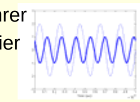
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer

Lecture #15: Applications of the Fourier Transform – Part I: Frequency Response and Ideal Filters



Overview

- We have discussed the Continuous Time Fourier Transform (CTFT) for a few lectures now.
- Today we examine several practical *applications* of the Fourier Transform
  - Frequency response and the Impulse response
  - Ideal filters
    - Lowpass, highpass, bandpass, bandstop
- What to read – Sections 6.1-6.3 in the text

Frequency Response

- Recall that an LTI system can be described by its *impulse response*
- Specifically

$$y(t) = \underbrace{x(t)}_{\text{input}} * \underbrace{h(t)}_{\text{impulse respone}}$$

- In the frequency domain we have

$$F\{y(t)\} = F\{x(t) * h(t)\}$$
$$Y(f) = X(f)H(f)$$

- The Fourier Transform of the impulse response, *H(f)*, is termed the *frequency response* of the system

System Representation

- We can draw the system using the frequency domain or the time domain

$$x(t) \longrightarrow \boxed{\begin{array}{c}\mathcal{H}\\ h(t)\end{array}} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

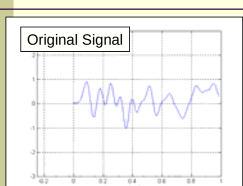
$$X(f) \longrightarrow \boxed{\begin{array}{c}\mathcal{H}\\ H(f)\end{array}} \longrightarrow Y(f)$$

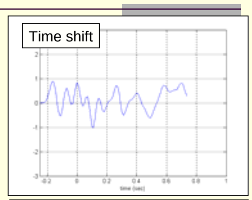
$$Y(f) = X(f)H(f)$$

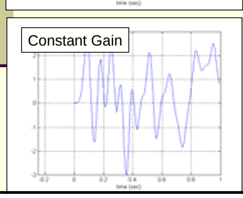
Distortion

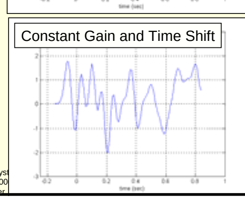
- *Distortion* is commonly construed to mean the shape of the signal has changed
- Amplifying the signal by a constant gain or delaying the signal is not distortion
- Examples of distortion
  - Non-constant (time-selective) gain
  - Frequency-selective gain
  - Clipping

Changes that are not distortion

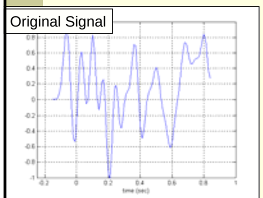


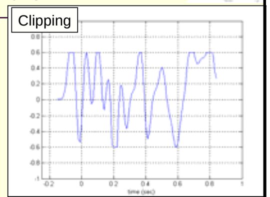


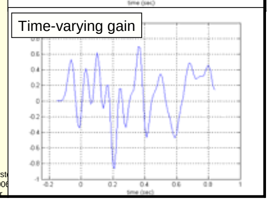




Examples of Distortion



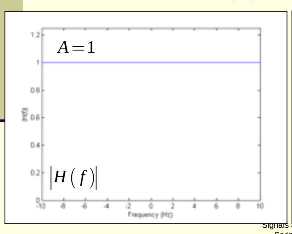


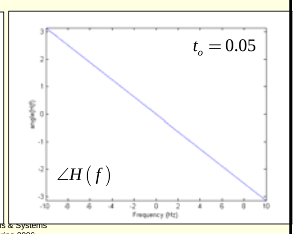


Distortionless System

- A distortionless system passes all frequencies with equal gain and has a linear phase response (i.e., simple time shift)

$$H(f) = Ae^{-j2\pi f t_o}$$





Distortionless System

- The distortionless system was defined in the frequency domain
- What does this mean in the time domain?

$$H(f) = Ae^{-j2\pi f t_o}$$

- The impulse response of the distortionless system is found by taking the inverse FFT

$$\begin{aligned} h(t) &= F^{-1}\{H(f)\} \\ &= F^{-1}\{Ae^{-j2\pi f t_o}\} \\ &= A\delta(t - t_o) \end{aligned}$$

Filters

- A filter is a system which passes certain frequencies and rejects other frequencies
- Types of filters
  - Low pass filter
  - High pass filter
  - Bandpass filter
  - Bandstop filter
- Ideal filter
  - An ideal filter is one which perfectly passes frequencies in a certain range (termed the *pass band*) and perfectly rejects frequencies in another range termed the *stop band*
  - An ideal filter doesn't *distort* the signal in the pass band

