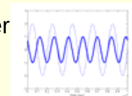
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer
Lecture #14: Examples of the Fourier Transform



Overview

- Today we continue our discussion of the Continuous Time Fourier Transform (CTFT)
- In this lecture we examine several examples of the Fourier Transform using both the properties described in the previous lecture as well as the definition discussed in the Lecture 12.
- What to read – Sections 5.2-5.5 in the text

Summary of Properties

Property	
Conjugation	$x^*(t) \xleftrightarrow{F} X^*(-k)$
Linearity	$\alpha x(t)+\beta y(t) \xleftrightarrow{F} \alpha X(f)+\beta Y(f)$
Time-shifting	$x(t-t_o) \xleftrightarrow{F} e^{-j2\pi f t_o} X(f)$
Frequency-shifting	$e^{j2\pi f_o t} x(t) \xleftrightarrow{F} X(f-f_o)$
Time reversal	$x(-t) \xleftrightarrow{F} X(-f)$
Time-differentiation	$\frac{d}{dt}\{x(t)\} \xleftrightarrow{F} (j2\pi f) X(f)$
Time-integration	$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FS} \frac{1}{j2\pi f} X(f)$
Time/freq-scaling	$x(at) \xleftrightarrow{F} \frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$
Multiplication	$x(t)y(t) \xleftrightarrow{F} X(f)*Y(f)$
Convolution	$x(t)*y(t) \xleftrightarrow{F} X(f)Y(f)$

Example 1

- Fine the energy in the pulse *sinc(t)*
- The energy for any signal can be defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

- Substituting for this signal gives

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |\text{sinc}(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} \left| \frac{\sin(\pi t)}{\pi t} \right|^2 dt \\ &= \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{(\pi t)^2} dt \end{aligned}$$

Example 1 – cont.

- Continuing

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{(\pi t)^2} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx \\ &= \frac{2}{\pi} \underbrace{\int_{0}^{\infty} \frac{\sin^2(x)}{x^2} dx}_{\pi/2} \\ &= \frac{2}{\pi} \frac{\pi}{2} \\ &= 1 \end{aligned}$$

Making a change of variables x = πt

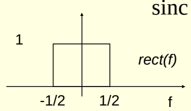
Example 1 – cont.

- Using our knowledge of Fourier Transforms and Parseval's Theorem we could have solved this very easily.
- Parseval's Theorem states

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

- Further, the Fourier Transform of *sinc(t)* is

$$\text{sinc}(t) \xleftrightarrow{F} rect(f)$$



The energy can easily be found to be 1 from inspection this plot.

Example 2

- Find the Fourier Transform of the signal

$$z(t) = rect\left(\frac{t}{T}\right)\cos(2\pi f_o t)$$

- Recall the modulation property:

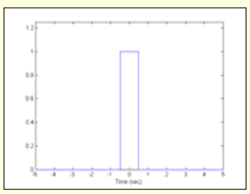
$$Z(f) = \frac{1}{2}X(f - f_c) + \frac{1}{2}X(f + f_c)$$

- Thus, we can write directly

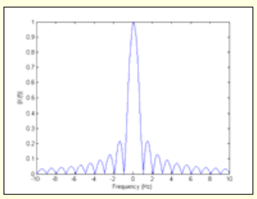
$$Z(f) = \frac{T}{2}\text{sinc}((f - f_o)T) + \frac{T}{2}\text{sinc}((f + f_o)T)$$

Example 2 – cont.

x(t) = rect(t)

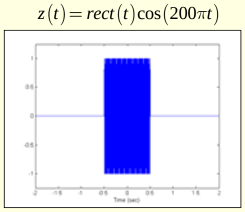


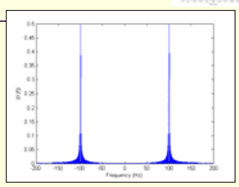
|X(f)| = |sinc(f)|

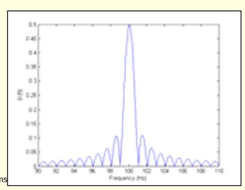


Example 2 – cont.

