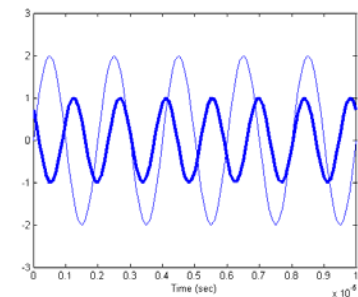


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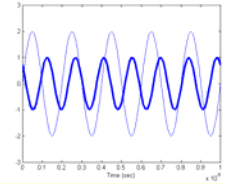
Signals and Systems

Spring 2006

Instructor: Dr. R. Michael Buehrer
Lecture #14: Examples of the
Fourier Transform

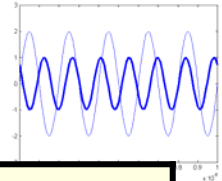


Overview



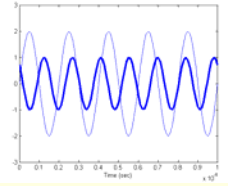
- Today we continue our discussion of the Continuous Time Fourier Transform (CTFT)
- In this lecture we examine several examples of the Fourier Transform using both the properties described in the previous lecture as well as the definition discussed in the Lecture 12.
- What to read – Sections 5.2-5.5 in the text

Summary of Properties



| Property | |
|----------------------|--|
| Conjugation | $x^*(t) \xleftrightarrow{F} X^*(-k)$ |
| Linearity | $\alpha x(t) + \beta y(t) \xleftrightarrow{F} \alpha X(f) + \beta Y(f)$ |
| Time-shifting | $x(t - t_o) \xleftrightarrow{F} e^{-j2\pi f t_o} X(f)$ |
| Frequency-shifting | $e^{j2\pi f_o t} x(t) \xleftrightarrow{F} X(f - f_o)$ |
| Time reversal | $x(-t) \xleftrightarrow{F} X(-f)$ |
| Time-differentiation | $\frac{d}{dt} \{x(t)\} \xleftrightarrow{F} (j2\pi f) X(f)$ |
| Time-integration | $\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{FS} \frac{1}{j2\pi f} X(f)$ |
| Time/freq-scaling | $x(at) \xleftrightarrow{F} \frac{1}{ a } X\left(\frac{f}{a}\right)$ |
| Multiplication | $x(t) y(t) \xleftrightarrow{F} X(f) * Y(f)$ |
| Convolution | $x(t) * y(t) \xleftrightarrow{F} X(f) Y(f)$ |

Example 1



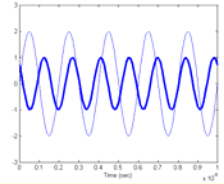
- Find the energy in the pulse $\text{sinc}(t)$
- The energy for any signal can be defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- Substituting for this signal gives

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |\text{sinc}(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \left| \frac{\sin(\pi t)}{\pi t} \right|^2 dt \\ &= \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{(\pi t)^2} dt \end{aligned}$$

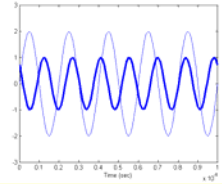
Example 1 – cont.



■ Continuing

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{(\pi t)^2} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx \quad \text{Making a change of variables } x = \pi t \\ &= \frac{2}{\pi} \underbrace{\int_0^{\infty} \frac{\sin^2(x)}{x^2} dx}_{\pi/2} \\ &= \frac{2}{\pi} \frac{\pi}{2} \\ &= 1 \end{aligned}$$

Example 1 – cont.

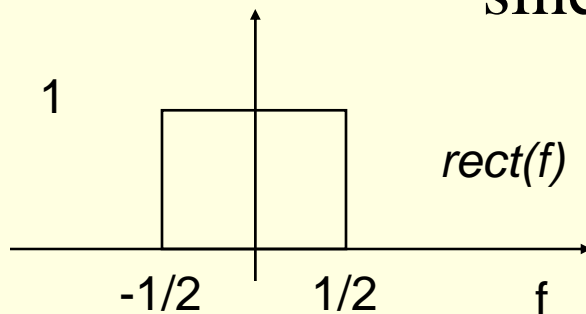


- Using our knowledge of Fourier Transforms and Parseval's Theorem we could have solved this very easily.
- Parseval's Theorem states

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

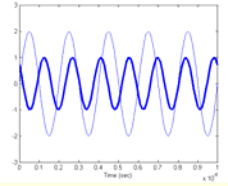
- Further, the Fourier Transform of $\text{sinc}(t)$ is

$$\text{sinc}(t) \xleftrightarrow{F} \text{rect}(f)$$



The energy can easily be found to be 1 from inspection this plot.

