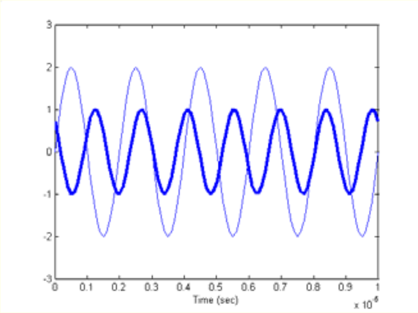
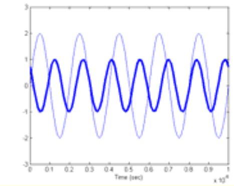
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer

Lecture #11: The Frequency Domain
CTFS of Common Functions and Using CTFS Tables

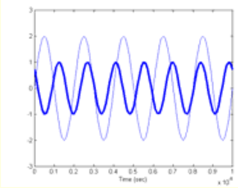


Overview



- Today we continue to discuss the concept of the Continuous Time Fourier Series (CTFS)
- Specific Topics include
  - The “Frequency Domain”
  - CTFS of common functions
  - Using tables and properties
- What to read – Sections 4.5-4.7 in the text

The Frequency Domain



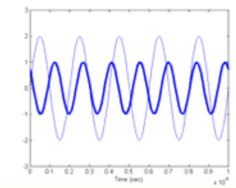
- Let us consider periodic functions where the CTFS is defined over one period T_o.
- The CTFS coefficients can be written in function form by using the discrete time unit impulse

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & else \end{cases}$$

- When we relate two representations of a function through the CTFS, we say that one representation is in the *time domain* and one is in the *frequency domain* since it represents the amount of the signal at each frequency

$$\underbrace{x(t)}_{\substack{time\\domain}} \overset{FS}{\longleftrightarrow} \underbrace{X[k]}_{\substack{frequency\\domain}}$$

Example



$$x(t) = e^{j2\pi f_o t} \qquad T_F = T_o$$

- The Fourier Series Coefficients are calculated as

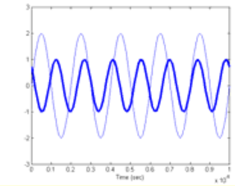
$$X[k] = \frac{1}{T_F} \int_{t_o}^{t_o + T_F} x(t) e^{-j2\pi k f_F t} dt$$

$$= \frac{1}{T_o} \int_{0}^{T_o} x(t) e^{-j2\pi k f_o t} dt$$

$$= \frac{1}{T_o} \int_{0}^{T_o} e^{j2\pi f_o t} e^{-j2\pi k f_o t} dt$$

$$= \frac{1}{T_o} \int_{0}^{T_o} e^{j2\pi f_o (1-k) t} dt$$

Example (cont.)



- Evaluating the integral:

$$X[k] = \frac{1}{j2\pi f_o (1-k) T_o} e^{j2\pi f_o (1-k) t} \Big|_0^{T_o}$$

$$= \frac{e^{j2\pi f_o (1-k) T_o} - 1}{j2\pi f_o (1-k) T_o}$$

$$= \begin{cases} \dfrac{0}{0} & k = 1 \\ 0 & k \neq 1 \end{cases}$$

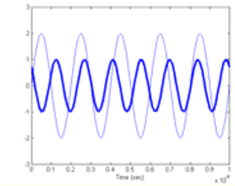
- Using L'Hopital's Rule

$$\left. \frac{\frac{d}{dk}\left\{ e^{j2\pi f_o (1-k) T_o} - 1 \right\}}{\frac{d}{dk}\left\{ j2\pi f_o (1-k) T_o \right\}} \right|_{k=1} = \left. \frac{-j2\pi f_o T_o e^{j2\pi f_o (1-k) T_o}}{-j2\pi f_o T_o} \right|_{k=1}$$

$$= e^{j2\pi f_o (1-k) T_o} \Big|_{k=1}$$

$$= 1$$

Example (final)



- Putting these together we have

$$X[k] = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases}$$

- This can be written as

$$X[k] = \delta[k-1]$$

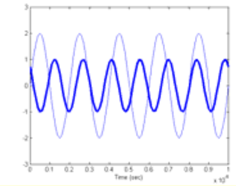
- Thus, we have the CTFS pair

$$e^{j2\pi f_o t} \overset{FS}{\longleftrightarrow} \delta[k-1] \qquad T_F = T_o$$