$$z(t) = rect(t)\cos(200\pi t)$$







$$Z(f) = \frac{1}{2}\text{sinc}(f - 100) + \frac{1}{2}\text{sinc}(f + 100)$$

Example 3

- Find the Fourier Transform of a 50% duty cycle square wave with period T_o and amplitude 1.
- The signal can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t - nT_o}{T_o/2}\right)$$

- Clearly this signal is not integrable and thus does not have a Fourier Transform in the strict sense.
- However, we can write this in terms of its Fourier Series coefficients as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} = \frac{1}{2}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_F t}$$

$$X[k] = f_o T_w \mathrm{sinc}(k f_o T_w) = \frac{1}{2}\mathrm{sinc}\left(\frac{k}{2}\right)$$

Example 3 (cont.)

- Writing the signal in terms of its Fourier Series coefficients (using $T_F = T_o$):

$$x(t) = \frac{1}{2}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_o t}$$

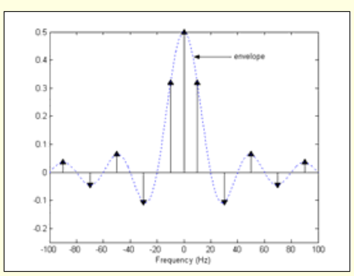
- Using the linearity property and the FT for a complex exponential: $F\left\{e^{j2\pi f_o t}\right\} = \delta(f - f_o)$

$$X(f) = F\left\{\frac{1}{2}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_o t}\right\} = \frac{1}{2}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k}{2}\right) F\left\{e^{j2\pi k f_o t}\right\} = \frac{1}{2}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k}{2}\right)\delta(f - k f_o)$$

Note that since the signal is *periodic* the spectrum (i.e., Fourier Transform) is *discrete*

Example 3 – cont.

- Plotting the spectrum:



Example 4

- Using Parseval's Theorem, determine the energy of

$$x(t) = \cos(2\pi f_o t)$$

- Solution: From Parseval's Theorem:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} \left|X(f)\right|^2 dt \\ &= \int_{-\infty}^{\infty} \left|\frac{1}{2}\delta(f - f_o) + \frac{1}{2}\delta(f + f_o)\right|^2 dt \\ &= \int_{-\infty}^{\infty} \frac{1}{4}\delta^2(f - f_o) + \frac{1}{4}\delta^2(f + f_o)\, dt \\ &= \infty \end{aligned}$$

Side Bar – $\delta^2(t)$

- How do we evaluate $\delta^2(t)$?

$$\begin{aligned} \int_{-\infty}^{\infty} \delta^2(t)dt &= \lim_{a \to 0} \int_{-\infty}^{\infty} \left\{\frac{1}{a} rect\left(\frac{t}{a}\right)\right\}^2 dt \\ &= \lim_{a \to 0} \frac{1}{a^2} \int_{-a/2}^{a/2} dt \\ &= \lim_{a \to 0} \frac{1}{a^2} a \\ &= \lim_{a \to 0} \frac{1}{a} \\ &= \infty \end{aligned}$$

Thus we have the following:

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_o)dt = g(t_o)$$

$$\int_{-\infty}^{\infty} g(t)\delta^2(t - t_o)dt = \infty$$

Example 4 – cont.

- We can double check the result using time domain integration:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt \\ &= \int_{-\infty}^{\infty} \left|\cos(2\pi f_o t)\right|^2 dt \\ &= \int_{-\infty}^{\infty} \cos^2(2\pi f_o t)\, dt \\ &= \infty \end{aligned}$$

Example 5

- Use the integration property to determine the Fourier Transform of *z(t) = tri(t)*
- Taking the derivative of *tri(t)* twice we have

$$\frac{d}{dt}\left\{tri(t)\right\} = rect\left(t+\frac{1}{2}\right) - rect\left(t-\frac{1}{2}\right)$$

$$\begin{aligned}\frac{d}{dt}\left\{rect\left(t+\frac{1}{2}\right) - rect\left(t-\frac{1}{2}\right)\right\} &= \delta(t+1) - \delta(t) - \left(\delta(t) - \delta(t-1)\right) \\ &= \delta(t+1) - 2\delta(t) + \delta(t-1)\end{aligned}$$

$$\begin{aligned}F\left\{\delta(t+1) - 2\delta(t) + \delta(t-1)\right\} &= \exp(j2\pi f) - 2 + \exp(-j2\pi f) \\ &= \underbrace{2\cos(2\pi f) - 2}_{X(f)}\end{aligned}$$

Example 5 – cont.

