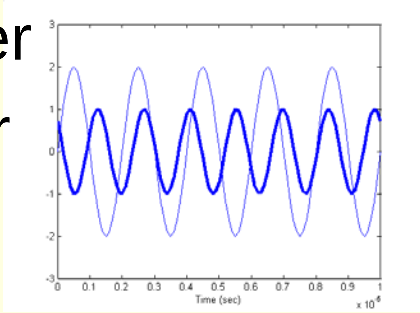
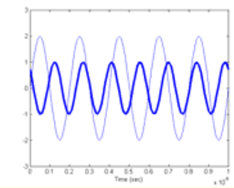
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer



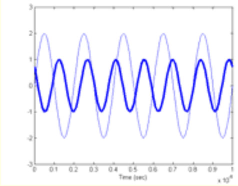
Lecture #15: Applications of the Fourier Transform – Part I: Frequency Response and Ideal Filters

Overview



- We have discussed the Continuous Time Fourier Transform (CTFT) for a few lectures now.
- Today we examine several practical *applications* of the Fourier Transform
  - Frequency response and the Impulse response
  - Ideal filters
    - Lowpass, highpass, bandpass, bandstop
- What to read – Sections 6.1-6.3 in the text

Frequency Response



- Recall that an LTI system can be described by its *impulse response*
- Specifically

$$y(t) = \underbrace{x(t)}_{\text{input}} * \underbrace{h(t)}_{\text{impulse respone}}$$

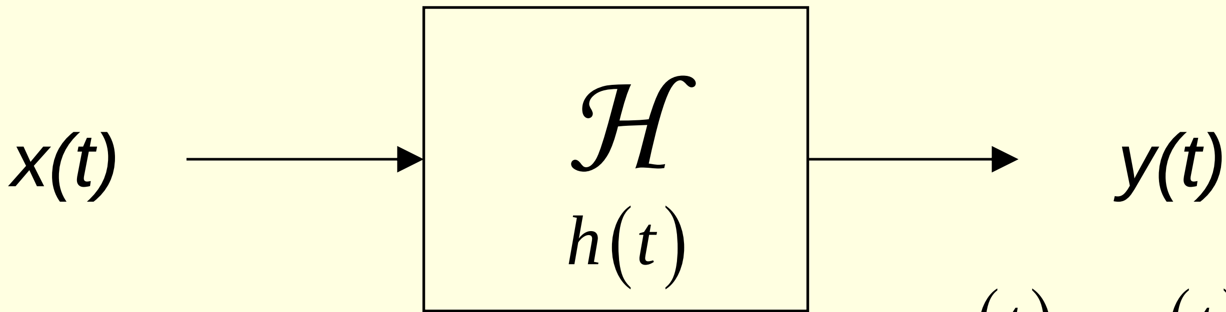
- In the frequency domain we have

$$F\left\{y(t)\right\} = F\left\{x(t) * h(t)\right\}$$
$$Y(f) = X(f)H(f)$$

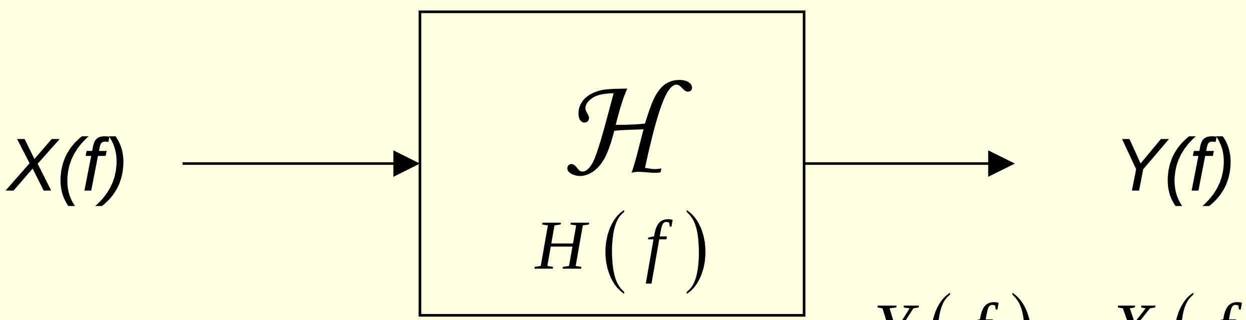
- The Fourier Transform of the impulse response, *H(f)*, is termed the *frequency response* of the system

System Representation

- We can draw the system using the frequency domain or the time domain

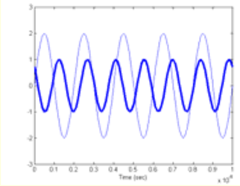


$$y(t) = x(t) * h(t)$$



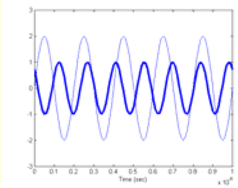
$$Y(f) = X(f)H(f)$$

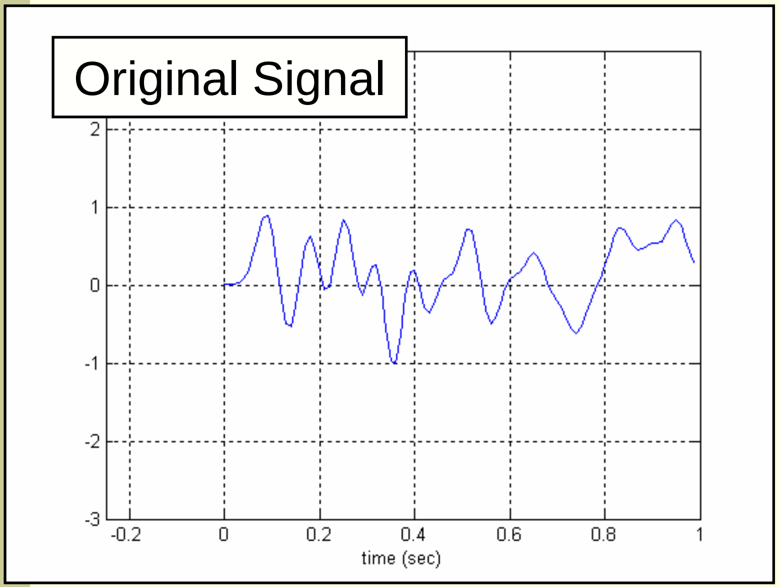
Distortion

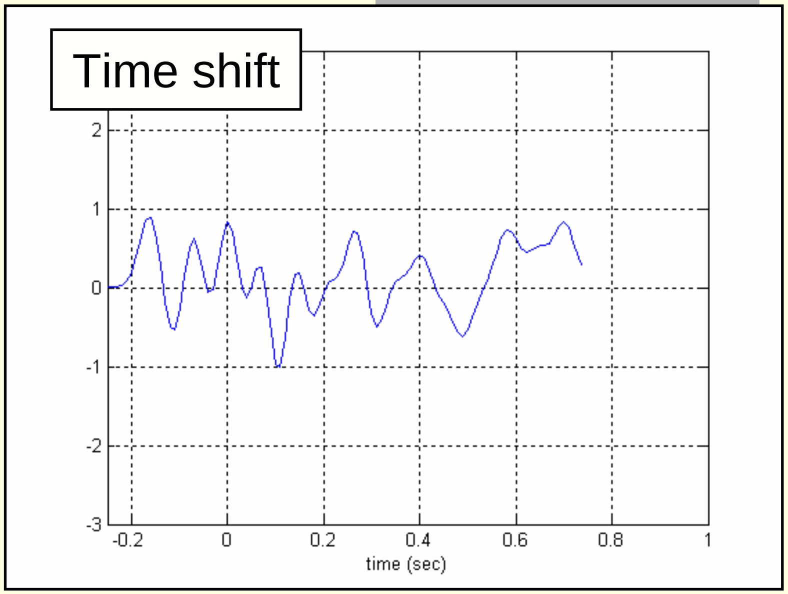


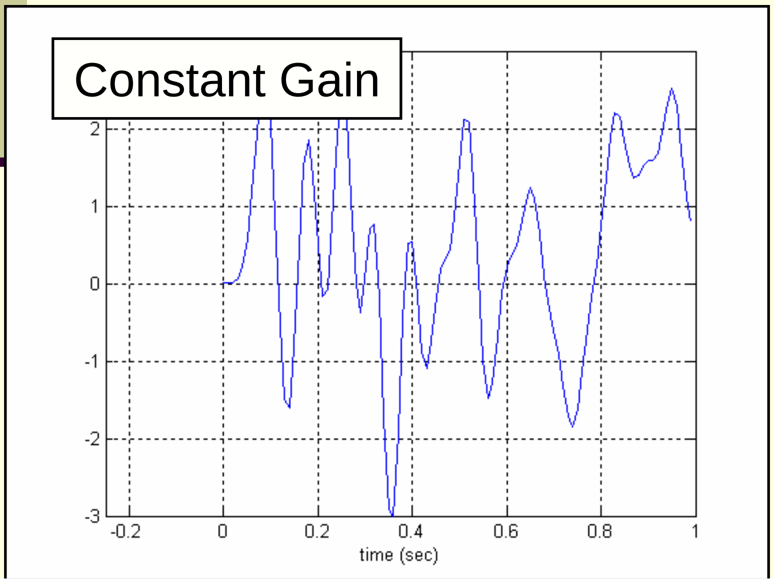
- *Distortion* is commonly construed to mean the shape of the signal has changed
- Amplifying the signal by a constant gain or delaying the signal is not distortion
- Examples of distortion
  - Non-constant (time-selective) gain
  - Frequency-selective gain
  - Clipping

