ECE 2704
Signals and Systems
Spring 2006

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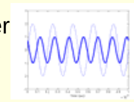
Lecture #12: Introduction to the Fourier Transform



Overview

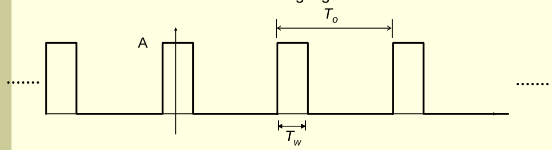
- Previously we examined a technique for representing a signal using an infinite sum of sinusoids
  - This representation is termed the Fourier Series.
  - If we represent the signal of interest with a sum of delta functions with each delta function weighted by the Fourier Series coefficients, we can view this representation as a *frequency domain representation*
- Today we expand on this idea by introducing the concept of the Continuous Time Fourier Transform (CTFT)
- What to read – Section 5.1-5.4 in the text

Limitations of the Fourier Series

- The continuous time Fourier Series (CTFS) is a useful analytical tool but has limitations:
  - It can represent periodic signals for all time and can represent aperiodic signals for a finite time, but cannot represent an aperiodic signal for all time
  - It inherently depends on the fundamental frequency (i.e., the observation interval) chosen – signals with different fundamental periods must be converted to a common observation interval.
- The Fourier Transform will overcome these limitations by allowing us to represent periodic *and* aperiodic signals without depending on the observation interval

Illustrative Example

- To help us understand the relationship between the CTFS and the CTFT consider a the following signal



- with $T_w = T_o/2$ and $t_o = 0$.

$$x(t) = A\sum_{n=-\infty}^{\infty} rect\left(\frac{t-2nT_w}{T_w}\right)$$

- We know that the Continuous Time Fourier Series is

$$X[k] = \frac{A}{2}\text{sinc}\left(\frac{k}{2}\right)$$

Reducing the Duty Cycle

- Now let us reduce the duty cycle such that
  - *T_w remains constant*
  - *$T_o=10T_w$*
  - The average power is kept constant (i.e., we increase the amplitude by T_o *to maintain constant average power)*

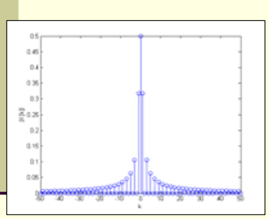
$$x(t) = 10A\sum_{n=-\infty}^{\infty} rect\left(\frac{t-10nT_w}{T_w}\right) \qquad X[k] = \frac{A}{2}\text{sinc}\left(\frac{k}{10}\right)$$

- If we further reduce the duty cycle such that
  - *T_w remains constant*
  - *$T_o=1000T_w$*
  - The average power is kept constant (i.e., we increase the amplitude by T_o

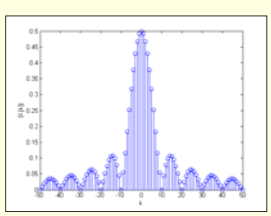
$$x(t) = 1000A\sum_{n=-\infty}^{\infty} rect\left(\frac{t-1000nT_w}{T_w}\right) \qquad X[k] = \frac{A}{2}\text{sinc}\left(\frac{k}{1000}\right)$$

The Magnitude Response

- Plotting for $T_o = 2T_w$ and $T_o = 10T_w$



$$X[k] = \frac{A}{2}\text{sinc}\left(\frac{k}{2}\right)$$

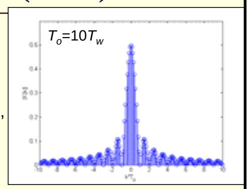


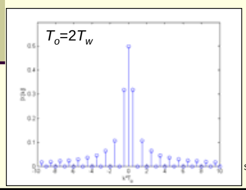
$$X[k] = \frac{A}{2}\text{sinc}\left(\frac{k}{10}\right)$$

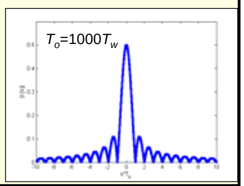
Decreasing duty cycle slows the CTFS in frequency and shows more detail of the underlying sinc function

Magnitude Response (cont.)

- If we normalize the *x*-axis by the fundamental period, we see that increasing the period increases the sampling rate of the underlying function
- If we let the period go to infinity (i.e., we let the square pulse train approach a single pulse), the Fourier Series representation approaches a continuous function







The Fourier Transform

- Let's examine the Fourier Series representation of a periodic signal with fundamental frequency

$$\Delta f = f_F = 1/T_F$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k\Delta f t} \qquad X[k] = \frac{1}{T_F}\int_{T_F} x(t) e^{-j2\pi k \Delta f t} dt$$

- Substituting $X[k]$ into the equation for $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{T_{Fx}} \int_{T_{Fx}} x(\tau) e^{-j2\pi k\Delta f \tau} d\tau \right\} e^{j2\pi k \Delta f t}$$

- Since the exact starting and stopping points for the integration are arbitrary (provided that we integrate over a period), let us define the integration period to be

$$-T_F/2 \le t \le T_F/2$$

The Fourier Transform (cont.)