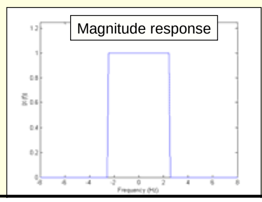
The Ideal Lowpass Filter

- The ideal lowpass filter has a frequency response

$$H(f)=\begin{cases}Ae^{-j2\pi f t_o} & |f|\le f_m\\ 0 & |f|>f_m\end{cases}$$

$$=\underbrace{A\,rect\left(\frac{f}{2f_m}\right)}_{\substack{magnitude\\response}}\underbrace{e^{-j2\pi f t_o}}_{\substack{time\\delay}}$$

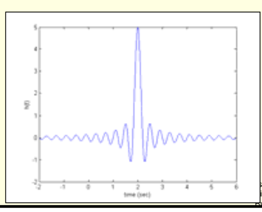
Example – f_m=2.5Hz



Impulse response of the Ideal LPF

- The impulse response of the ideal LPF can be found by taking the inverse Fourier Transform of the ideal LPF frequency response

$$h(t)=F^{-1}\{H(f)\}$$
$$=F^{-1}\left\{A\,rect\left(\frac{f}{2f_m}\right)e^{-j2\pi f t_o}\right\}$$
$$=2Af_m\mathrm{sinc}\left(2f_m(t-t_o)\right)$$



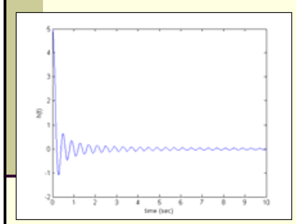
• Example: f_m = 5
• t_o = 2
• Is there any problem with this impulse response?

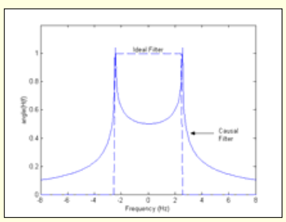
Causality

- A causal system is one that does not have a response prior to the time of the applied input
- The impulse response is the response of a system to an impulse applied at time *t=0.*
- A system whose impulse response is nonzero for t<0 is thus *non-causal.*
- The ideal lowpass filter is thus non-causal and for that reason is not physically realizable.
- We can attempt to approximate the ideal LPF by introducing a delay into our system

Causal LPF

- We can make the filter causal by simply truncating the impulse response before *t=0.*

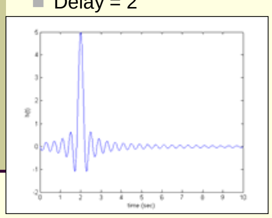


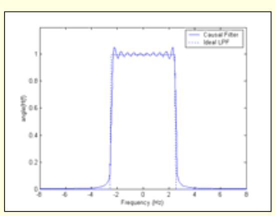


- The resulting filter is far from ideal.

Causal LPF

- A second option is to delay the impulse response and truncate it.
- Delay = 2

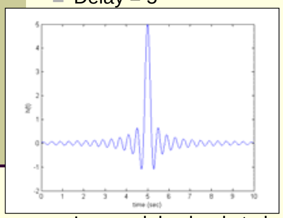


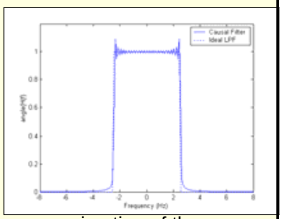


- This makes the system closer to ideal, but requires a delay which some applications may not tolerate

Causal LPF

- A second option is to delay the impulse response and truncate it.
- Delay = 5

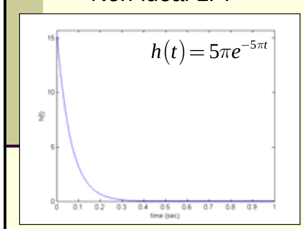


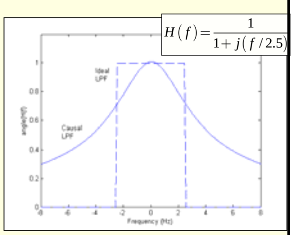


- Larger delay leads to better approximation of the ideal LPF

Causal LPF

- RC filter
- Causal, exponential impulse response
- Non-ideal LPF





Ideal High Pass Filter

- The ideal HPF has a frequency response

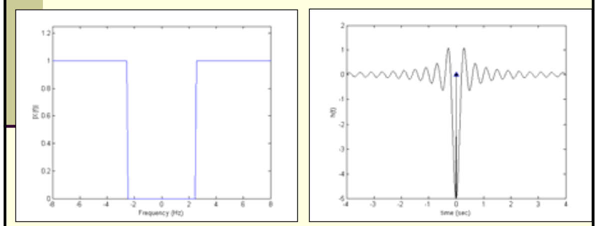
$$H(f) = \begin{cases} 0 & |f| \le f_m \\ Ae^{-j2\pi f t_o} & |f| > f_m \end{cases}$$
$$= \underbrace{\left[A - A\,rect\left(\frac{f}{2f_m}\right)\right]}_{\substack{magnitude \\ response}} \underbrace{e^{-j2\pi f t_o}}_{\substack{time \\ delay}}$$