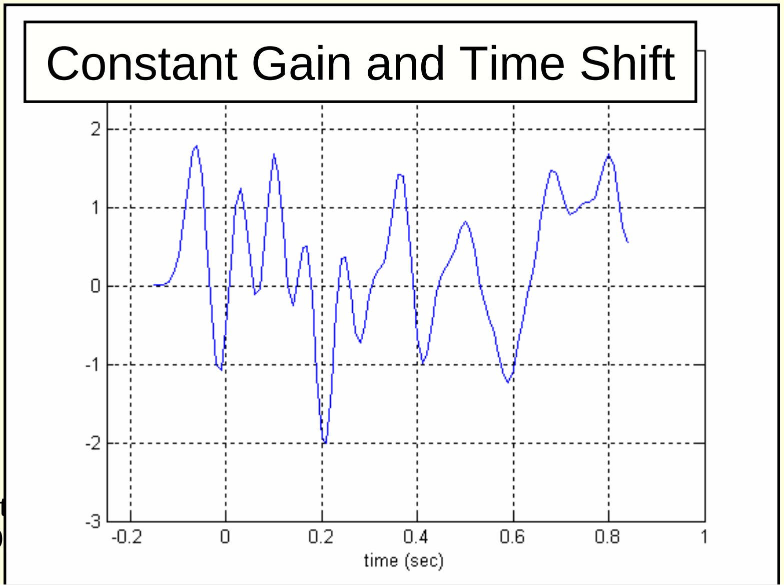
Changes that are not distortion



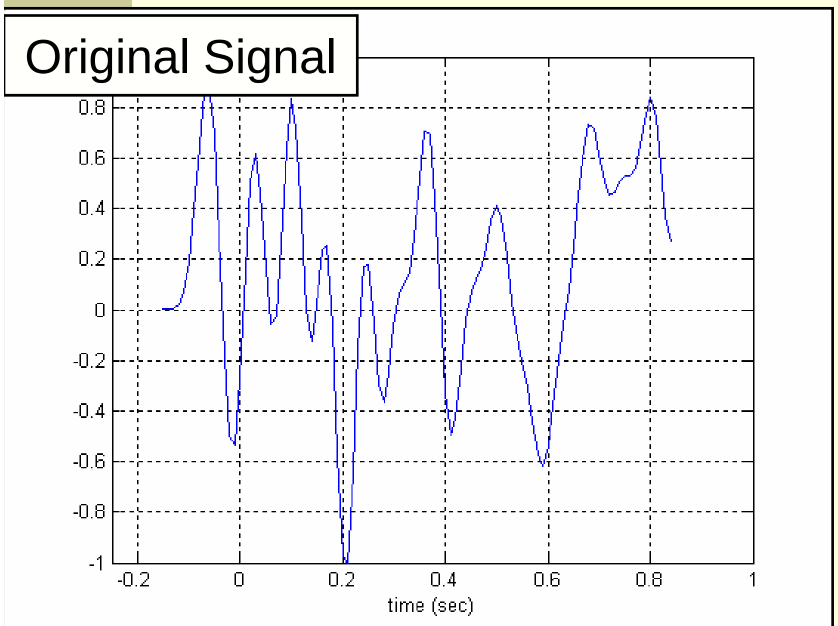


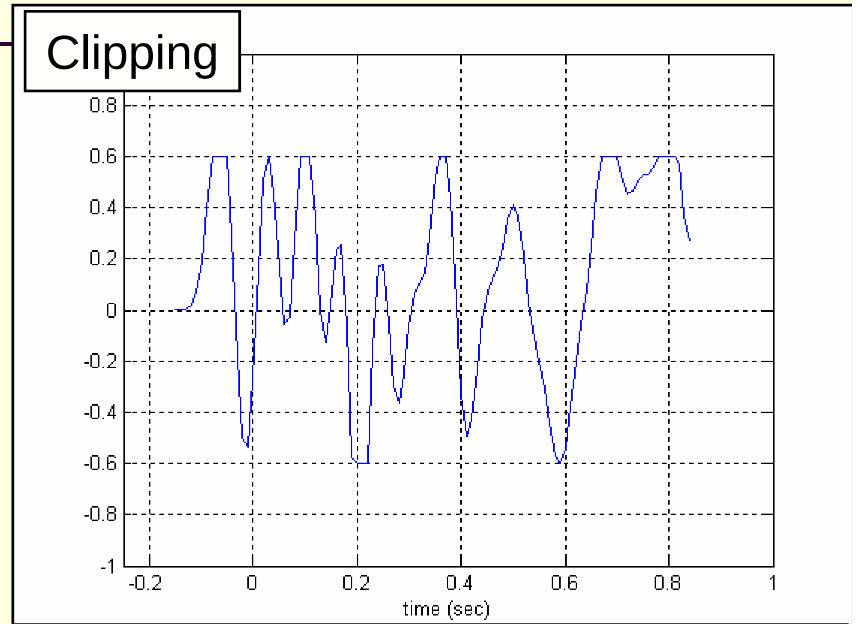


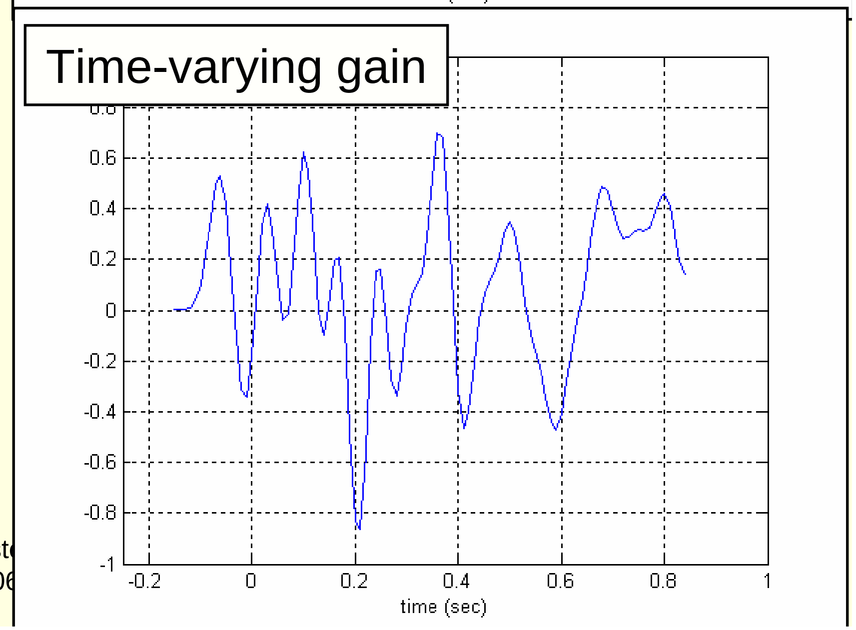




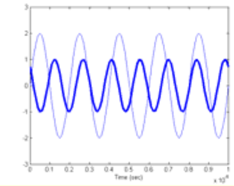
Examples of Distortion





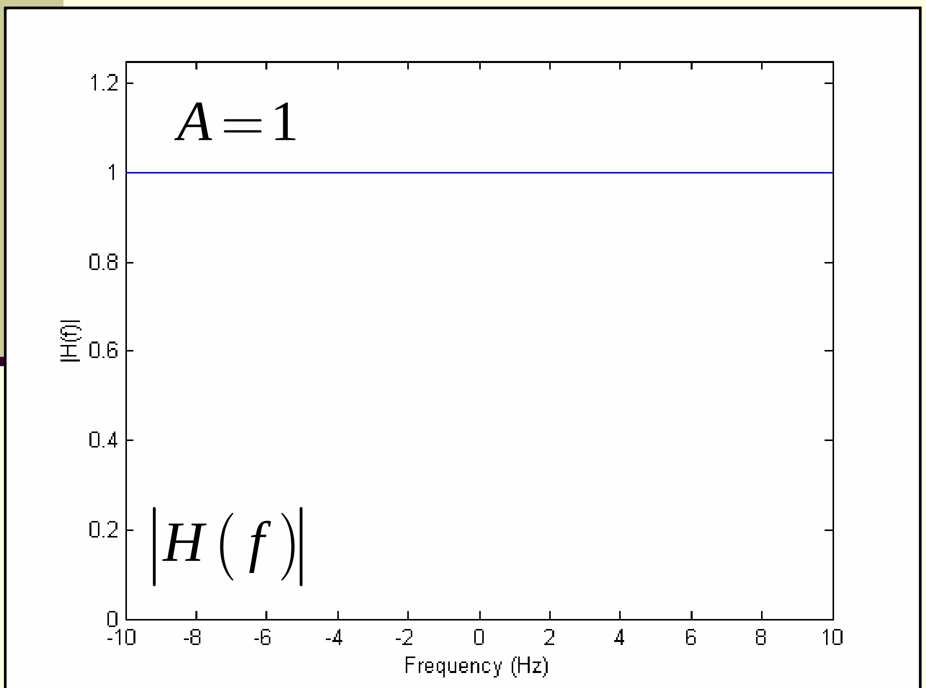


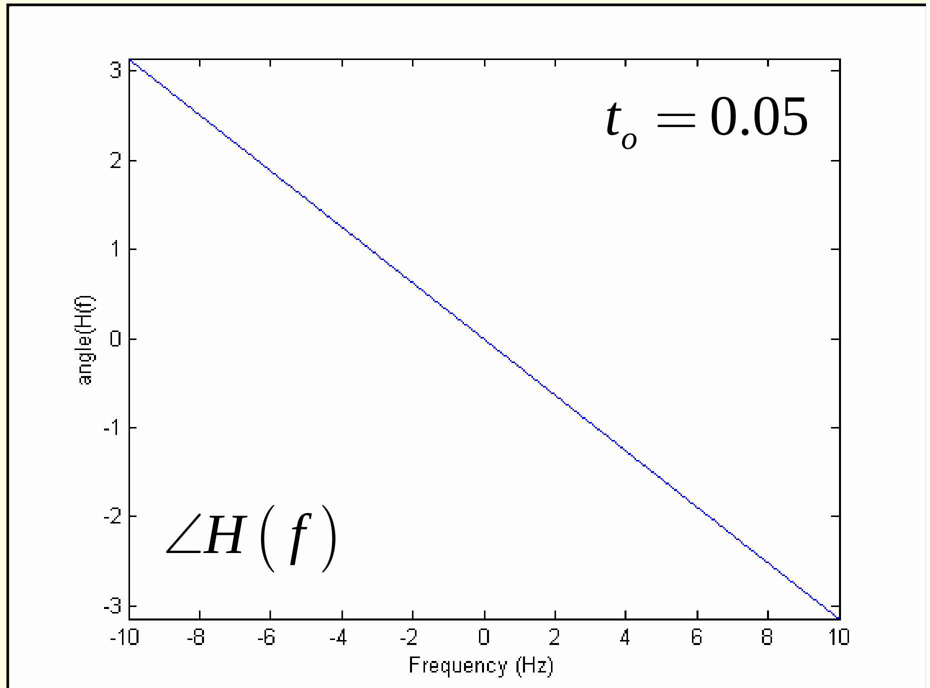
Distortionless System



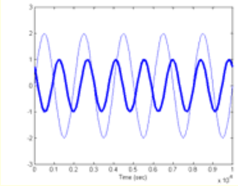
- A distortionless system passes all frequencies with equal gain and has a linear phase response (i.e., simple time shift)

$$H\left(f\right) = Ae^{-j2\pi f t_o}$$





Distortionless System



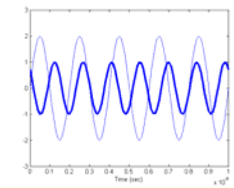
- The distortionless system was defined in the frequency domain
- What does this mean in the time domain?

$$H(f) = Ae^{-j2\pi f t_o}$$

- The impulse response of the distortionless system is found by taking the inverse FFT

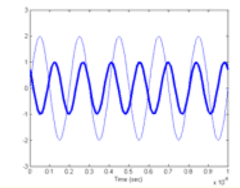