- Using this region of integration

$$x(t) = \sum_{k=-\infty}^{\infty} \left\{ \Delta f \int_{-T_F/2}^{T_F/2} x(\tau) e^{-j2\pi k\Delta f \tau} d\tau \right\} e^{j2\pi k \Delta f t}$$

- Now in the limit as T_F approaches infinity, the periodic signal becomes aperiodic, Δf approaches the differential df, $k\Delta f$ becomes a continuous variable f

$$x(t) = \lim_{T_F \to \infty} \left\{ \sum_{k=-\infty}^{\infty} \left\{ \Delta f \int_{-T_F/2}^{T_F/2} x(\tau) e^{-j2\pi k\Delta f \tau} d\tau \right\} e^{j2\pi k \Delta f t} \right\}$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau \right\} e^{j2\pi f t} df$$

The Fourier Transform

- We then define the Continuous Time Fourier Transform as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$
$$= F\{x(t)\}$$

and the original signal can be written in terms of the Fourier Transform as

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$
$$= F^{-1}\{X(f)\}$$

This is sometimes called the *inverse Fourier Transform*

The Frequency Domain

- The original signal *x(t)* is said to be in the *time domain* since its argument represents time
- The Fourier Transform *X(f)* representation is said to be in the *frequency domain* since its argument *f* represents frequency
- Notes:
  - Frequency is the reciprocal of time
  - The Fourier Transform is referred to as an *analysis* of the signal *x(t)* since it extracts the components of *x(t)* at each value of *f*
  - The Inverse Fourier Transform is referred to as *synthesis* since it recombines the components *X(f)* to obtain the original signal *x(t)*
  - The physical meaning of X(f) depends on the meaning of x(t). If x(t) has units of volts, X(f) has units volts/Hz.
    - Thus it represents how much of the over all voltage signal is present at each frequency.

Further Notes

- The function *X(f)* is also sometimes referred to as the *amplitude spectral density* or the *spectrum* of *x(t)*
- We often represent Fourier Transform pairs using the notation

$$x(t) \overset{F}{\leftrightarrow} X(f)$$

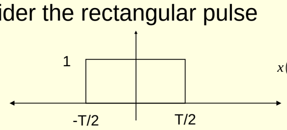
and we refer to *x(t)* and *X(f)* as a Fourier Transform pair

- Sometimes the Fourier Transform is defined in terms of radian frequency:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} df$$

Example

- Consider the rectangular pulse



$$x(t) = rect\left(\frac{t}{T}\right)$$

- Find the Continuous Time Fourier Transform

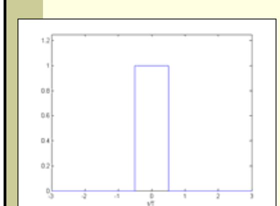
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$
$$= \int_{-T/2}^{T/2} e^{-j2\pi ft} dt$$

Example (cont.)

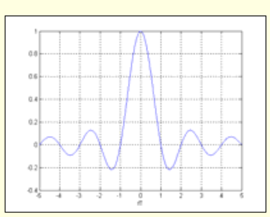
$$X(f) = \int_{-T/2}^{T/2} \left(\cos(2\pi ft) - j\sin(2\pi ft)\right) dt$$
$$= \left[\frac{1}{2\pi f}\sin(2\pi ft) + j\frac{1}{2\pi f}\cos(2\pi ft)\right]\Bigg|_{-T/2}^{T/2}$$
$$= \frac{1}{2\pi f}\left[\sin(2\pi fT/2) - \sin(-2\pi fT/2)\right] + \ldots$$
$$j\frac{1}{2\pi f}\left[\cos(2\pi fT/2) - \cos(-2\pi fT/2)\right]$$
$$= \frac{1}{\pi f}\sin(\pi fT)$$
$$= T\mathrm{sinc}(fT)$$

$$\boxed{rect\left(\frac{t}{T}\right) \overset{F}{\leftrightarrow} T\mathrm{sinc}(fT)}$$ Fourier Transform Pair

Plots



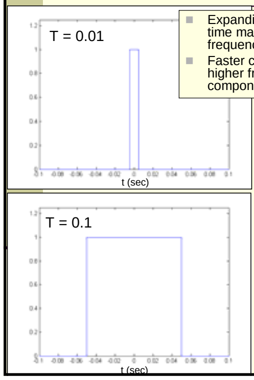
$rect\left(\frac{t}{T}\right)$

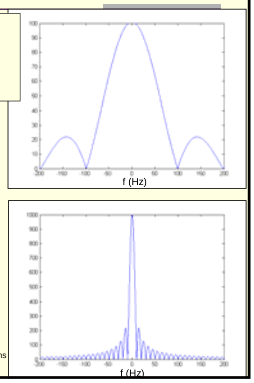


$T\mathrm{sinc}(fT)$

Time vs. Frequency

- Expanding a signal in time makes it narrow in frequency
- Faster changes require higher frequency components