- If $T_F = mT_o$

$$e^{j2\pi f_o t} \overset{FS}{\longleftrightarrow} \delta[k-m] \qquad T_F = mT_o$$

Example



$$x(t) = \cos(2\pi f_o t) \qquad T_F = T_o$$

- Using Euler's relationship

$$\cos(2\pi f_o t) = \frac{e^{j2\pi f_o t} + e^{-j2\pi f_o t}}{2}$$

- Using the property of linearity and the previous result:

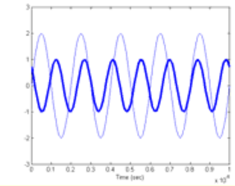
$$X[k] = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \qquad T_F = T_o$$

- And

$$\cos(2\pi f_o t) \overset{FS}{\longleftrightarrow} \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \qquad T_F = T_o$$

$$\cos(2\pi f_o t) \overset{FS}{\longleftrightarrow} \frac{1}{2}\delta[k-m] + \frac{1}{2}\delta[k+m] \qquad T_F = mT_o$$

Example



$$x(t) = \sin(2\pi f_o t) \qquad T_F = T_o$$

- Using Euler's relationship

$$\sin(2\pi f_o t) = \frac{e^{j2\pi f_o t} - e^{-j2\pi f_o t}}{2j}$$

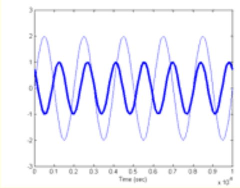
- Using the property of linearity and the previous result:

$$X[k] = \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1] \qquad T_F = T_o$$

- And

$$\sin(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1] \qquad T_F = T_o$$

$$\sin(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2j}\delta[k-m] - \frac{1}{2j}\delta[k+m] \qquad T_F = mT_o$$



Example

$$x(t) = 1$$

- The Fourier Series Coefficients are calculated as

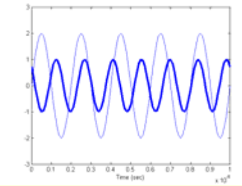
$$X[k] = \frac{1}{T_F}\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi k f_F t}dt$$

$$= \frac{1}{T_F}\int_{0}^{T_F} e^{-j2\pi k f_F t}dt$$

$$= \frac{1}{-j2\pi k f_F T_F} e^{-j2\pi k f_F t}\Bigg|_{0}^{T_F}$$

$$= \frac{e^{-j2\pi k f_F T_F} - 1}{-j2\pi k f_F T_F}$$

Example (cont.)



- Continuing

$$X[k] = \frac{e^{-j2\pi k f_F T_F} - 1}{-j2\pi k f_F T_F}$$

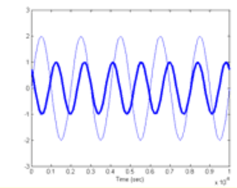
$$= \frac{e^{-j2\pi k} - 1}{-j2\pi k}$$

$$= \begin{cases} \frac{0}{0} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$\left. \frac{\frac{d}{dk}\left\{e^{-j2\pi k} - 1\right\}}{\frac{d}{dk}\left\{-j2\pi k\right\}} \right|_{k=0} = \left. \frac{-j2\pi e^{j2\pi k}}{-j2\pi} \right|_{k=0}$$

$$= \left. e^{j2\pi k} \right|_{k=0}$$

$$= 1$$



Example (solution)

- Thus, we have

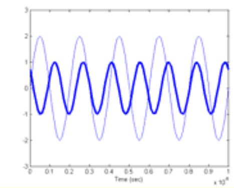
$$X[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$X[k] = \delta[k]$$