$$\begin{aligned} h(t) &= F^{-1}\{H(f)\} \\ &= F^{-1}\{Ae^{-j2\pi f t_o}\} \\ &= A\delta(t - t_o) \end{aligned}$$

Filters



- A filter is a system which passes certain frequencies and rejects other frequencies
- Types of filters
  - Low pass filter
  - High pass filter
  - Bandpass filter
  - Bandstop filter
- Ideal filter
  - An ideal filter is one which perfectly passes frequencies in a certain range (termed the *pass band*) and perfectly rejects frequencies in another range termed the *stop band*
  - An ideal filter doesn’t *distort* the signal in the pass band

The Ideal Lowpass Filter

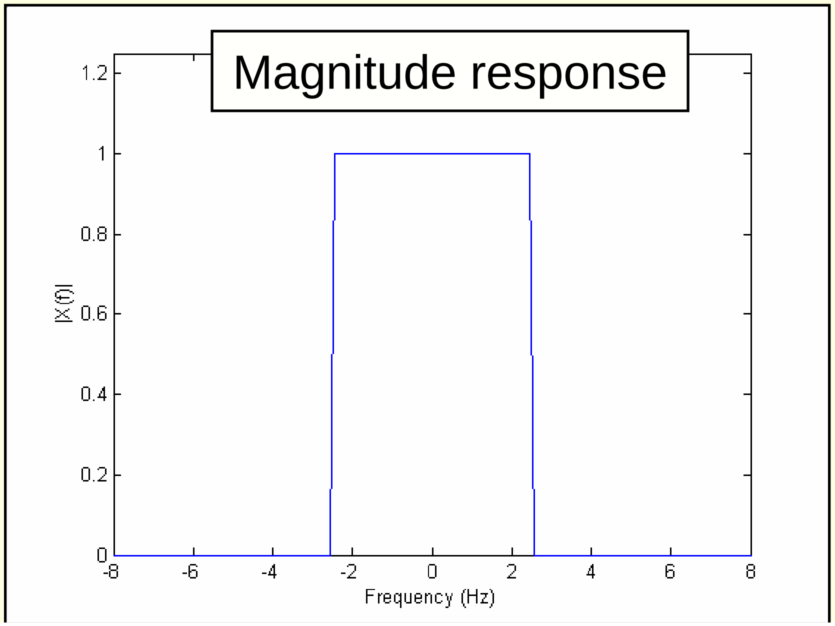


- The ideal lowpass filter has a frequency response

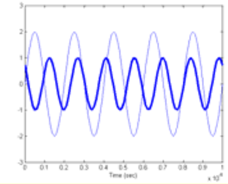
$$H(f)=\begin{cases} Ae^{-j2\pi f t_o} & |f|\le f_m \\ 0 & |f|>f_m \end{cases}$$

$$=\underbrace{A\,rect\left(\frac{f}{2f_m}\right)}_{\substack{magnitude\\response}}\underbrace{e^{-j2\pi f t_o}}_{\substack{time\\delay}}$$

