ECE 2704
Signals and Systems
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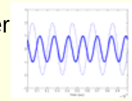
Lecture #13: Properties of the Fourier Transform



Overview

- Today we continue our discussion of the Continuous Time Fourier Transform (CTFT)
- The current lecture focuses on the properties of the Fourier Transform
  - These properties are very similar to the properties that we studied earlier for the CTFS
  - Understanding the properties and a handful of basic Fourier Transforms allows us to determine most of the Fourier Transforms of interest
- What to read – Section 5.5 in the text

Linearity

- If $$z(t)=\alpha x(t)+\beta y(t)$$
- Then
$$\begin{aligned}Z(f) &= \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty}\{\alpha x(t)+\beta y(t)\}e^{-j2\pi ft}dt \\ &= \alpha\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt+\beta\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt \\ &= \alpha X(f)+\beta Y(f)\end{aligned}$$
- In other words
$$\boxed{\alpha x(t)+\beta y(t)\overset{F}{\longleftrightarrow}\alpha X(f)+\beta Y(f)}$$

Time Shifting

- Let $z(t) = x(t - t_o)$
- Then

$$
\begin{aligned}
Z(f) &= \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} x(t - t_o) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi kf(\tau + t_o)} d\tau \\
&= e^{-j2\pi ft_o} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \\
&= e^{-j2\pi ft_o} X(f)
\end{aligned}
$$

let $\tau = t\text{-}t_o$

$$x(t - t_o) \overset{FS}{\longleftrightarrow} e^{-j2\pi ft_o} X(f)$$

Frequency Shifting

- Let $z(t) = e^{j2\pi f_o t} x(t)$
- Then

$$
\begin{aligned}
Z(f) &= \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} e^{j2\pi f_o t} x(t) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} x(t) e^{-j2\pi(f - f_o)t} dt \\
&= X(f - f_o)
\end{aligned}
$$

$$e^{j2\pi f_o t} x(t) \overset{FS}{\longleftrightarrow} X(f - f_o)$$

Interpretation

- Shifting by a constant in the time domain results in a phase shift in the frequency domain
  - Note that the phase shift changes with frequency since a constant time results in different phase values at each frequency
    - $\theta = 2\pi f t_o$
  - The magnitude of the Fourier Transform remains the same – This should make sense since shifting in time does not change the properties of the signal
- Shifting by a constant in the frequency domain results in a multiplication by a complex sinusoid in the time domain

Time Scaling

- Let $z(t) = x(at)$
- Then the Fourier Transform is

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt$$
$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda / a} d\lambda$$
$$= \frac{1}{a} X\left(\frac{f}{a}\right)$$

Let λ=at

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{a} X\left(\frac{f}{a}\right)$$

Frequency Scaling

- Let $Z(f) = X(af)$

$$z(t) = \int_{-\infty}^{\infty} Z(f) e^{j2\pi ft} df$$
$$= \int_{-\infty}^{\infty} X(af) e^{j2\pi ft} df$$
$$= \frac{1}{a} \int_{-\infty}^{\infty} X(\lambda) e^{j2\pi(\lambda/a)t} d\lambda$$
$$= \frac{1}{a} x\left(\frac{t}{a}\right)$$

Let λ=*af*

$$\frac{1}{a} x\left(\frac{t}{a}\right) \overset{F}{\longleftrightarrow} X(af)$$

Scaling - Interpretation

- Scaling a signal in time by α scales the Fourier transform (i.e., the signal in frequency) by $1/\alpha$.
- Scaling a signal in frequency by α scales the time domain signal by $1/\alpha$.
- Does this make sense? Recall our previous discussion that time and frequency are reciprocal.
- Let assume that $\alpha > 1$. Scaling a signal in time by α speeds the signal up in time.
  - The resulting transform is scaled by $1/\alpha$ which slows the transform down in frequency – this means that more of the larger frequency values are present to accomplish faster changes.
- Scaling a signal in time by $1/\alpha$ slows the signal down in time.
  - The resulting transform is scaled by α which speeds it up in frequency – this means that more low frequency values are present to account for slower changes.