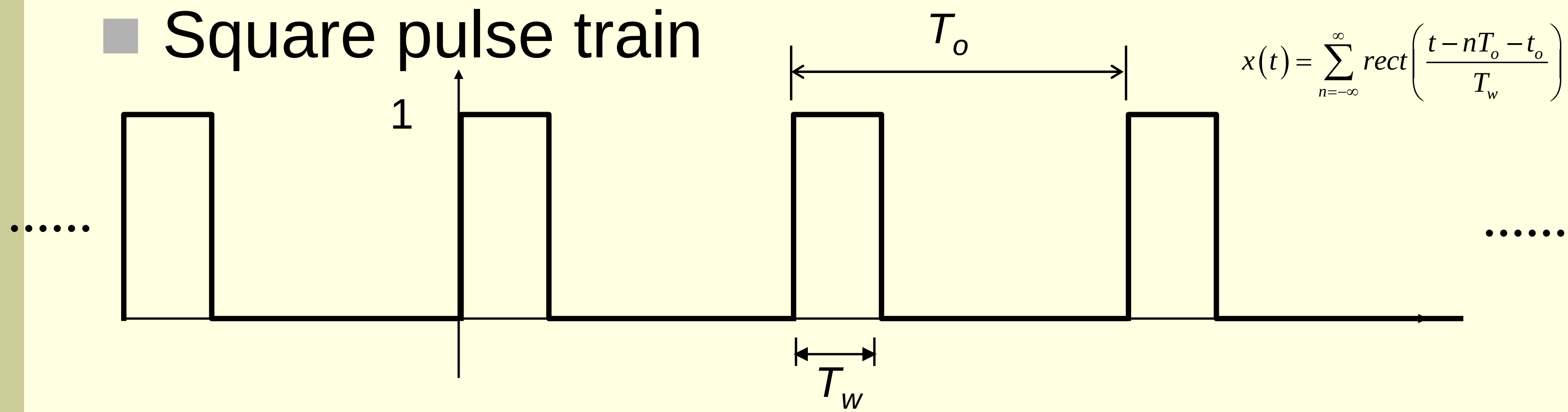
- and our CTFS pair is

$$1 \overset{FS}{\leftrightarrow} \delta[k]$$

Example



- Square pulse train

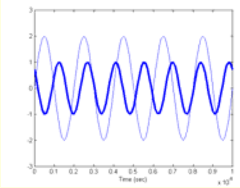


$$x(t) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t - nT_o - t_o}{T_w}\right)$$

$$X[k] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt$$

$$= \frac{1}{T_o} \int_{0}^{T_0} x(t) e^{-j2\pi k f_o t} dt$$

$$= \frac{1}{T_o} \int_{0}^{T_w} e^{-j2\pi k f_o t} dt$$



Example (cont.)

- Continuing

$$X[k] = \frac{e^{-j2\pi k f_o T_w} - e^0}{-j2\pi k f_o T_o}$$

$$= e^{-j2\pi k f_o T_w/2} \frac{e^{j2\pi k f_o T_w/2} - e^{-j2\pi k f_o T_w/2}}{j2\pi k}$$

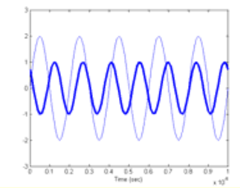
$$= e^{-j2\pi k f_o T_w/2} \frac{\sin\left(\pi k f_o T_w\right)}{\pi k}$$

$$= e^{-j\pi k f_o T_w} f_o T_w \mathrm{sinc}\left(k f_o T_w\right)$$

- In general:

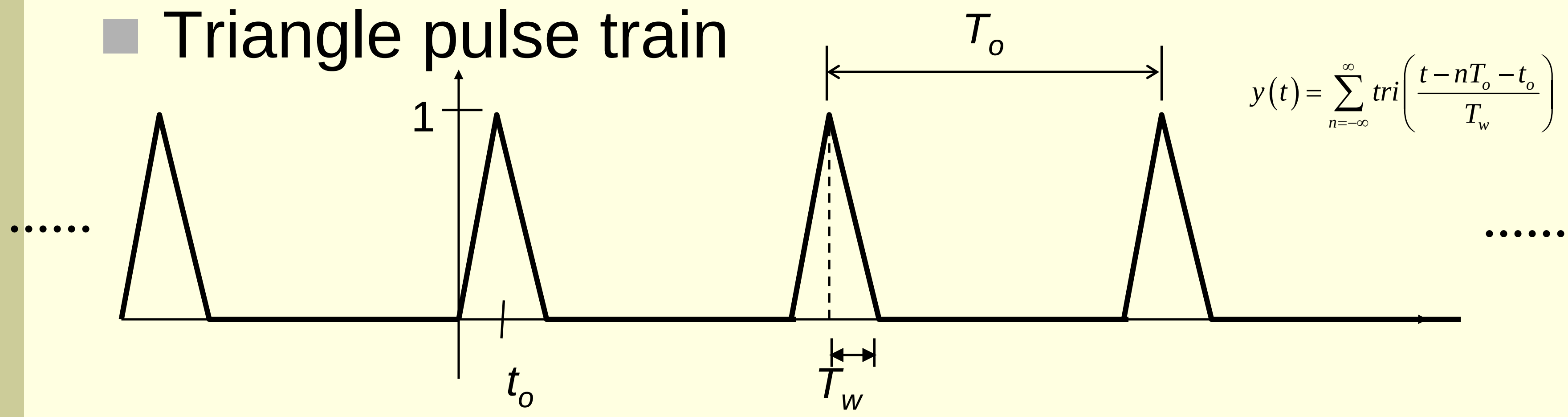
$$\sum_{n=-\infty}^{\infty} rect\left(\frac{t - nT_o - t_o}{T_w}\right) \overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o} f_o T_w \mathrm{sinc}\left(k f_o T_w\right)$$

