FS}{\leftrightarrow} \delta[k-m] \qquad T_F = mT_o$$

Example

$$x(t) = \cos(2\pi f_o t) \qquad T_F = T_o$$

- Using Euler's relationship

$$\cos(2\pi f_o t) = \frac{e^{j2\pi f_o t} + e^{-j2\pi f_o t}}{2}$$

- Using the property of linearity and the previous result:

$$X[k] = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \qquad T_F = T_o$$

- And

$$\cos(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \qquad T_F = T_o$$

$$\cos(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-m] + \frac{1}{2}\delta[k+m] \qquad T_F = mT_o$$

Example

$$x(t) = \sin(2\pi f_o t) \qquad T_F = T_o$$

- Using Euler's relationship

$$\sin(2\pi f_o t) = \frac{e^{j2\pi f_o t} - e^{-j2\pi f_o t}}{2j}$$

- Using the property of linearity and the previous result:

$$X[k] = \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1] \qquad T_F = T_o$$

- And

$$\sin(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1] \qquad T_F = T_o$$

$$\sin(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2j}\delta[k-m] - \frac{1}{2j}\delta[k+m] \qquad T_F = mT_o$$

Example

$$x(t) = 1$$

- The Fourier Series Coefficients are calculated as

$$\begin{aligned} X[k] &= \frac{1}{T_F}\int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt \\ &= \frac{1}{T_F}\int_{0}^{T_F} e^{-j2\pi k f_F t} dt \\ &= \frac{1}{-j2\pi k f_F T_F} e^{-j2\pi k f_F t}\Big|_0^{T_F} \\ &= \frac{e^{-j2\pi k f_F T_F} - 1}{-j2\pi k f_F T_F} \end{aligned}$$

Example (cont.)

- Continuing

$$
\begin{aligned}
X[k] &= \frac{e^{-j2\pi k f_F T_F} - 1}{-j2\pi k f_F T_F} \\
&= \frac{e^{-j2\pi k} - 1}{-j2\pi k} \\
&= \begin{cases} \frac{0}{0} & k = 0 \\ 0 & k \neq 0 \end{cases}
\end{aligned}
$$

$$
\left.\frac{\frac{d}{dk}\left\{e^{-j2\pi k} - 1\right\}}{\frac{d}{dk}\left\{-j2\pi k\right\}}\right|_{k=0} = \left.\frac{-j2\pi e^{j2\pi k}}{-j2\pi}\right|_{k=0}
$$

$$
= \left. e^{j2\pi k}\right|_{k=0}
$$

$$
= 1
$$

Example (solution)

- Thus, we have

$$
X[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}
$$

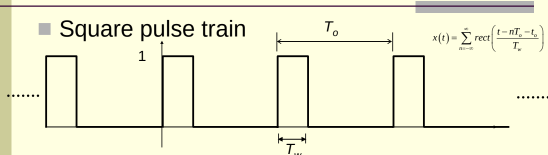
$$
X[k] = \delta[k]
$$

- and our CTFS pair is

$$
1 \overset{FS}{\longleftrightarrow} \delta[k]
$$

Example

- Square pulse train



$$
x(t) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t - nT_o - t_o}{T_w}\right)
$$

$$
\begin{aligned}
X[k] &= \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt \\
&= \frac{1}{T_o} \int_{0}^{T_0} x(t) e^{-j2\pi k f_o t} dt \\
&= \frac{1}{T_o} \int_{0}^{T_w} e^{-j2\pi k f_o t} dt
\end{aligned}
$$

Example (cont.)

- Continuing

$$X[k] = \frac{e^{-j2\pi k f_o T_w} - e^0}{-j2\pi k f_o T_o}$$

$$= e^{-j2\pi k f_o T_w/2} \frac{e^{j2\pi k f_o T_w/2} - e^{-j2\pi k f_o T_w/2}}{j2\pi k}$$

$$= e^{-j2\pi k f_o T_w/2} \frac{\sin(\pi k f_o T_w)}{\pi k}$$

$$= e^{-j\pi k f_o T_w} f_o T_w \operatorname{sinc}(k f_o T_w)$$

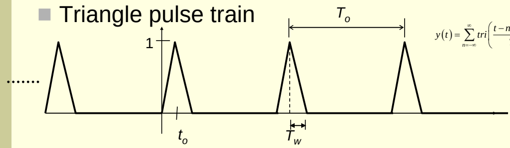
- In general:

$$\sum_{n=-\infty}^{\infty} rect\left(\frac{t - nT_o - t_o}{T_w}\right) \overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o} f_o T_w \operatorname{sinc}(k f_o T_w)$$

t_o = time delay
($T_w/2$ in this case
$T_F = T_o$

Example

- Triangle pulse train



$$y(t) = \sum_{n=-\infty}^{\infty} tri\left(\frac{t - nT_o - t_o}{T_w}\right)$$

- Now, we could simply plug into the equations to derive the CTFS. However, we can see that the triangular pulse train is the result of a periodic convolution of a square pulse train

Example (cont.)