Example 2



- Find the Fourier Transform of the signal

$$z(t) = \text{rect}\left(\frac{t}{T}\right) \cos(2\pi f_o t)$$

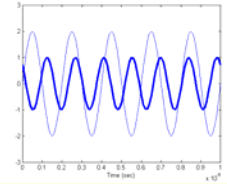
- Recall the modulation property:

$$Z(f) = \frac{1}{2} X(f - f_c) + \frac{1}{2} X(f + f_c)$$

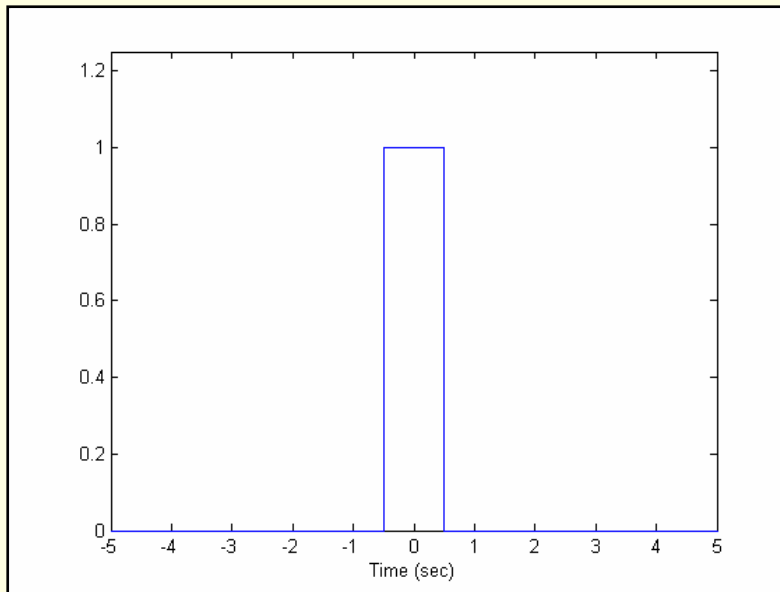
- Thus, we can write directly

$$Z(f) = \frac{T}{2} \text{sinc}((f - f_o)T) + \frac{T}{2} \text{sinc}((f + f_o)T)$$

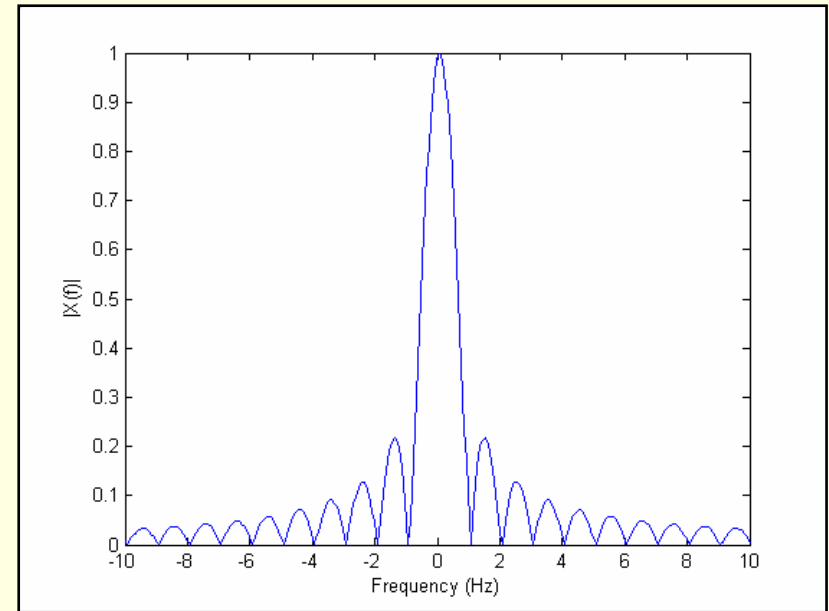
Example 2 – cont.