Transform of a Conjugate

- Let $z(t) = x^*(t)$

$$z(t) = \left[\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df\right]^*$$
$$= \int_{-\infty}^{\infty} X^*(f) e^{-j2\pi ft} df$$
$$= -\int_{\infty}^{-\infty} X^*(-\lambda) e^{j2\pi\lambda t} d\lambda$$
$$= \int_{-\infty}^{\infty} X^*(-\lambda) e^{j2\pi\lambda t} d\lambda$$

$$\boxed{x^*(t) \xleftrightarrow{F} X^*(-f)}$$

Convolution in Time

- Let $z(t) = x(t) * y(t)$
- Then

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} \left\{x(t) * y(t)\right\} e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau\right\} e^{-j2\pi ft} dt$$

- Changing the order of integration:

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\left\{\int_{-\infty}^{\infty} y(t-\tau) e^{-j2\pi ft} dt\right\}}_{F\{y(t-\tau)\}} d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} Y(f) d\tau$$

Convolution (cont.)

- Finishing ….

$$Z(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} Y(f) d\tau$$
$$= Y(f) \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau$$
$$= X(f) Y(f)$$

$$\boxed{x(t) * y(t) \xleftrightarrow{F} X(f) Y(f)}$$

Multiplication in Time

- Now let $z(t) = x(t)y(t)$

$$Z(f) = \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} x(t)\left\{\int_{-\infty}^{\infty} Y(\lambda)e^{j2\pi\lambda t}d\lambda\right\}e^{-j2\pi ft}dt$$

- Changing the order of integration

$$= \int_{-\infty}^{\infty} Y(\lambda)\underbrace{\left\{\int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-\lambda)t}dt\right\}}_{X(f-\lambda)}d\lambda$$

Multiplication (cont.)

- Continuing …

$$Z(f) = \int_{-\infty}^{\infty} Y(\lambda)X(f-\lambda)d\lambda$$

$$= X(f)*Y(f)$$

$$\boxed{x(t)y(t) \stackrel{F}{\longleftrightarrow} X(f)*Y(f)}$$

- Thus, convolution in the time domain results in multiplication in the frequency domain while multiplication in the time domain results in convolution in the frequency domain.
- This can greatly simply some system analysis

Time Differentiation

- Using the Fourier Transform representation of *x(t)* and taking the derivative

$$\frac{d}{dt}\{x(t)\} = \frac{d}{dt}\left\{\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df\right\}$$

$$= \int_{-\infty}^{\infty} j2\pi fX(f)e^{j2\pi ft}df$$

$$= F^{-1}\{j2\pi fX(f)\}$$

- Thus,

$$\boxed{\frac{d}{dt}\{x(t)\} \stackrel{F}{\longleftrightarrow} j2\pi fX(f)}$$

Modulation

- A common operation in communication systems is *modulation* or the multiplication of a signal by a high frequency sinusoid:

$$z(t) = x(t)\cos(2\pi f_c t)$$

- We can find the Fourier Transform of *z(t)* using the multiplication-convolution property

$$Z(f) = X(f) * F\{\cos(2\pi f_c t)\}$$
$$= X(f) * \left\{\frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)\right\}$$

- Using the sifting property of the impulse

$$Z(f) = \frac{1}{2}X(f - f_c) + \frac{1}{2}X(f + f_c)$$

Parseval's Theorem

- While the time domain signal *x(t)* and the frequency domain signal *X(f)* appear quite different they do have the same energy.
- That is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, dt$$

- In other words, it doesn't matter whether I calculate the energy of a signal in the time domain or in the frequency domain, I get the same result.
  - This should make sense since the two representations are equivalent

Parseval's Theorem - Proof

- First recall that $|x(t)|^2 = x(t)x^*(t)$. Then, from the multiplication and conjugate properties of the Fourier Transform:

$$x(t)x^*(t) \overset{F}{\longleftrightarrow} X(f) * X^*(-f)$$

- Rewriting the Fourier Transform relationship:

$$\int_{-\infty}^{\infty} x(t)x^*(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} X(\lambda)X^*(-(f-\lambda))\,d\lambda$$
$$= \int_{-\infty}^{\infty} X(\lambda)X^*(\lambda - f)\,d\lambda$$

- Since this must hold for any value of *f*, let us choose *f = 0:*

$$\int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \int_{-\infty}^{\infty} X(\lambda)X^*(\lambda)\,d\lambda$$
$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(\lambda)|^2 \, d\lambda$$

Q.E.D.