Time vs. Frequency

- As mentioned earlier, time and frequency are reciprocal
- If a function speeds up in time, it slows down in frequency
  - If a signal changes rapidly it requires more high-frequency components
  - Signals which change rapidly in time are said to have a *large bandwidth* (a measure of the frequency content)
- If a function slows down time, it speeds up in frequency
  - If a signal changes slowly in time it requires less high-frequency components and more low-frequency components
  - Signals which change slowly in time are said to have a *small bandwidth*

Generalized Fourier Transform

- Consider the Fourier Transform of a constant *A*

$$x(t) = A$$

$$X(f) = \int_{-\infty}^{\infty} A e^{-j2\pi ft} dt$$

$$= A \int_{-\infty}^{\infty} e^{-j2\pi ft} dt$$

- Unfortunately, this integral does not converge. Thus, the Fourier Transform does not technically exist. However, we can determine a *generalized Fourier Transform*

Generalized FT (cont.)

- Consider a signal $x(t) = Ae^{-\sigma|t|}$ for σ > 0

$$X(f) = \int_{-\infty}^{\infty} Ae^{-\sigma|t|}e^{-j2\pi ft}dt$$
$$= \int_{-\infty}^{0} Ae^{\sigma t}e^{-j2\pi ft}dt + \int_{0}^{\infty} Ae^{-\sigma t}e^{-j2\pi ft}dt$$
$$= A\frac{2\sigma}{\sigma^2 + (2\pi f)^2}$$

- Now, let σ approach zero

$$\lim_{\sigma \to 0}\left\{A\frac{2\sigma}{\sigma^2 + (2\pi f)^2}\right\} = \begin{cases} 0 & f \neq 0 \\ 0/0 & f = 0 \end{cases}$$

Generalized FT (cont.)

- Area

$$\int_{-\infty}^{\infty} A\frac{2\sigma}{\sigma^2 + (2\pi f)^2}df = A\left[\frac{2\sigma}{2\pi\sigma}\tan^{-1}\left(\frac{2\pi f}{\sigma}\right)\right]\Bigg|_{-\infty}^{\infty} = \frac{A}{\pi}\left\{\frac{\pi}{2} + \frac{\pi}{2}\right\} = A$$

- Thus, as σ → 0, the resulting function
  - is zero everywhere except at f=0.
  - has unit area regardless of σ
- This is exactly the definition of the unit impulse.
- Thus, we have the Fourier Transform pair

$$A \overset{F}{\leftrightarrow} \delta(f)$$

Frequency Shift Property

- Let $z(t) = e^{j2\pi f_o t}x(t)$
- Then

$$Z(f) = \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}dt$$
$$= \int_{-\infty}^{\infty} e^{j2\pi f_o t}x(t)e^{-j2\pi ft}dt$$
$$= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_o)t}dt$$
$$= X(f - f_o)$$

$$e^{j2\pi f_o t}x(t) \xleftrightarrow{FS} X(f - f_o)$$

FT of Complex Exponentials

- Using the generalized Fourier Transform of a constant and the frequency shift property we can find the FT of a complex exponential:

$$z(t) = e^{j2\pi f_o t}$$

- This is just a "frequency shift" of a constant, thus

$$Z(f) = \delta(f - f_o)$$

$$\boxed{F\left\{e^{j2\pi f_o t}\right\} = \delta(f - f_o)}$$

- This also results in

$$\sin(2\pi f_o t) \overset{F}{\leftrightarrow} \frac{1}{2j}\delta(f - f_o) - \frac{1}{2j}\delta(f + f_o)$$

$$\cos(2\pi f_o t) \overset{F}{\leftrightarrow} \frac{1}{2}\delta(f - f_o) + \frac{1}{2}\delta(f + f_o)$$

Fourier Transform of Periodic Signals

- Using this same approach we can develop the Fourier Transform for any periodic signal.
- Specifically, we can define a CTFS for any periodic signal that is valid over all time by making $T_F = T_o$.
- Using the linearity property of the Fourier Transform (to be shown next class) we can write the FT of any periodic signal:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{-j2\pi k f_o t}$$

$$X(f) = F\left\{\sum_{k=-\infty}^{\infty} X[k] e^{-j2\pi k f_o t}\right\}$$

$$= \sum_{k=-\infty}^{\infty} X[k]\delta(f + kf_o)$$

Summary

- In this lecture we have introduced a new tool termed the *Fourier Transform.*
- The Fourier Transform is useful for providing a *frequency domain* representation of periodic and aperiodic signals that is valid for *all time.*
- The Fourier Transform is an incredibly useful tool in many fields of engineering.
- Understanding the relationship between time and frequency is perhaps one of the most important concepts in this course.