Example – f_m=2.5Hz

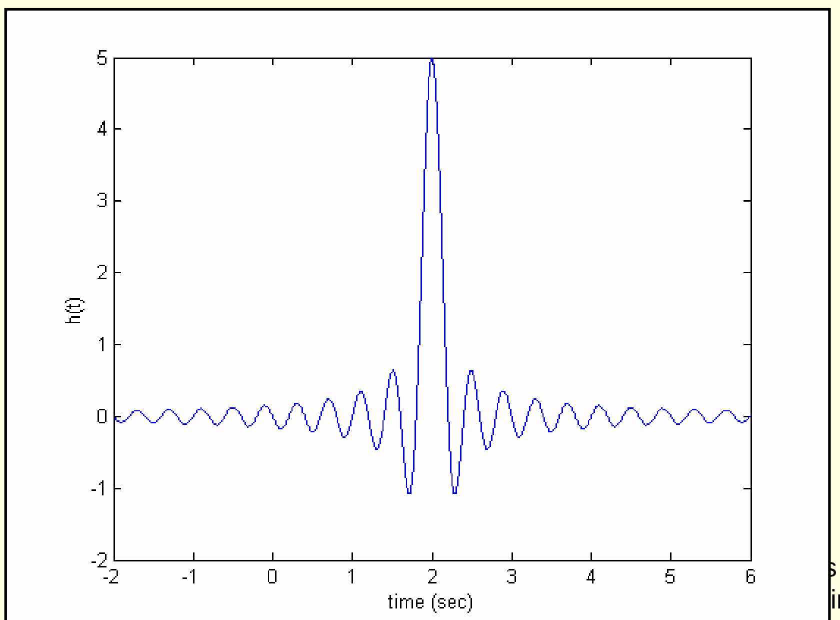


Impulse response of the Ideal LPF

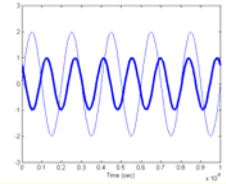


- The impulse response of the ideal LPF can be found by taking the inverse Fourier Transform of the ideal LPF frequency response

$$
\begin{aligned}
h(t) &= F^{-1}\{H(f)\} \\
&= F^{-1}\left\{A\, rect\left(\frac{f}{2f_m}\right)e^{-j2\pi f t_o}\right\} \\
&= 2Af_m \mathrm{sinc}\left(2f_m\left(t-t_o\right)\right)
\end{aligned}
$$

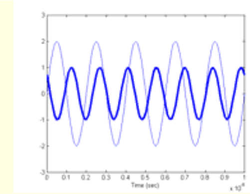


- Example: $f_m = 5$
- $t_o = 2$
- Is there any problem with this impulse response?



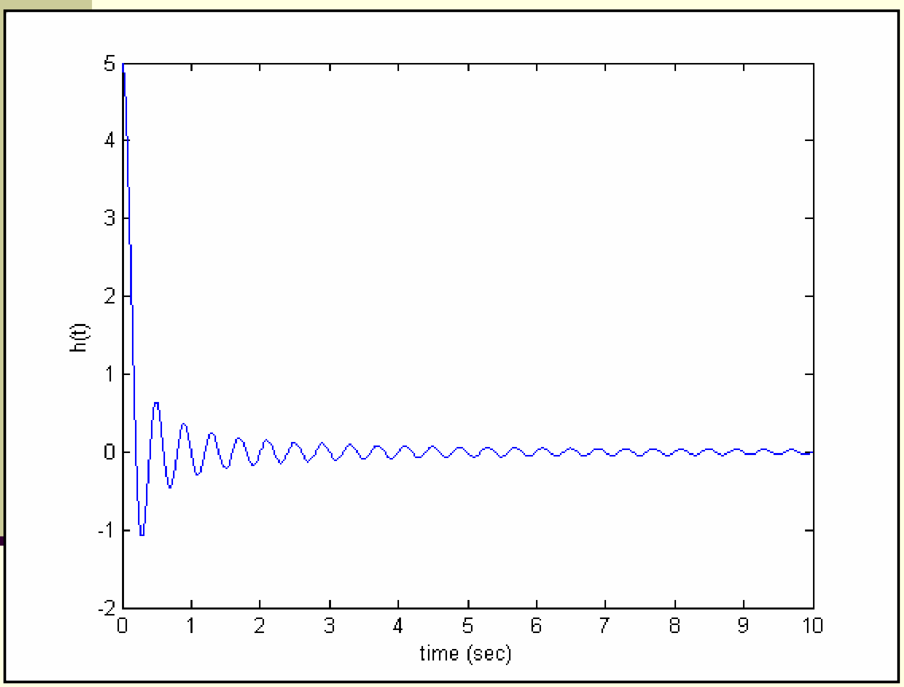
Causality

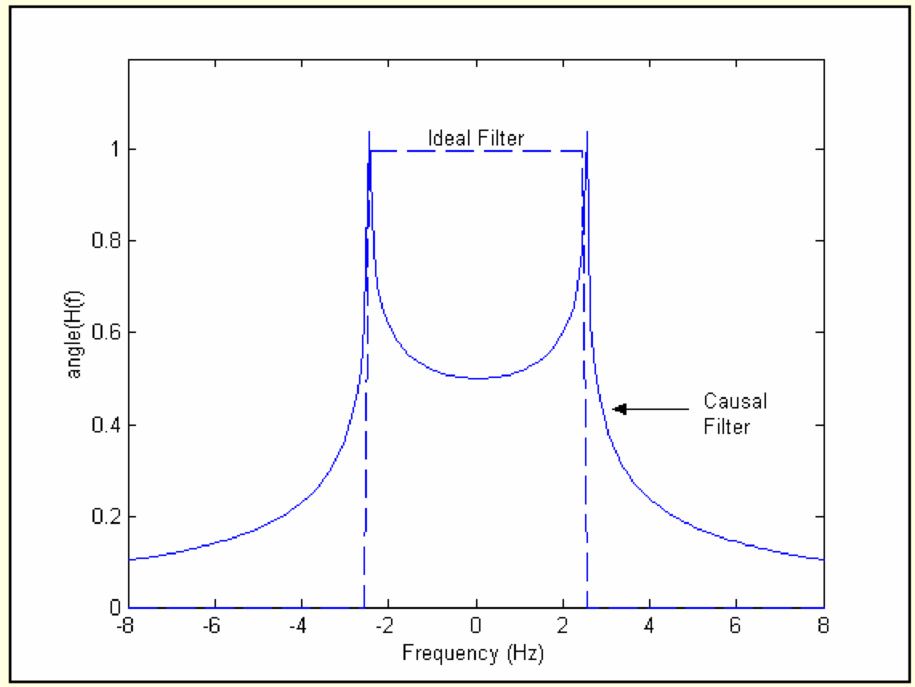
- A causal system is one that does not have a response prior to the time of the applied input
- The impulse response is the response of a system to an impulse applied at time $t=0$.
- A system whose impulse response is nonzero for $t<0$ is thus *non-causal.*
- The ideal lowpass filter is thus non-causal and for that reason is not physically realizable.
- We can attempt to approximate the ideal LPF by introducing a delay into our system



Causal LPF

- We can make the filter causal by simply truncating the impulse response before *t=0.*

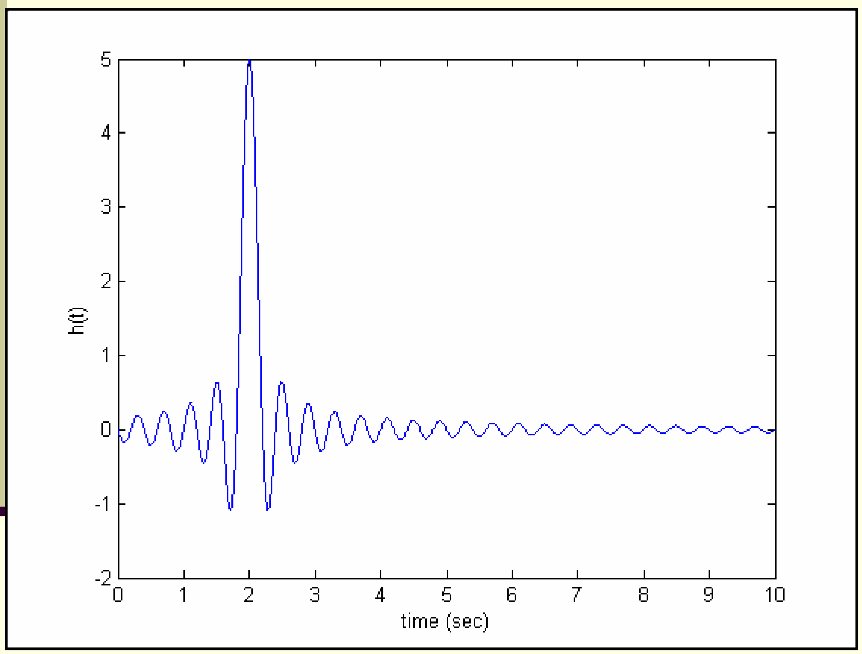


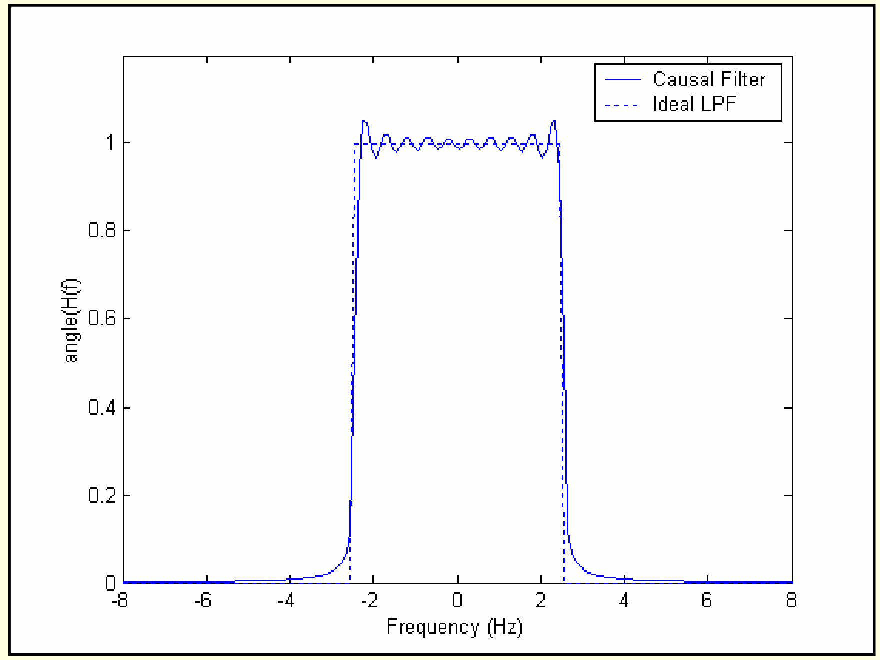


- The resulting filter is far from ideal.