$$x(t) = \text{rect}(t)$$

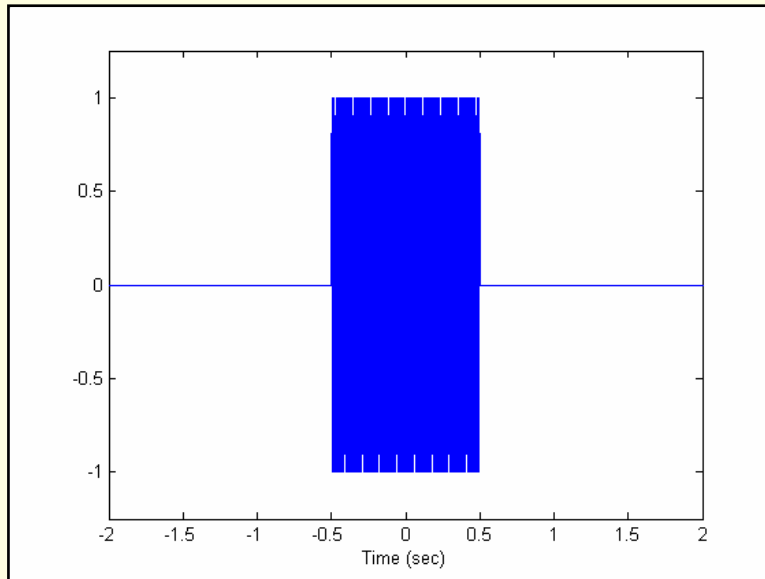


$$|X(f)| = |\text{sinc}(f)|$$

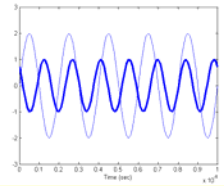
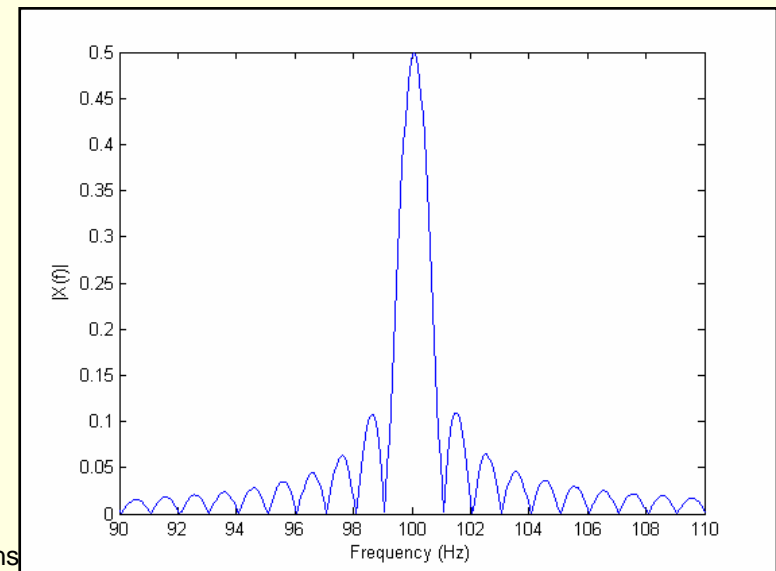
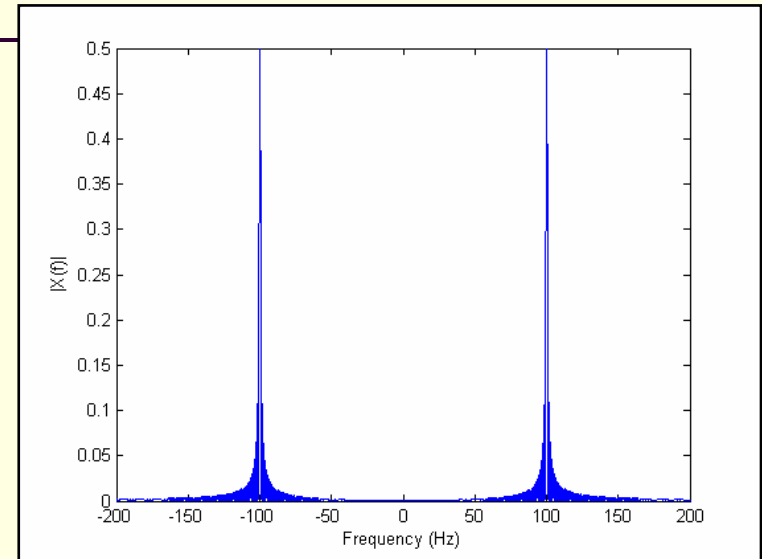


Example 2 – cont.