t_o = time delay
(T_w/2 in this case
$T_F = T_o$



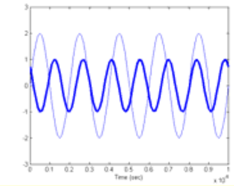
Example

- Triangle pulse train



$$y(t)=\sum_{n=-\infty}^{\infty} tri\left(\frac{t-nT_o-t_o}{T_w}\right)$$

- Now, we could simply plug into the equations to derive the CTFS. However, we can see that the triangular pulse train is the result of a periodic convolution of a square pulse train



Example (cont.)

- Specifically

$$y(t)=\frac{1}{T_w}\left\{x(t)\otimes x(t)\right\}$$

- Thus,

$$\begin{aligned}Y[k]&=\frac{1}{T_w}\left\{T_o X[k]X[k]\right\}\\&=\frac{1}{T_w}T_o\left\{e^{-j2\pi k f_o T_w/2} f_o T_w \mathrm{sinc}(k f_o T_w)e^{-j2\pi k f_o T_w/2} f_o T_w \mathrm{sinc}(k f_o T_w)\right\}\\&=f_o T_w e^{-j2\pi k f_o T_w}\mathrm{sinc}^2(k f_o T_w)\end{aligned}$$

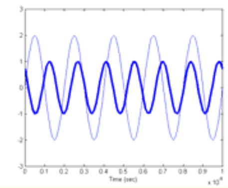
t_o for the rectangular pulse train is $T_w/2$ in this example

- In general:

$$\sum_{n=-\infty}^{\infty} tri\left(\frac{t-nT_o-t_o}{T_w}\right)\overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o} f_o T_w \mathrm{sinc}^2(k f_o T_w)$$

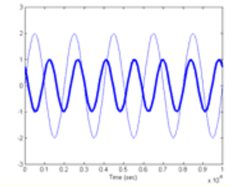
t_o = time delay
= T_w in this example

Use of Tables



- We could continue to find CTFS pairs, but in general it is easier to use properties and tables of pairs to find new pairs.

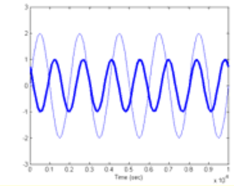
Property	
Conjugation	$x^*(t) \xleftrightarrow{FS} X^*[-k]$
Linearity	$\alpha x(t) + \beta y(t) \xleftrightarrow{FS} \alpha X[k] + \beta Y[k]$
Time-shifting	$x(t - t_o) \xleftrightarrow{FS} e^{-j2\pi k f_{Fx} t_o} X[k]$
Frequency-shifting	$e^{j2\pi k_o f_{Fx} t} x(t) \xleftrightarrow{FS} X[k - k_o]$
Time reversal	$x(-t) \xleftrightarrow{FS} X[-k]$
Time-derivative	$\frac{d}{dt}\{x(t)\} \xleftrightarrow{FS} (j2\pi k f_{Fx}) X[k]$
Time-integration	$\int_{-\infty}^{t} x(\tau) d\tau \xleftrightarrow{FS} \frac{1}{j2\pi k f_{Fx}} X[k]$



Properties (cont.)