Causal LPF

- A second option is to delay the impulse response and truncate it.
- Delay = 2

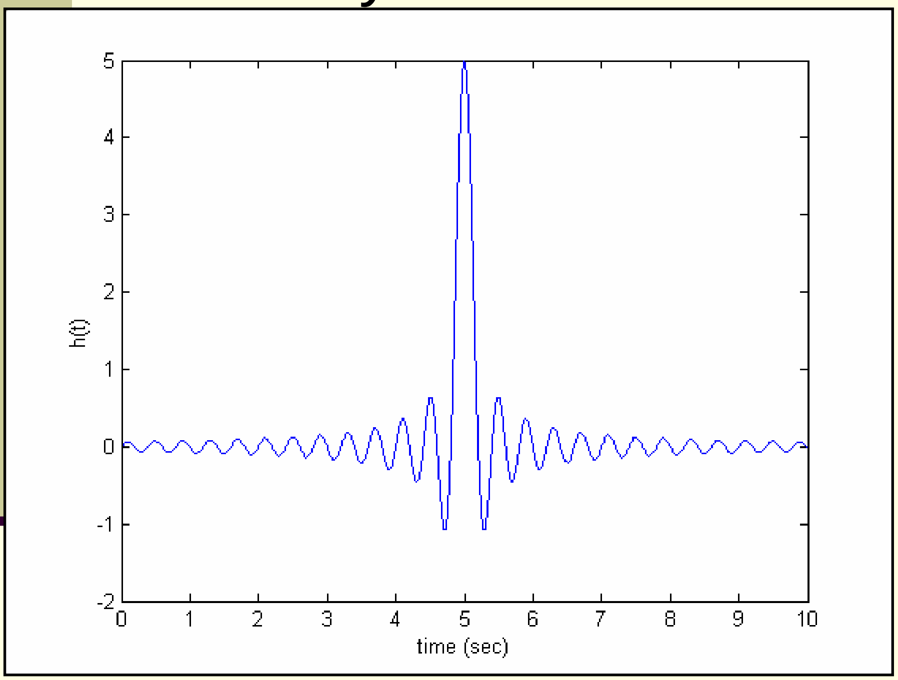


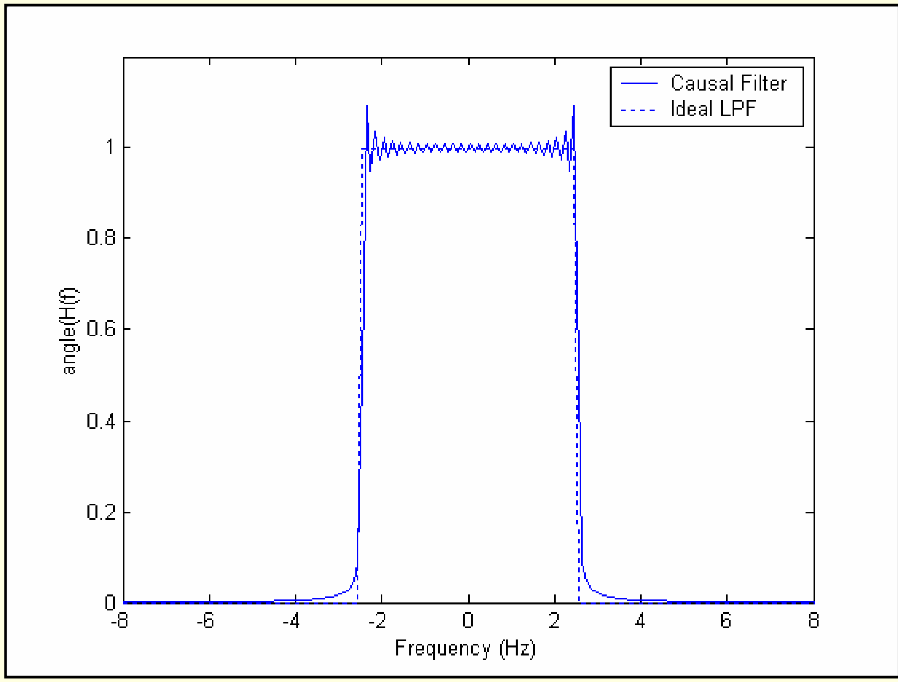


- This makes the system closer to ideal, but requires a delay which some applications may not tolerate