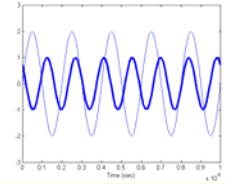
$$z(t) = \text{rect}(t) \cos(200\pi t)$$



$$Z(f) = \frac{1}{2} \text{sinc}(f - 100) + \frac{1}{2} \text{sinc}(f + 100)$$



Example 3



- Find the Fourier Transform of a 50% duty cycle square wave with period T_o and amplitude 1.
- The signal can be written as

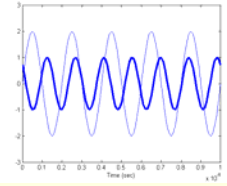
$$x(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - nT_o}{T_o/2}\right)$$

- Clearly this signal is not integrable and thus does not have a Fourier Transform in the strict sense.
- However, we can write this in terms of its Fourier Series coefficients as

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_F t} \end{aligned}$$

$$\begin{aligned} X[k] &= f_o T_w \text{sinc}(k f_o T_w) \\ &= \frac{1}{2} \text{sinc}\left(\frac{k}{2}\right) \end{aligned}$$

Example 3 (cont.)



- Writing the signal in terms of its Fourier Series coefficients (using $T_F = T_o$):

$$x(t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_o t}$$

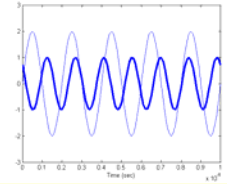
- Using the linearity property and the FT for a complex exponential:

$$F \left\{ e^{j2\pi f_o t} \right\} = \delta(f - f_o)$$

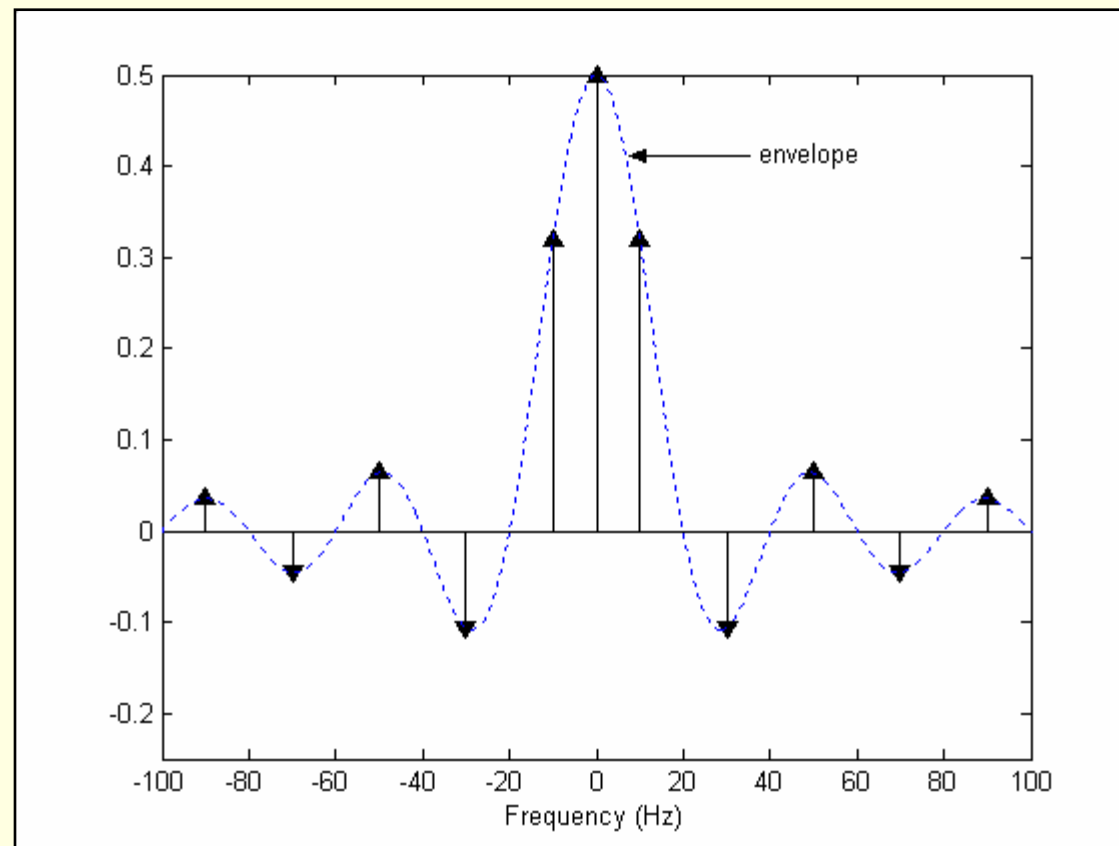
$$\begin{aligned} X(f) &= F \left\{ \frac{1}{2} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_o t} \right\} \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{2}\right) F \left\{ e^{j2\pi k f_o t} \right\} \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{2}\right) \delta(f - k f_o) \end{aligned}$$