- Specifically

$$y(t) = \frac{1}{T_w}\{x(t) \otimes x(t)\}$$

- Thus,

$$Y[k] = \frac{1}{T_w}\{T_o X[k] X[k]\}$$

$$= \frac{1}{T_w} T_o \left\{e^{-j2\pi k f_o T_w/2} f_o T_w \operatorname{sinc}(k f_o T_w) e^{-j2\pi k f_o T_w/2} f_o T_w \operatorname{sinc}(k f_o T_w)\right\}$$

$$= f_o T_w e^{-j2\pi k f_o T_w} \operatorname{sinc}^2(k f_o T_w)$$

t_o for the rectangular pulse train is $T_w/2$ in this example

- In general:

$$\sum_{n=-\infty}^{\infty} tri\left(\frac{t - nT_o - t_o}{T_w}\right) \overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o} f_o T_w \operatorname{sinc}^2(k f_o T_w)$$

t_o = time delay
= T_w in this example

Use of Tables

- We could continue to find CTFS pairs, but in general it is easier to use properties and tables of pairs to find new pairs.

Property	
Conjugation	$x^*(t)\xleftrightarrow{FS} X^*[-k]$
Linearity	$\alpha x(t)+\beta y(t)\xleftrightarrow{FS}\alpha X[k]+\beta Y[k]$
Time-shifting	$x(t-t_o)\xleftrightarrow{FS} e^{-j2\pi k f_{Fx} t_o} X[k]$
Frequency-shifting	$e^{j2\pi k_o f_{Fx} t} x(t)\xleftrightarrow{FS} X[k-k_o]$
Time reversal	$x(-t)\xleftrightarrow{FS} X[-k]$
Time-derivative	$\frac{d}{dt}\{x(t)\}\xleftrightarrow{FS}(j2\pi k f_{Fx})X[k]$
Time-integration	$\int_{-\infty}^{t} x(\tau)d\tau\xleftrightarrow{FS}\frac{1}{j2\pi k f_{Fx}}X[k]$

Properties (cont.)

Property	
Time-scaling (integer a) $T_F = T_o$	$x(at)\overset{FS}{\leftrightarrow}\begin{cases} X\left[\frac{k}{a}\right] & \frac{k}{a}=\text{integer} \\ 0 & \text{else}\end{cases}$
Multiplication	$x(t)y(t)\xleftrightarrow{FS}\sum_{q=-\infty}^{\infty} Y[q]X[k-q]$
Convolution	$x(t)\otimes y(t)\xleftrightarrow{FS} T_F X[k]Y[k]$
Conjugation	$x^*(t)\xleftrightarrow{FS} X^*[-k]$

Example

- Find the Continuous Time Fourier Series of

$$x(t)=\cos(50\pi t-\pi/4)$$

for $T_F = T_o$

We can use

$$\cos(2\pi f_o t)\overset{FS}{\leftrightarrow}\frac{1}{2}\delta[k-1]+\frac{1}{2}\delta[k+1]$$

and

$$x(t-t_o)\xleftrightarrow{FS} e^{-j2\pi k f_{Fx} t_o} X[k]$$

Example (cont.)

- Re-writing the original equation

$$x(t) = \cos(50\pi t - \pi/4)$$
$$= \cos\left(50\pi\left(t - \frac{\pi/4}{50\pi}\right)\right)$$
$$= \cos\left(50\pi\left(t - \frac{1}{200}\right)\right)$$

$$X[k] = e^{-j2\pi k*25(1/200)}\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$
$$= e^{-j2\pi k/8}\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$

Example (cont.)

- Continuing

$$X[k] = e^{-j2\pi k/8}\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$
$$= e^{-j2\pi/8}\frac{1}{2}\delta[k-1] + e^{j2\pi/8}\frac{1}{2}\delta[k+1]$$
$$= e^{-j\pi/4}\frac{1}{2}\delta[k-1] + e^{j\pi/4}\frac{1}{2}\delta[k+1]$$
$$= \frac{(1-j)}{\sqrt{2}}\frac{1}{2}\delta[k-1] + \frac{(1+j)}{\sqrt{2}}\frac{1}{2}\delta[k+1]$$

$$\cos(50\pi t - \pi/4) \xleftrightarrow{FS} \frac{(1-j)}{\sqrt{2}}\frac{1}{2}\delta[k-1] + \frac{(1+j)}{\sqrt{2}}\frac{1}{2}\delta[k+1]$$