- From the integration property we have

$$\begin{aligned}Z(f) &= \frac{1}{(j2\pi f)^2} X(f) \\ &= \frac{1}{(j2\pi f)^2}\left\{2\cos(2\pi f) - 2\right\} \\ &= \frac{-1}{(j\pi f)^2}\sin^2(\pi f) \\ &= \frac{\sin^2(\pi f)}{(\pi f)^2} \\ &= \mathrm{sinc}^2(f)\end{aligned}$$

Example 6

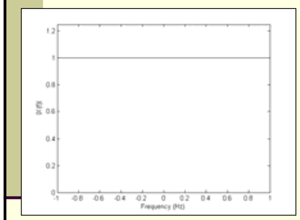
- Plot the magnitude and phase of the signal

$$\delta(t-2)$$

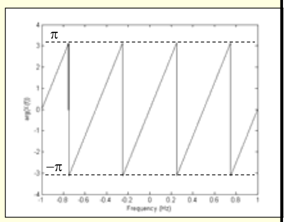
- Solution:
  - From our previous examples we know that
    $$F\left\{\delta(t)\right\} = 1$$
  - Further, we know from the time shift property that
    $$x(t-t_o) \overset{FS}{\longleftrightarrow} e^{-j2\pi f t_o} X(f)$$
  - Thus, we have
    $$\begin{aligned}F\left\{\delta(t-2)\right\} &= e^{j2\pi f*2} \\ &= e^{j4\pi f}\end{aligned}$$

Example 6 – cont.

Magnitude Plot |X(f)|



Phase Plot of X(f)



$$\left| e^{j4\pi f} \right| = 1$$

$$angle\left(e^{j4\pi f}\right) = 4\pi f$$

Example 7

- Find and plot the Fourier Transform of

$$x(t) = 2e^{-3t}u(t)$$

- Solution: We do not have a previous result that can be used, so we must go back to the definition of the Fourier Transform:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} 2e^{-3t}u(t)e^{-j2\pi ft}dt$$

Example 7 – cont.

$$X(f) = \int_{-\infty}^{\infty} 2e^{-3t}u(t)e^{-j2\pi ft}dt$$

$$= 2\int_{0}^{\infty} e^{-3t}e^{-j2\pi ft}dt$$

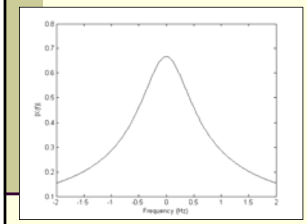
$$= 2\int_{0}^{\infty} e^{-t(3+j2\pi f)}dt$$

$$= \frac{-2}{3+j2\pi f} e^{-t(3+j2\pi f)}\Big|_{0}^{\infty}$$

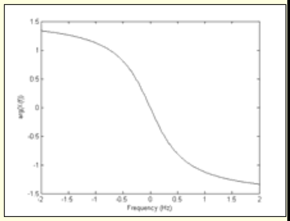
$$= \frac{2}{3+j2\pi f}$$

Example 7 – cont.

Magnitude Plot |X(f)|



Phase Plot of X(f)



$$\left|\frac{2}{3+j2\pi f}\right| = \frac{2}{|3+j2\pi f|} = \frac{2}{\sqrt{9+4\pi^2 f^2}}$$

$$angle\left(\frac{2}{3+j2\pi f}\right) = angle(2) - angle(3+j2\pi f) = 0 - \tan^{-1}\left(\frac{2\pi f}{3}\right)$$

Summary

- In this lecture we have gone through several examples of the Fourier Transform
- In the next class we will examine several applications of the Fourier Transform to system theory including
  - Filtering
  - System transfer function
  - Log-magnitude plots
  - Communication system analysis