Causal LPF

- A second option is to delay the impulse response and truncate it.
- Delay = 5

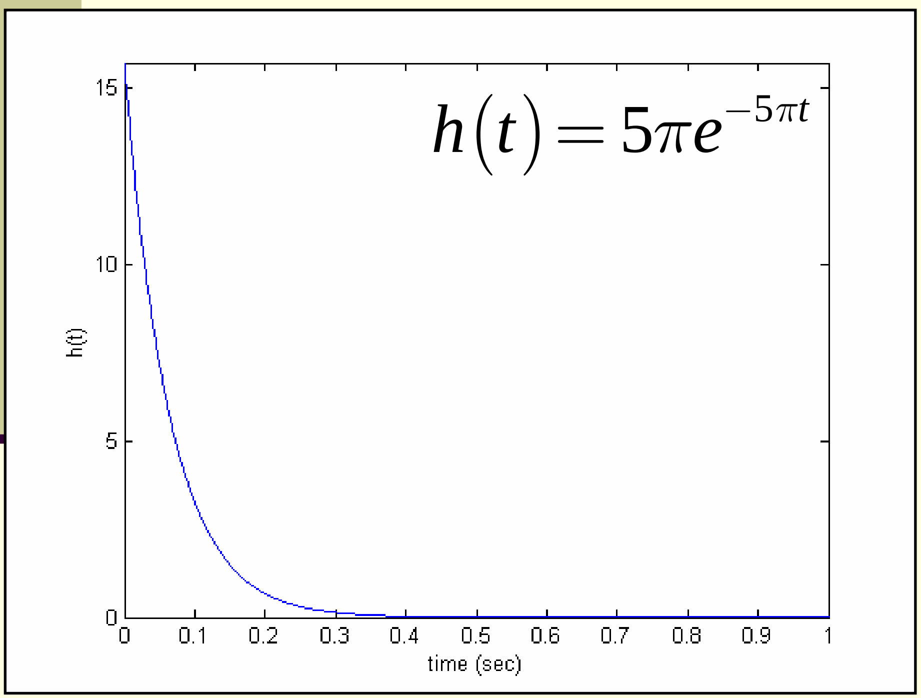


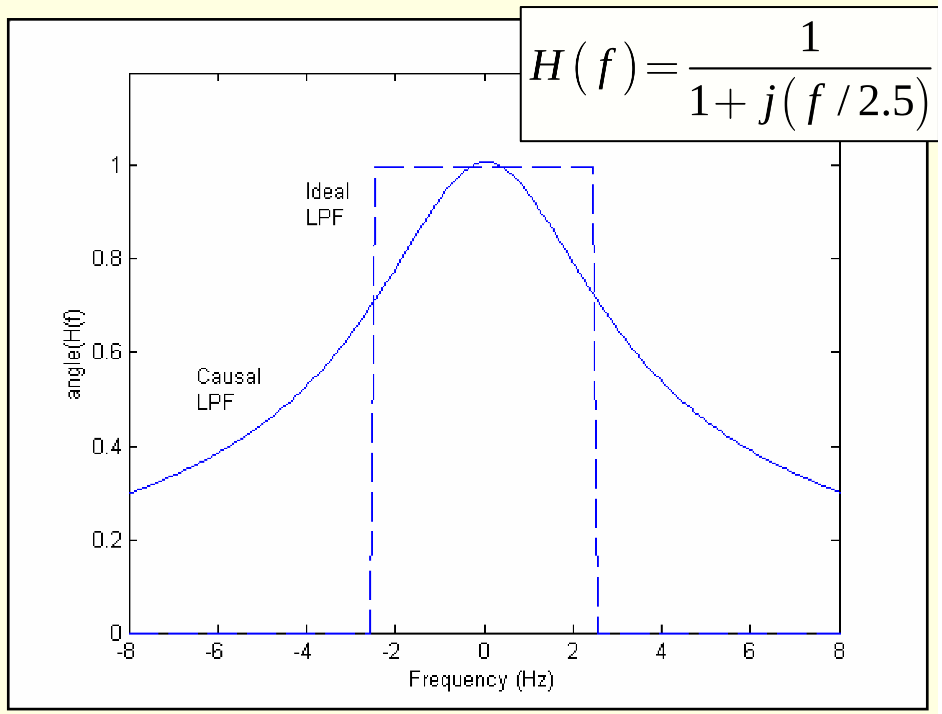


- Larger delay leads to better approximation of the ideal LPF

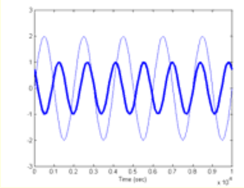
Causal LPF

- RC filter
- Causal, exponential impulse response
- Non-ideal LPF





Ideal High Pass Filter



- The ideal HPF has a frequency response

$$H(f)=\begin{cases} 0 & |f|\le f_m \\ Ae^{-j2\pi f t_o} & |f|> f_m \end{cases}$$

$$=\underbrace{\left[A-A\,rect\left(\frac{f}{2f_m}\right)\right]}_{\substack{magnitude\\response}}\underbrace{e^{-j2\pi f t_o}}_{\substack{time\\delay}}$$

- The corresponding impulse response is

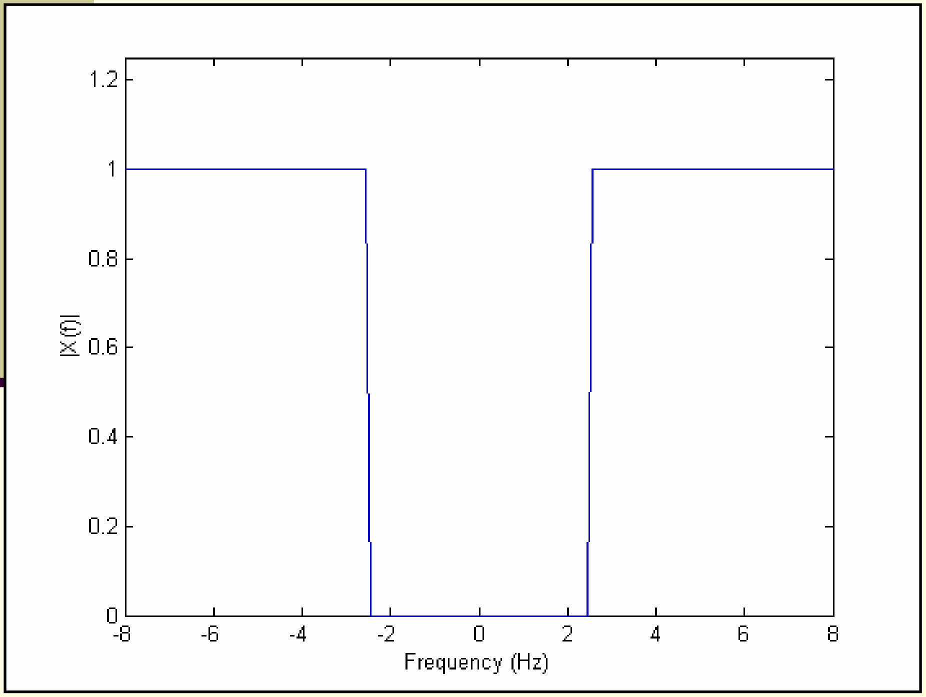
$$h(t)=F^{-1}\{H(f)\}$$

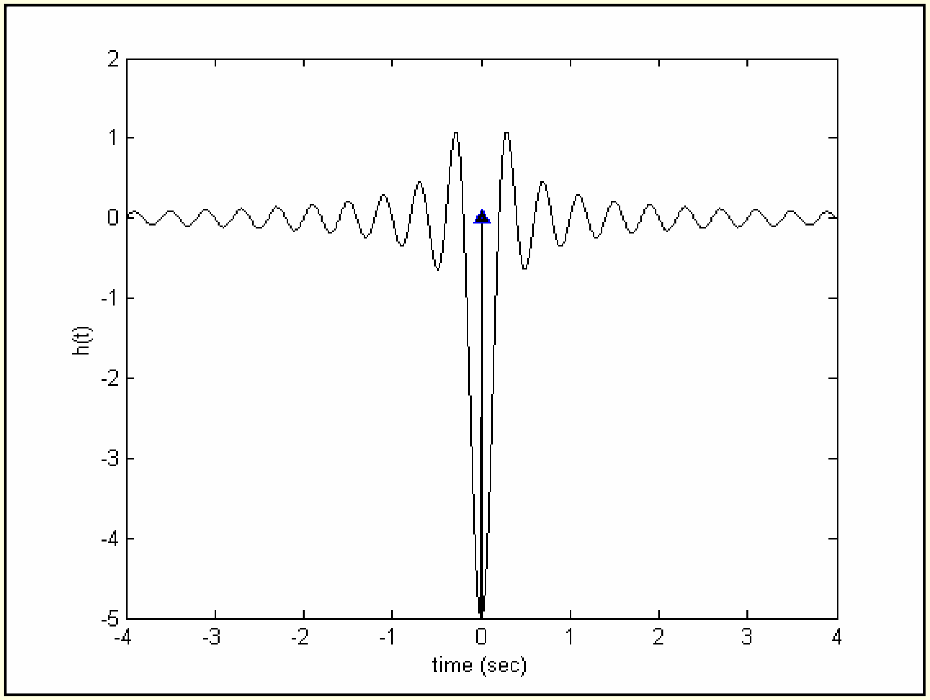
$$=F^{-1}\left\{\left[A-A\,rect\left(\frac{f}{2f_m}\right)\right]e^{-j2\pi f t_o}\right\}$$

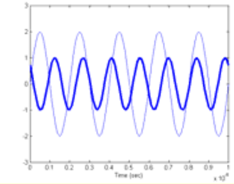
$$=A\delta(f)-2Af_m\mathrm{sinc}\left(2f_m\left(t-t_o\right)\right)$$

Ideal High Pass Filter

- The ideal HPF is non-causal, just like the ideal lowpass filter







Ideal Bandpass Filter

- The ideal bandpass filter can be written as

$$H(f)=\begin{cases} Ae^{-j2\pi f t_o} & f_L \le |f| \le f_H \\ 0 & else \end{cases}$$

$$=\underbrace{A\left[rect\left(\frac{f-f_o}{\Delta f}\right)+rect\left(\frac{f+f_o}{\Delta f}\right)\right]}_{\substack{magnitude\\response}}\underbrace{e^{-j2\pi f t_o}}_{\substack{time\\delay}}$$

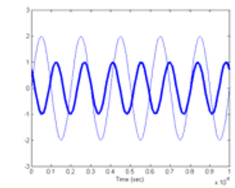
$$h(t)=F^{-1}\left\{H(f)\right\}$$

$$=F^{-1}\left\{A\left[rect\left(\frac{f-f_o}{\Delta f}\right)+rect\left(\frac{f+f_o}{\Delta f}\right)\right]e^{-j2\pi f t_o}\right\}$$