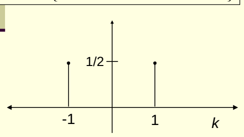
Graphing the Solution

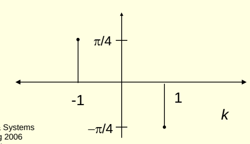
- When plotting the CTFS, due to the fact that the solution is generally *complex,* we typically plot the magnitude and phase separately.

$$\cos(50\pi t - \pi/4) \xleftrightarrow{FS} \frac{(1-j)}{\sqrt{2}}\frac{1}{2}\delta[k-1] + \frac{(1+j)}{\sqrt{2}}\frac{1}{2}\delta[k+1]$$

$$|X[k]| = \left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$

$$\angle X[k] = \left\{-\frac{\pi}{4}\delta[k-1] + \frac{\pi}{4}\delta[k+1]\right\}$$





Example

- Find the Continuous Time Fourier Series of

$$x(t) = 5\cos(10\pi t)\cos(10000\pi t)$$

for $T_F = 1/5$

- Note that we have two sinusoids at different multiples of the fundamental frequency f_F = 5Hz
- Further, we know that

$$\cos(2\pi m f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-m] + \frac{1}{2}\delta[k+m]$$

Example (cont.)

- Re-writing the function in terms of f_F:

$$x(t) = 5\cos(2\pi f_F t)\cos(2\pi 1000 f_F t)$$

$$\cos(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$$

$$\cos(2\pi 1000 f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-1000] + \frac{1}{2}\delta[k+1000]$$

- Further, recalling our multiplication-convolution property:

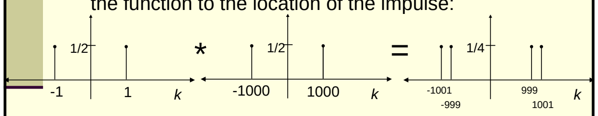
$$y(t)z(t) \overset{FS}{\leftrightarrow} Y[k] * Z[k]$$

Example - Convolving

- Thus, we have

$$X[k] = 5\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\} * \left\{\frac{1}{2}\delta[k-1000] + \frac{1}{2}\delta[k+1000]\right\}$$

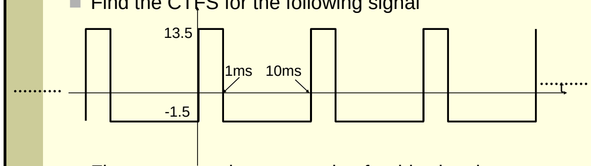
- Recall that convolving with an impulse simply moves the function to the location of the impulse:



$$X[k] = \frac{5}{4}\left\{\delta[k-1001] + \delta[k-999] + \delta[k+1001] + \delta[k+999]\right\}$$

Example

- Find the CTFS for the following signal



- First, we can write an equation for this signal as

$$x(t) = 15\sum_{n=-\infty}^{\infty} rect\left(\frac{t - n \bullet 0.01 - 0.0005}{0.001}\right) - \frac{3}{2}$$

- Recall the CTFS pair $\sum_{n=-\infty}^{\infty} rect\left(\frac{t - nT_o - t_o}{T_w}\right) \overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o}\frac{T_w}{T_o}\text{sinc}\left(k\frac{T_w}{T_o}\right)$

Example (cont.)

- Thus we have

$$15\sum_{n=-\infty}^{\infty} rect\left(\frac{t - n \bullet 0.01 - 0.0005}{0.001}\right) \overset{FS}{\leftrightarrow} 15e^{-j2\pi k 100 * 0.0005}\frac{0.001}{0.01}\text{sinc}\left(k\frac{0.001}{0.01}\right)$$

$$X_1[k] = 15e^{-j2\pi k 100 * 0.0005}\frac{0.001}{0.01}\text{sinc}\left(k\frac{0.001}{0.01}\right)$$

$$= e^{-j2\pi k/200}\frac{3}{2}\text{sinc}\left(\frac{k}{10}\right)$$

- The second component can be found using the pair

$$1 \overset{FS}{\leftrightarrow} \delta[k]$$

- Thus,

$$X_2[k] = -\frac{3}{2}\delta[k]$$

- Using linearity

$$X[k] = e^{-j2\pi k/200}\frac{3}{2}\text{sinc}\left(\frac{k}{10}\right) - \frac{3}{2}\delta[k]$$

Summary

- In this lecture we have continued our discussion of the Continuous Time Fourier Series (CTFS)
- The CTFS is most useful for periodic signals since it is a valid representation over all time
- We can view the CTFS coefficients as a discrete function in the *frequency domain*
- By having tables of a small number of common functions and transform properties we can determine the CTFS for most useful functions