Property	
Time-scaling (integer a) $T_F = T_o$	$x(at) \overset{FS}{\leftrightarrow} \begin{cases} X\left[\frac{k}{a}\right] & \frac{k}{a} = \text{integer} \\ 0 & \text{else} \end{cases}$
Multiplication	$x(t)y(t) \overset{FS}{\longleftrightarrow} \sum_{q=-\infty}^{\infty} Y[q]X[k-q]$
Convolution	$x(t) \otimes y(t) \overset{FS}{\longleftrightarrow} T_F X[k]Y[k]$
Conjugation	$x^*(t) \overset{FS}{\longleftrightarrow} X^*[-k]$

Example



- Find the Continuous Time Fourier Series of

$$x(t) = \cos(50\pi t - \pi/4)$$

for $T_F = T_o$

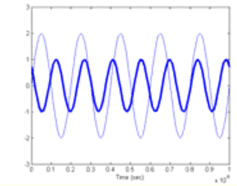
We can use

$$\cos(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$$

and

$$x(t - t_o) \xleftrightarrow{FS} e^{-j2\pi k f_{Fx} t_o} X[k]$$

Example (cont.)



- Re-writing the original equation

$$x(t) = \cos(50\pi t - \pi/4)$$

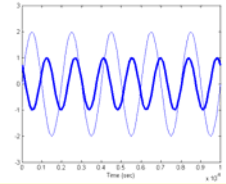
$$= \cos\left(50\pi\left(t - \frac{\pi/4}{50\pi}\right)\right)$$

$$= \cos\left(50\pi\left(t - \frac{1}{200}\right)\right)$$

$$X[k] = e^{-j2\pi k*25(1/200)}\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$

$$= e^{-j2\pi k/8}\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$

Example (cont.)



- Continuing

$$X[k] = e^{-j2\pi k/8}\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$

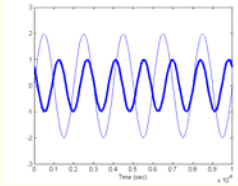
$$= e^{-j2\pi/8}\frac{1}{2}\delta[k-1] + e^{j2\pi/8}\frac{1}{2}\delta[k+1]$$

$$= e^{-j\pi/4}\frac{1}{2}\delta[k-1] + e^{j\pi/4}\frac{1}{2}\delta[k+1]$$

$$= \frac{(1-j)}{\sqrt{2}}\frac{1}{2}\delta[k-1] + \frac{(1+j)}{\sqrt{2}}\frac{1}{2}\delta[k+1]$$

$$\boxed{\cos(50\pi t - \pi/4) \xleftrightarrow{FS} \frac{(1-j)}{\sqrt{2}}\frac{1}{2}\delta[k-1] + \frac{(1+j)}{\sqrt{2}}\frac{1}{2}\delta[k+1]}$$

Graphing the Solution

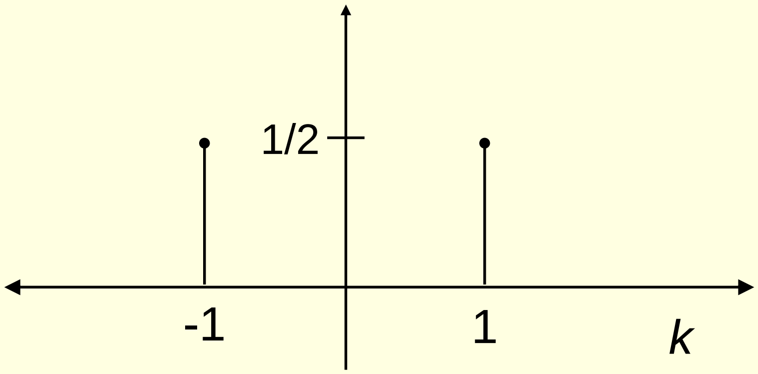


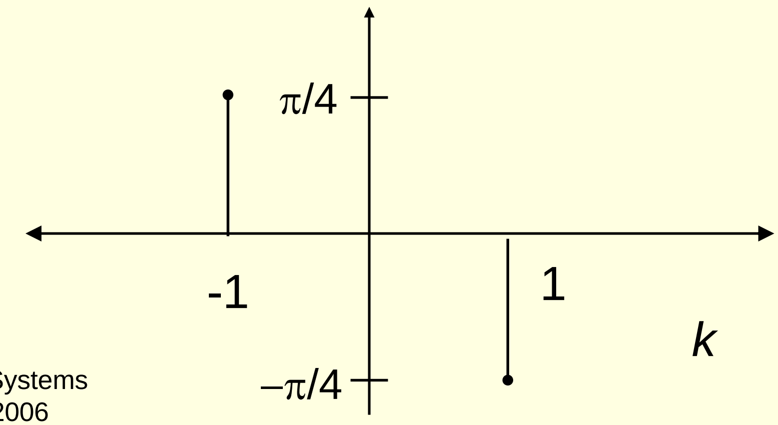
- When plotting the CTFS, due to the fact that the solution is generally *complex*, we typically plot the magnitude and phase separately.