Note that since the signal is *periodic* the spectrum (i.e., Fourier Transform) is *discrete*

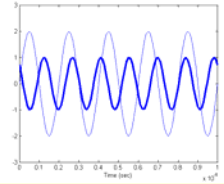
Example 3 – cont.



- Plotting the spectrum:



Example 4



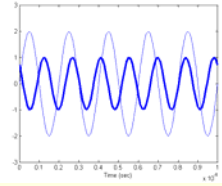
- Using Parseval's Theorem, determine the energy of

$$x(t) = \cos(2\pi f_0 t)$$

- Solution: From Parseval's Theorem:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |X(f)|^2 dt \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0) \right|^2 dt \\ &= \int_{-\infty}^{\infty} \frac{1}{4} \delta^2(f - f_0) + \frac{1}{4} \delta^2(f + f_0) dt \\ &= \infty \end{aligned}$$

Side Bar – $\delta^2(t)$



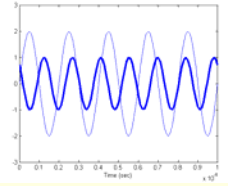
- How do we evaluate $\delta^2(t)$?

$$\begin{aligned}\int_{-\infty}^{\infty} \delta^2(t) dt &= \lim_{a \rightarrow 0} \int_{-\infty}^{\infty} \left\{ \frac{1}{a} \text{rect} \left(\frac{t}{a} \right) \right\}^2 dt \\ &= \lim_{a \rightarrow 0} \frac{1}{a^2} \int_{-a/2}^{a/2} dt \\ &= \lim_{a \rightarrow 0} \frac{1}{a^2} a \\ &= \lim_{a \rightarrow 0} \frac{1}{a} \\ &= \infty\end{aligned}$$

Thus we have the following:

$$\begin{aligned}\int_{-\infty}^{\infty} g(t) \delta(t - t_o) dt &= g(t_o) \\ \int_{-\infty}^{\infty} g(t) \delta^2(t - t_o) dt &= \infty\end{aligned}$$

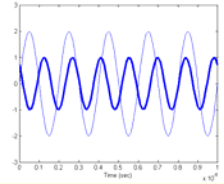
Example 4 – cont.



- We can double check the result using time domain integration:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} |\cos(2\pi f_o t)|^2 dt \\ &= \int_{-\infty}^{\infty} \cos^2(2\pi f_o t) dt \\ &= \infty \end{aligned}$$