Duality

- Due to the similar nature of the Fourier Transform and the Inverse Fourier Transform, the CTFT exhibits the *duality property.*
- The duality property says that if we have the Fourier Transform pair

$$x(t) \stackrel{F}{\longleftrightarrow} X(f)$$

then we also have the Fourier Transform pair

$$X(t) \stackrel{F}{\longleftrightarrow} x(-f)$$

Example

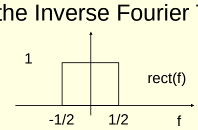
- From our previous development we know the following FT pair

$$rect(t) \stackrel{F}{\longleftrightarrow} \mathrm{sinc}(f)$$

- The duality property says that

$$\mathrm{sinc}(t) \stackrel{F}{\longleftrightarrow} rect(-f) = rect(f)$$

Check: Find the Inverse Fourier Transform for



Example (cont.)

- Check:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\ &= \int_{-1/2}^{1/2} e^{j2\pi ft} df \\ &= \left. \frac{e^{j2\pi ft}}{j2\pi t} \right|_{-1/2}^{1/2} \\ &= \frac{e^{j\pi t}}{j2\pi t} - \frac{e^{-j\pi t}}{j2\pi t} \\ &= \frac{1}{\pi t} \frac{e^{j\pi t} - e^{-j\pi t}}{2j} \\ &= \frac{\sin(\pi t)}{\pi t} \\ &= \mathrm{sinc}(t) \end{aligned}$$

Example (cont.)

More generally, consider the function *X(f) = rect(f/B)*:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\
&= \int_{-B/2}^{B/2} e^{j2\pi ft} df \\
&= \frac{e^{j2\pi ft}}{j2\pi t}\Bigg|_{-B/2}^{B/2} \\
&= \frac{e^{j\pi tB}}{j2\pi t} - \frac{e^{-j\pi tB}}{j2\pi t} \\
&= \frac{1}{\pi t}\frac{e^{j\pi tB} - e^{-j\pi tB}}{2j} \\
&= \frac{\sin(\pi tB)}{\pi t} \\
&= B\text{sinc}(tB)
\end{aligned}
$$

Example 2

- From our previous development we know the following FT pair

$$A \overset{F}{\longleftrightarrow} \delta(f)$$

- The duality property says that

$$\delta(t) \overset{F}{\longleftrightarrow} 1$$

- Check: Find the Fourier Transform for

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt \\
&= e^{-j2\pi ft}\Big|_{t=0} \\
&= 1
\end{aligned}
$$

The FT and the Total Area Integral

- The total area under a function can be determined by evaluating the Fourier Transform at *f=0:*

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\
X(0) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi t*0} dt \\
&= \int_{-\infty}^{\infty} x(t) dt
\end{aligned}
$$

Integration

- Just as we have the property

$$\frac{d}{dt}\left\{x(t)\right\} \xleftrightarrow{F} j2\pi f X(f)$$

- Similarly we can show that

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

Summary of Properties

Property	
Conjugation	$x^*(t) \xleftrightarrow{F} X^*(-k)$
Linearity	$\alpha x(t) + \beta y(t) \xleftrightarrow{F} \alpha X(f) + \beta Y(f)$
Time-shifting	$x(t - t_o) \xleftrightarrow{F} e^{-j2\pi f t_o} X(f)$
Frequency-shifting	$e^{j2\pi f_o t} x(t) \xleftrightarrow{F} X(f - f_o)$
Time reversal	$x(-t) \xleftrightarrow{F} X(-f)$
Time-differentiation	$\frac{d}{dt}\left\{x(t)\right\} \xleftrightarrow{F} (j2\pi f) X(f)$
Time-integration	$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FS} \frac{1}{j2\pi f} X(f)$
Time/freq-scaling	$x(at) \xleftrightarrow{F} \frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$
Multiplication	$x(t)\,y(t) \xleftrightarrow{F} X(f) * Y(f)$
Convolution	$x(t) * y(t) \xleftrightarrow{F} X(f)\,Y(f)$

Summary

- In this lecture we have examined several properties of the Fourier Transform
- The properties are very similar to those for the Fourier Series
- We will find these properties very useful in determining the Fourier Transform of arbitrary signals.
  - Using a simple table of Fourier Transforms and FT properties, we can determine the FT of most signals of interest.