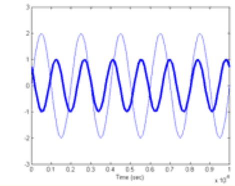
$$\cos\left(50\pi t - \pi/4\right) \xleftrightarrow{FS} \frac{(1-j)}{\sqrt{2}}\frac{1}{2}\delta[k-1] + \frac{(1+j)}{\sqrt{2}}\frac{1}{2}\delta[k+1]$$

$$\left|X[k]\right| = \left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\}$$

$$\angle X[k] = \left\{-\frac{\pi}{4}\delta[k-1] + \frac{\pi}{4}\delta[k+1]\right\}$$







Example

- Find the Continuous Time Fourier Series of

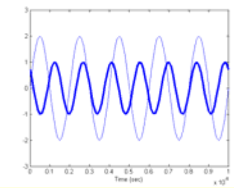
$$x(t) = 5\cos(10\pi t)\cos(10000\pi t)$$

for $T_F = 1/5$

- Note that we have two sinusoids at different multiples of the fundamental frequency f_F = 5Hz
- Further, we know that

$$\cos(2\pi m f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-m] + \frac{1}{2}\delta[k+m]$$

Example (cont.)



- Re-writing the function in terms of f_F:

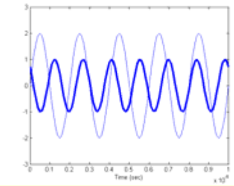
$$x(t) = 5\cos(2\pi f_F t)\cos(2\pi 1000 f_F t)$$

$$\cos(2\pi f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$$

$$\cos(2\pi 1000 f_o t) \overset{FS}{\leftrightarrow} \frac{1}{2}\delta[k-1000] + \frac{1}{2}\delta[k+1000]$$

- Further, recalling our multiplication-convolution property:

$$y(t)z(t) \overset{FS}{\leftrightarrow} Y[k] * Z[k]$$

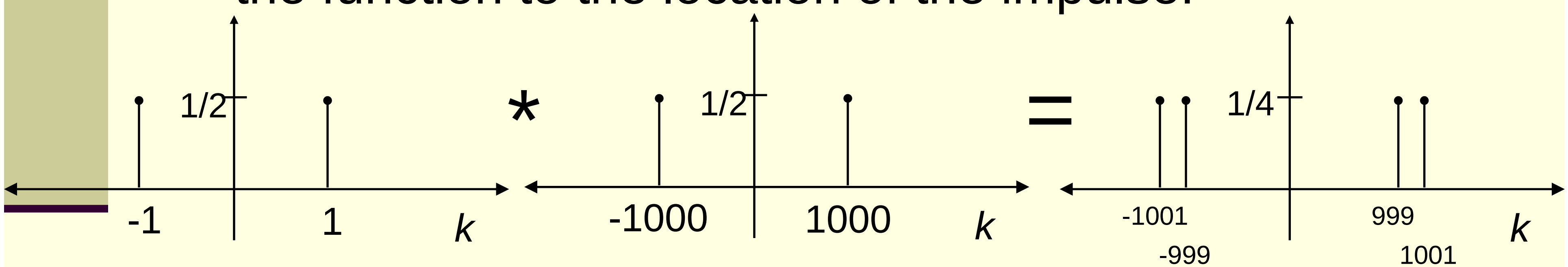


Example - Convolving

- Thus, we have

$$X[k] = 5\left\{\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right\} * \left\{\frac{1}{2}\delta[k-1000] + \frac{1}{2}\delta[k+1000]\right\}$$