Example 5



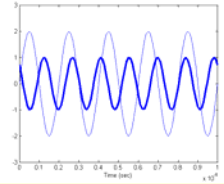
- Use the integration property to determine the Fourier Transform of $z(t) = \text{tri}(t)$
- Taking the derivative of $\text{tri}(t)$ twice we have

$$\frac{d}{dt}\{\text{tri}(t)\} = \text{rect}\left(t + \frac{1}{2}\right) - \text{rect}\left(t - \frac{1}{2}\right)$$

$$\begin{aligned}\frac{d}{dt}\left\{\text{rect}\left(t + \frac{1}{2}\right) - \text{rect}\left(t - \frac{1}{2}\right)\right\} &= \delta(t+1) - \delta(t) - (\delta(t) - \delta(t-1)) \\ &= \delta(t+1) - 2\delta(t) + \delta(t-1)\end{aligned}$$

$$\begin{aligned}F\{\delta(t+1) - 2\delta(t) + \delta(t-1)\} &= \exp(j2\pi f) - 2 + \exp(-j2\pi f) \\ &= \underbrace{2\cos(2\pi f)}_{X(f)} - 2\end{aligned}$$

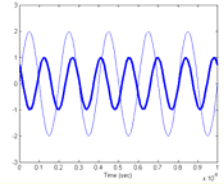
Example 5 – cont.



- From the integration property we have

$$\begin{aligned} Z(f) &= \frac{1}{(j2\pi f)^2} X(f) \\ &= \frac{1}{(j2\pi f)^2} \{2\cos(2\pi f) - 2\} \\ &= \frac{-1}{(j\pi f)^2} \sin^2(\pi f) \\ &= \frac{\sin^2(\pi f)}{(\pi f)^2} \\ &= \text{sinc}^2(f) \end{aligned}$$

Example 6



- Plot the magnitude and phase of the signal

$$\delta(t-2)$$

- Solution:

- From our previous examples we know that

$$F\{\delta(t)\} = 1$$

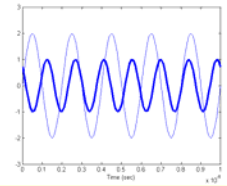
- Further, we know from the time shift property that

$$x(t-t_o) \xleftrightarrow{FS} e^{-j2\pi ft_o} X(f)$$

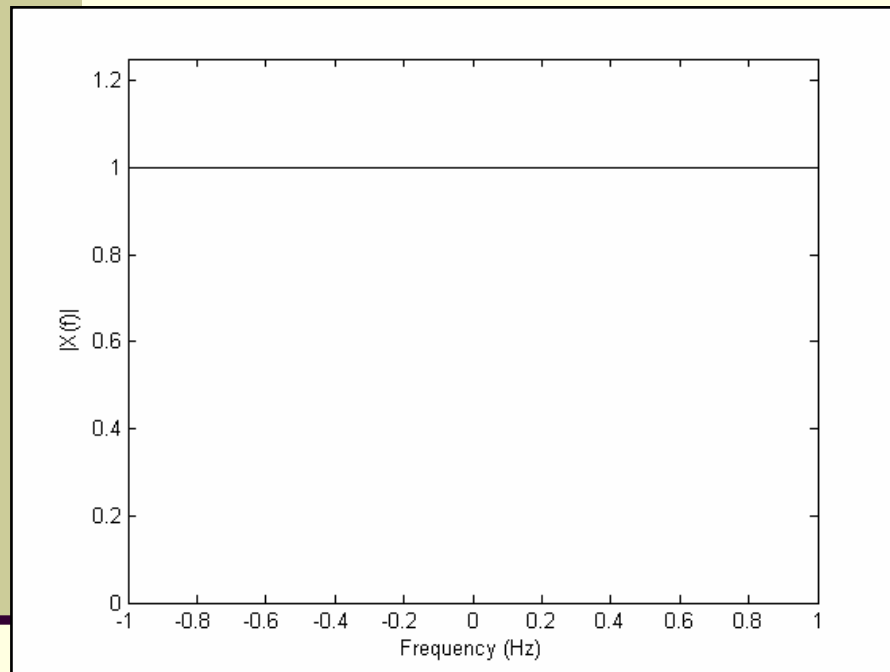
- Thus, we have

$$\begin{aligned} F\{\delta(t-2)\} &= e^{j2\pi f*2} \\ &= e^{j4\pi f} \end{aligned}$$

Example 6 – cont.