- The corresponding impulse response is

$$h(t) = F^{-1}\{H(f)\}$$
$$= F^{-1}\left\{\left[A - A\,rect\left(\frac{f}{2f_m}\right)\right] e^{-j2\pi f t_o}\right\}$$
$$= A\delta(f) - 2Af_m \mathrm{sinc}(2f_m(t - t_o))$$

Ideal High Pass Filter

- The ideal HPF is non-causal, just like the ideal lowpass filter



Ideal Bandpass Filter

- The ideal bandpass filter can be written as

$$H(f)=\begin{cases} Ae^{-j2\pi f t_o} & f_L \le |f| \le f_H \\ 0 & else \end{cases}$$

$$= \underbrace{A\left[rect\left(\frac{f-f_o}{\Delta f}\right) + rect\left(\frac{f+f_o}{\Delta f}\right)\right]}_{\substack{magnitude \\ response}} \underbrace{e^{-j2\pi f t_o}}_{\substack{time \\ delay}}$$

$$h(t) = F^{-1}\{H(f)\}$$

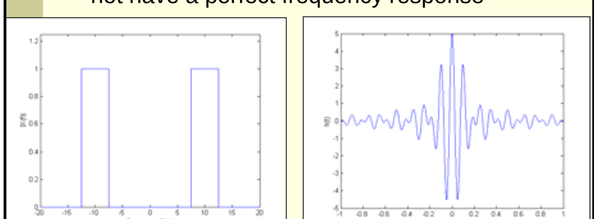
$$= F^{-1}\left\{ A\left[rect\left(\frac{f-f_o}{\Delta f}\right) + rect\left(\frac{f+f_o}{\Delta f}\right)\right] e^{-j2\pi f t_o}\right\}$$

$$= 2A\Delta f \,\mathrm{sinc}\left(\Delta f_m (t-t_o)\right)\cos\left(2\pi f_o (t-t_o)\right)$$

f_L = lower frequency limit
f_H = upper frequency limit
$\Delta f = f_H - f_L$
$f_o = (f_H - f_L)/2$

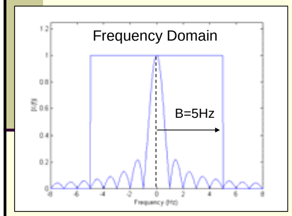
Ideal Bandpass Filter

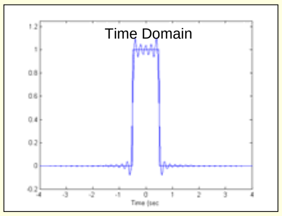
- Ideal BPF is also non-causal
- Realistic bandpass filters will be causal and not have a perfect frequency response



Example

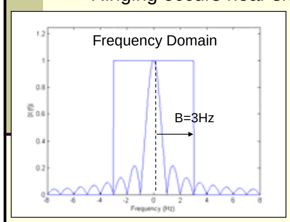
- Consider a time-domain square pulse of width 1 second which is passed through a filter with a bandwidth of 5Hz
- The bandwidth restriction does not cause a substantial change in the pulse shape

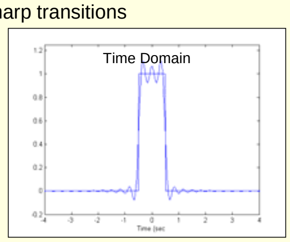




Example – cont.

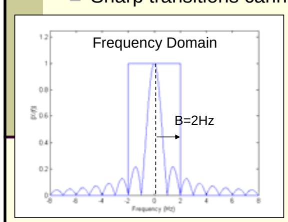
- Now consider a a filter with a bandwidth of 3Hz
- The bandwidth restriction still does not cause a substantial change in the pulse shape
- Ringing occurs near sharp transitions

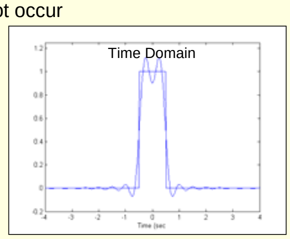




Example – cont.