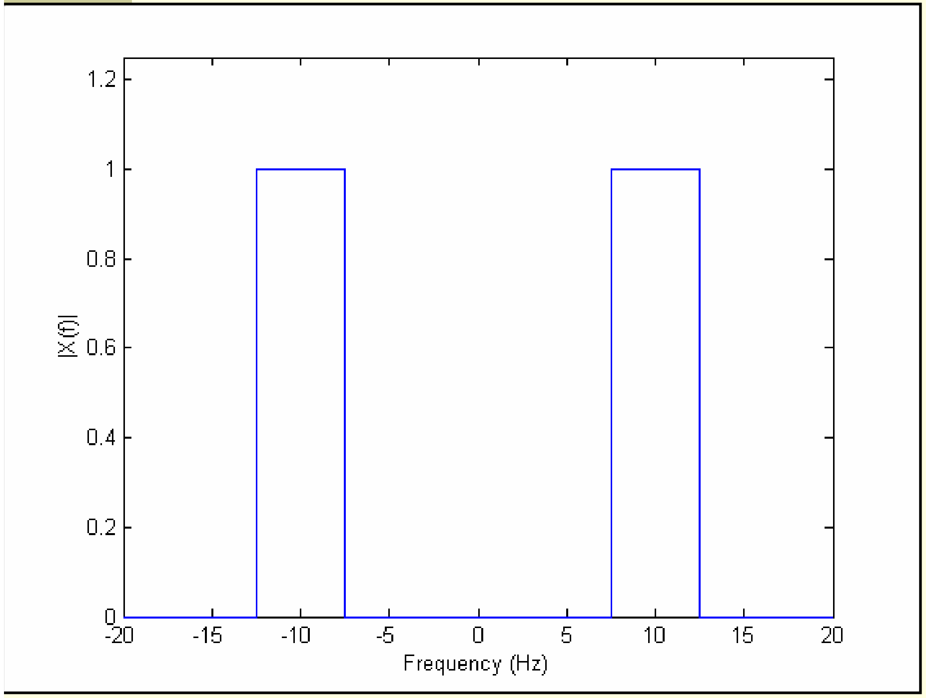
$$=2A\Delta f\ \mathrm{sinc}\left(\Delta f_m\left(t-t_o\right)\right)\cos\left(2\pi f_o\left(t-t_o\right)\right)$$

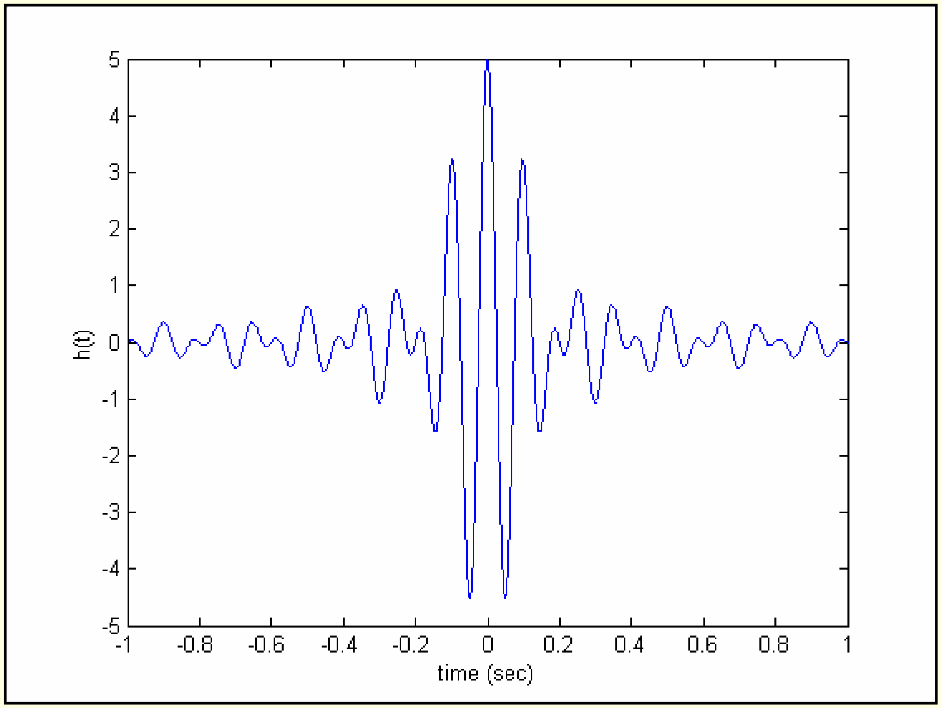
f_L = lower frequency limit
f_H = upper frequency limit
$\Delta f = f_H - f_L$
$f_o = (f_H - f_L)/2$

Ideal Bandpass Filter

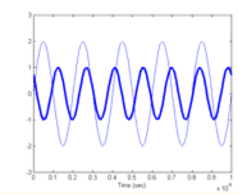


- Ideal BPF is also non-causal
- Realistic bandpass filters will be causal and not have a perfect frequency response