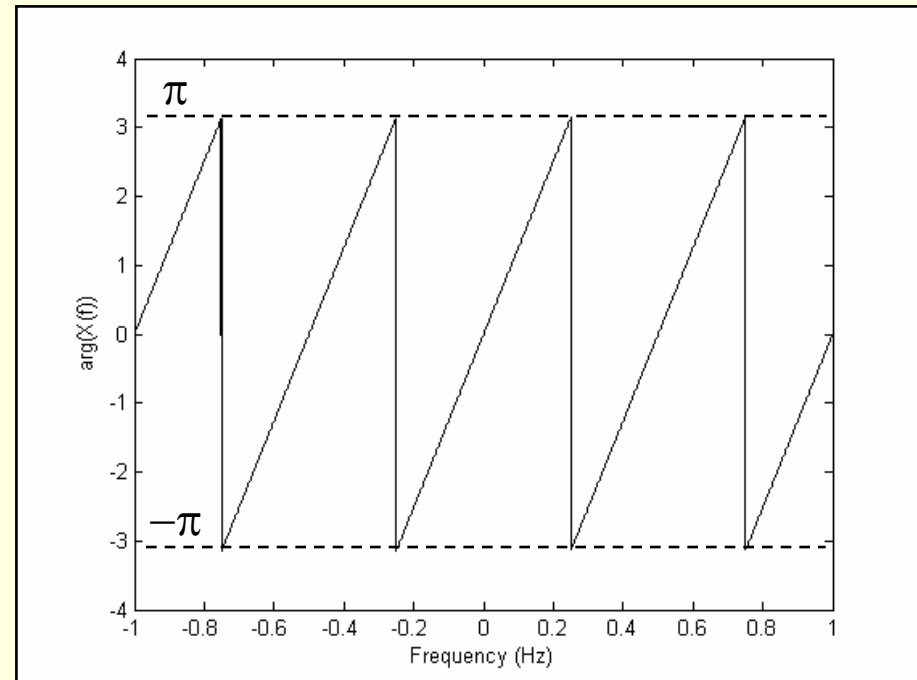


Magnitude Plot $|X(f)|$



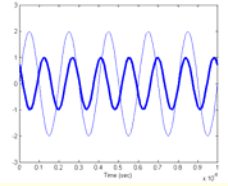
$$\left| e^{j4\pi f} \right| = 1$$

Phase Plot of $X(f)$



$$\text{angle}\left(e^{j4\pi f}\right) = 4\pi f$$

Example 7



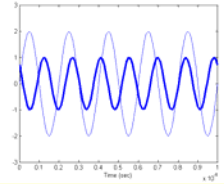
- Find and plot the Fourier Transform of

$$x(t) = 2e^{-3t}u(t)$$

- Solution: We do not have a previous result that can be used, so we must go back to the definition of the Fourier Transform:

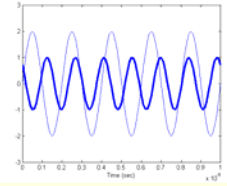
$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} 2e^{-3t}u(t)e^{-j2\pi ft} dt \end{aligned}$$

Example 7 – cont.