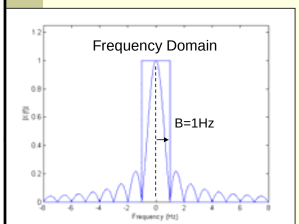
- Now consider a a filter with a bandwidth of 2Hz
- The bandwidth restriction now begins to cause a more substantial change in the pulse shape
- Sharp transitions cannot occur

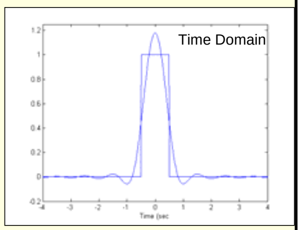




Example – cont.

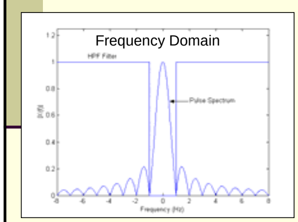
- Now consider a a filter with a bandwidth of 1Hz
- The bandwidth restriction now changes the shape considerably

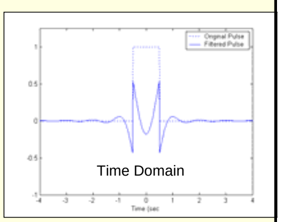




Example 2

- Consider a time-domain square pulse of width 1 second which is passed through a HP filter with a bandstop bandwidth of 1Hz
- The loss of the low-frequency energy causes a large change in the pulse. Note that the pulse now has zero average value.



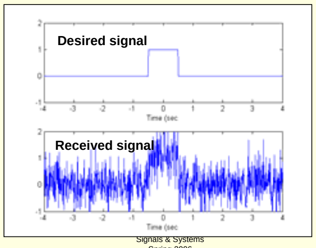


Example 3 - Noise

- A major use of filters is the elimination of noise.
- Noise typically has a much larger bandwidth than the signal of interest.
  - Filtering the received signal with a bandpass or lowpass filter can reduce the amount of noise
- We typically consider *additive noise* where the received signal *r(t)* is equal to the desired signal *x(t)* plus noise *n(t)*
  - r(t) = x(t) + n(t)

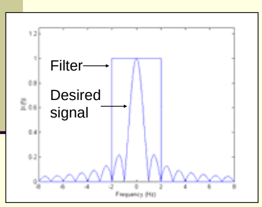
Example 3 (cont.)

- Consider a square pulse with duration one second that is received with the addition of noise
- The ratio of the received desired signal power to the noise power, signal-to-noise ratio or SNR, is 2

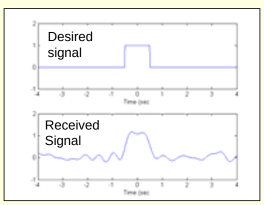


Example 3 (cont.)

- If we filter the signal with an ideal LPF with bandwidth B = 2Hz, we know that we will introduce some distortion to the desired signal, but we can also eliminate much of the noise



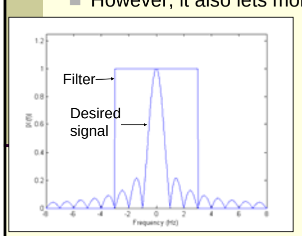
Frequency Domain



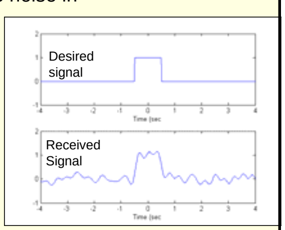
Time Domain

Example 3 (cont.)

- We know that increasing the bandwidth to B = 3Hz will reduce the amount of distortion to the original signal
- However, it also lets more noise in



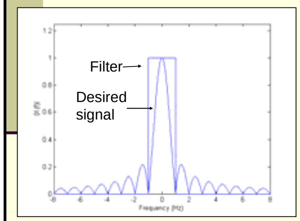
Frequency Domain



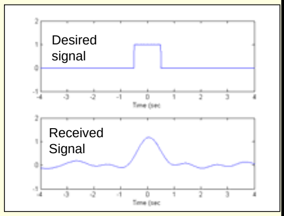
Time Domain

Example 3 (cont.)

- If we descrease the bandwidth to B = 1Hz will reduce the amount of noise in the received signal
- However, it introduces more distortion to the desired signal



Frequency Domain



Time Domain

Summary

- In this lecture we have examined the application of the Fourier Transform to system analysis, specifically through the system frequency response and through the application of filters
- We defined several ideal filters:
  - LPF, HPF, BPF
- Ideal filters require a non-causal impulse response
  - Real filters must use a delay or non-ideal response
  - Next class we will discuss practical filters