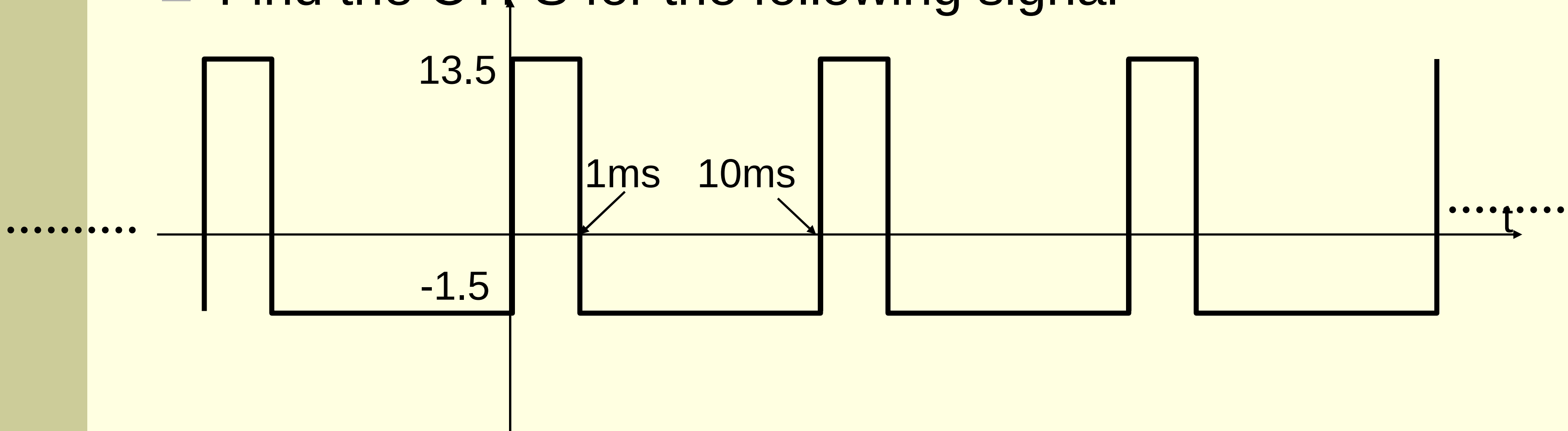
- Recall that convolving with an impulse simply moves the function to the location of the impulse:



$$X[k] = \frac{5}{4}\left\{\delta[k-1001] + \delta[k-999] + \delta[k+1001] + \delta[k+999]\right\}$$

Example

- Find the CTFS for the following signal

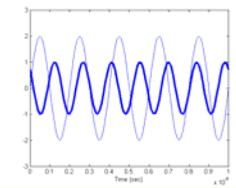


- First, we can write an equation for this signal as

$$x(t)=15\sum_{n=-\infty}^{\infty} rect\left(\frac{t-n\bullet 0.01-0.0005}{0.001}\right)-\frac{3}{2}$$

- Recall the CTFS pair $\sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT_o-t_o}{T_w}\right)\overset{FS}{\leftrightarrow} e^{-j2\pi k f_o t_o}\frac{T_w}{T_o}\text{sinc}\left(k\frac{T_w}{T_o}\right)$

Example (cont.)



- Thus we have

$$15\sum_{n=-\infty}^{\infty} rect\left(\frac{t-n\bullet 0.01-0.0005}{0.001}\right)\overset{FS}{\leftrightarrow}15e^{-j2\pi k100*0.0005}\frac{0.001}{0.01}\text{sinc}\left(k\frac{0.001}{0.01}\right)$$

$$X_1[k]=15e^{-j2\pi k100*0.0005}\frac{0.001}{0.01}\text{sinc}\left(k\frac{0.001}{0.01}\right)$$

$$=e^{-j2\pi k/200}\frac{3}{2}\text{sinc}\left(\frac{k}{10}\right)$$

- The second component can be found using the pair

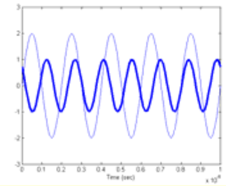
$$1\overset{FS}{\leftrightarrow}\delta[k]$$

- Thus,

$$X_2[k]=-\frac{3}{2}\delta[k]$$

- Using linearity

$$X[k]=e^{-j2\pi k/200}\frac{3}{2}\text{sinc}\left(\frac{k}{10}\right)-\frac{3}{2}\delta[k]$$



Summary

- In this lecture we have continued our discussion of the Continuous Time Fourier Series (CTFS)
- The CTFS is most useful for periodic signals since it is a valid representation over all time
- We can view the CTFS coefficients as a discrete function in the *frequency domain*
- By having tables of a small number of common functions and transform properties we can determine the CTFS for most useful functions