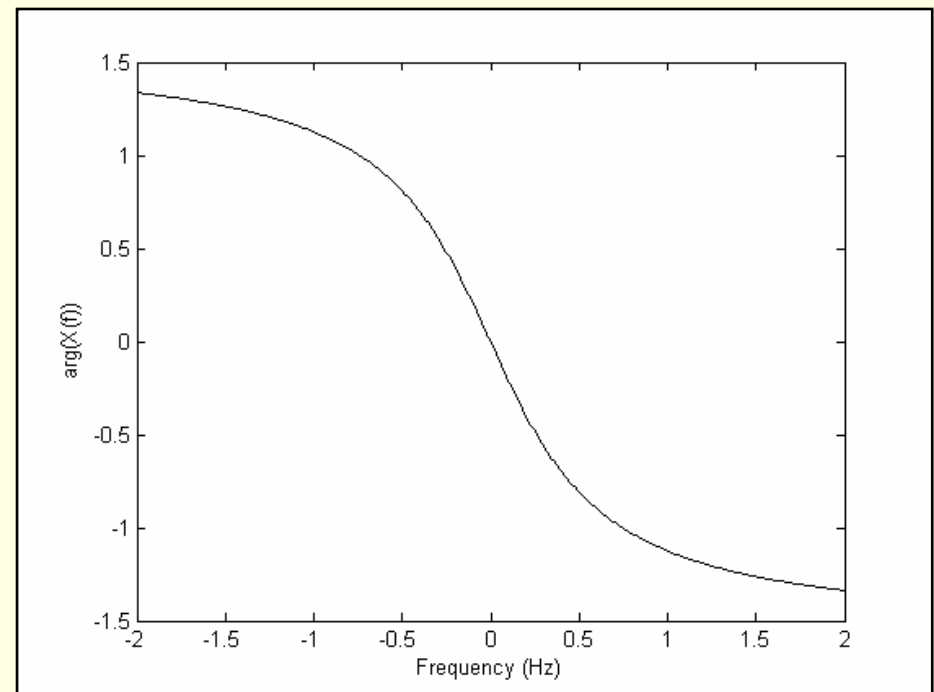
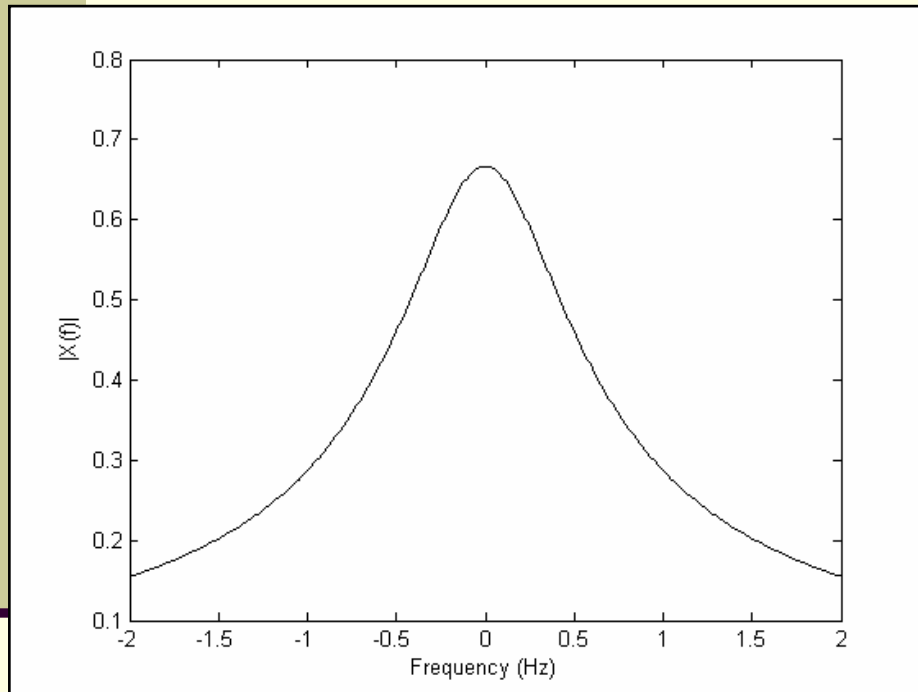
$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} 2e^{-3t} u(t) e^{-j2\pi ft} dt \\ &= 2 \int_0^{\infty} e^{-3t} e^{-j2\pi ft} dt \\ &= 2 \int_0^{\infty} e^{-t(3+j2\pi f)} dt \\ &= \frac{-2}{3+j2\pi f} e^{-t(3+j2\pi f)} \Big|_0^{\infty} \\ &= \frac{2}{3+j2\pi f} \end{aligned}$$

Example 7 – cont.



Magnitude Plot $|X(f)|$

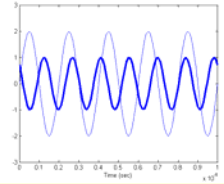
Phase Plot of $X(f)$



$$\begin{aligned} \left| \frac{2}{3 + j2\pi f} \right| &= \frac{2}{|3 + j2\pi f|} \\ &= \frac{2}{\sqrt{9 + 4\pi^2 f^2}} \end{aligned}$$

$$\begin{aligned} \text{angle} \left(\frac{2}{3 + j2\pi f} \right) &= \text{angle}(2) - \text{angle}(3 + j2\pi f) \\ &= 0 - \tan^{-1} \left(\frac{2\pi f}{3} \right) \end{aligned}$$

Summary



- In this lecture we have gone through several examples of the Fourier Transform
- In the next class we will examine several applications of the Fourier Transform to system theory including
 - Filtering
 - System transfer function
 - Log-magnitude plots
 - Communication system analysis