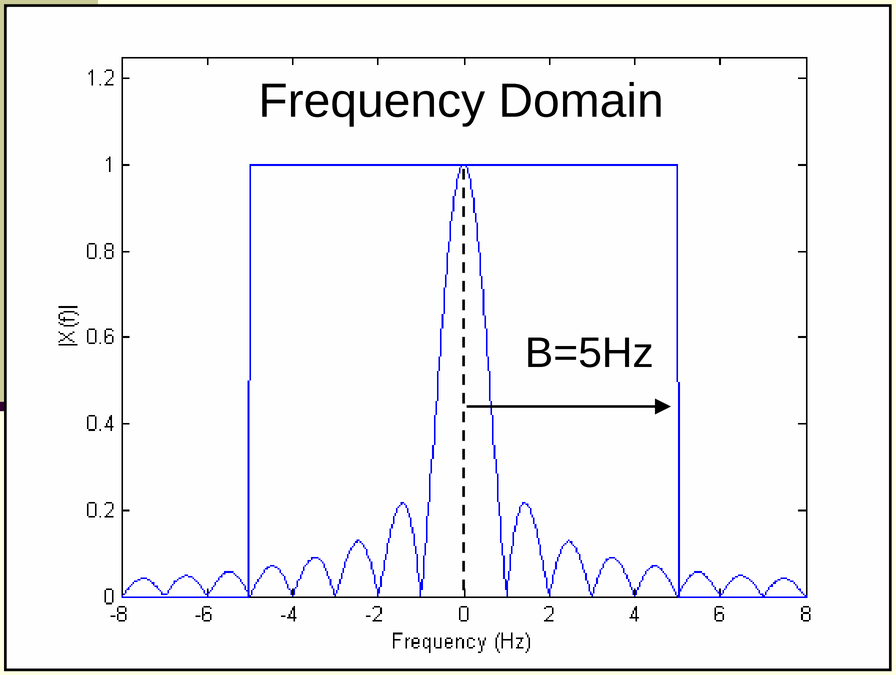


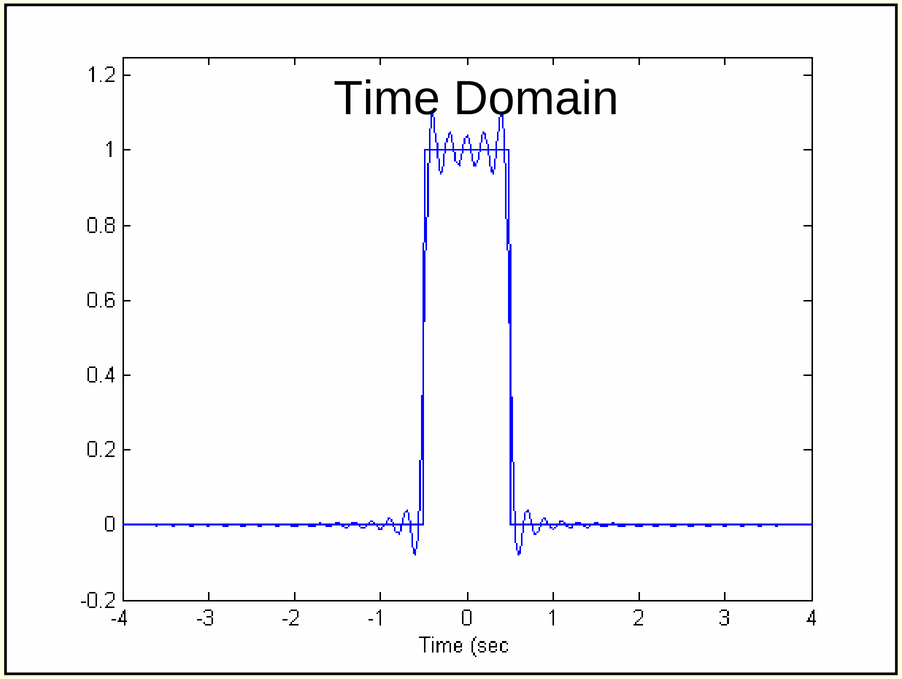


Example



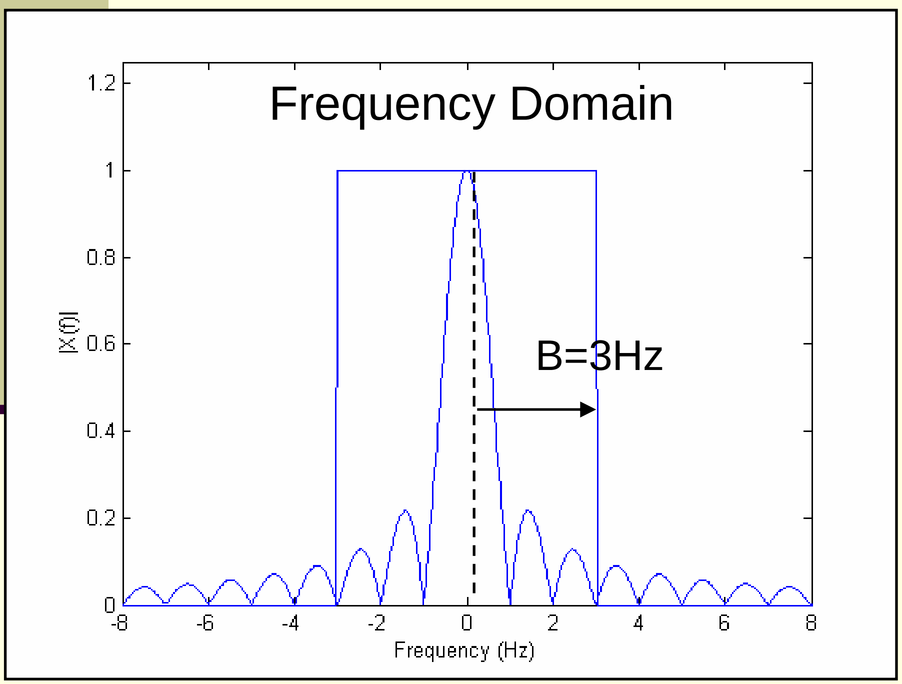
- Consider a time-domain square pulse of width 1 second which is passed through a filter with a bandwidth of 5Hz
- The bandwidth restriction does not cause a substantial change in the pulse shape

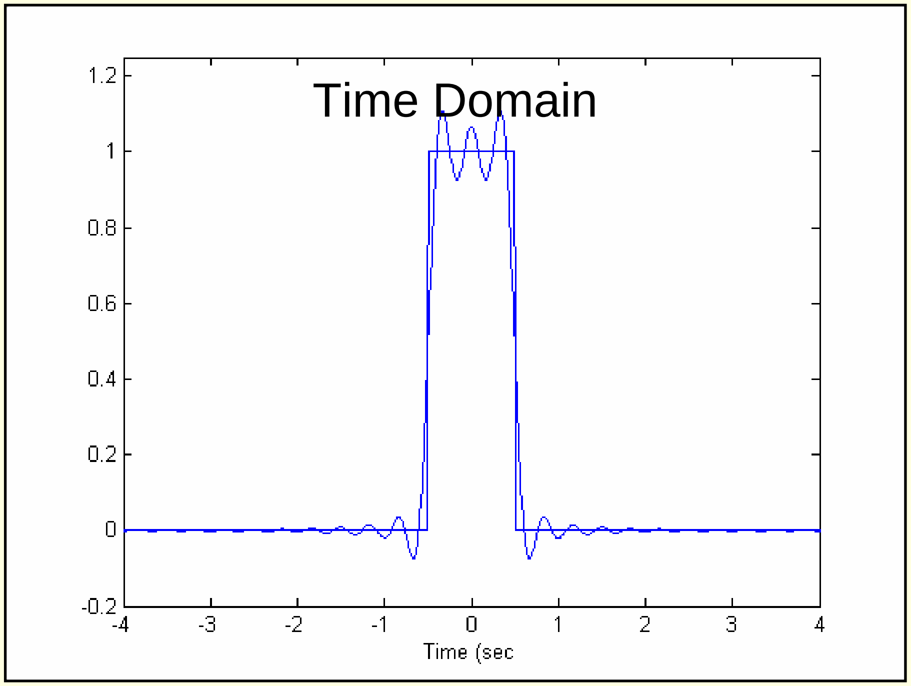




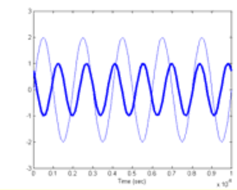
Example – cont.

- Now consider a a filter with a bandwidth of 3Hz
- The bandwidth restriction still does not cause a substantial change in the pulse shape
- Ringing occurs near sharp transitions

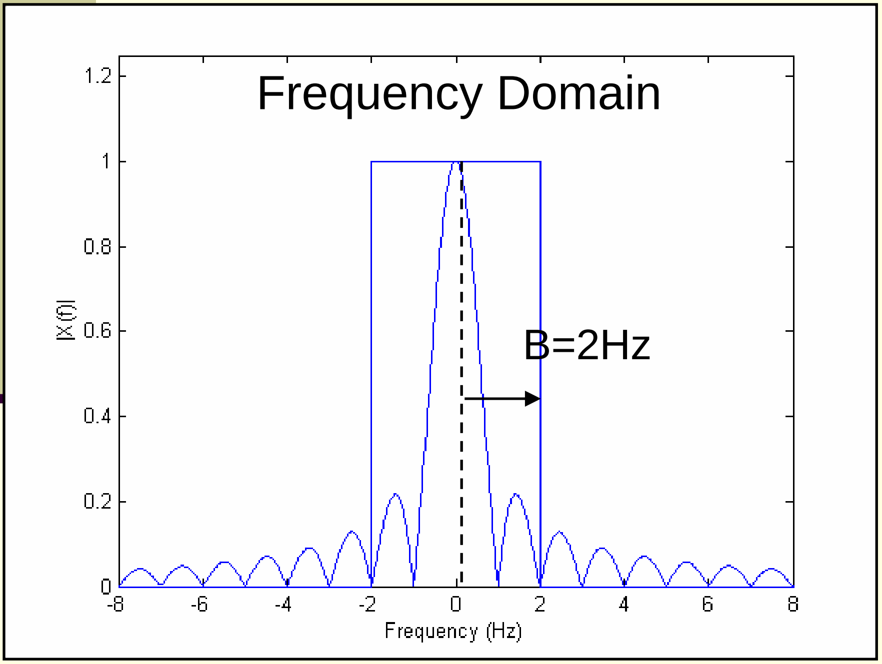


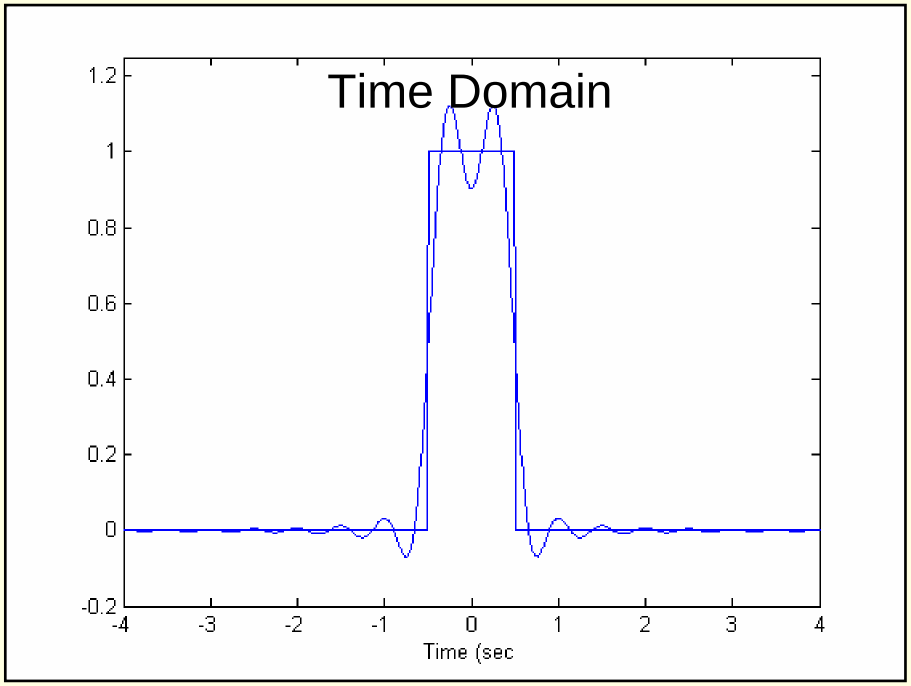


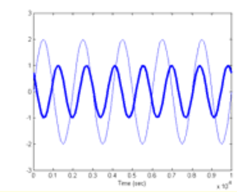
Example – cont.



- Now consider a a filter with a bandwidth of 2Hz
- The bandwidth restriction now begins to cause a more substantial change in the pulse shape
- Sharp transitions cannot occur

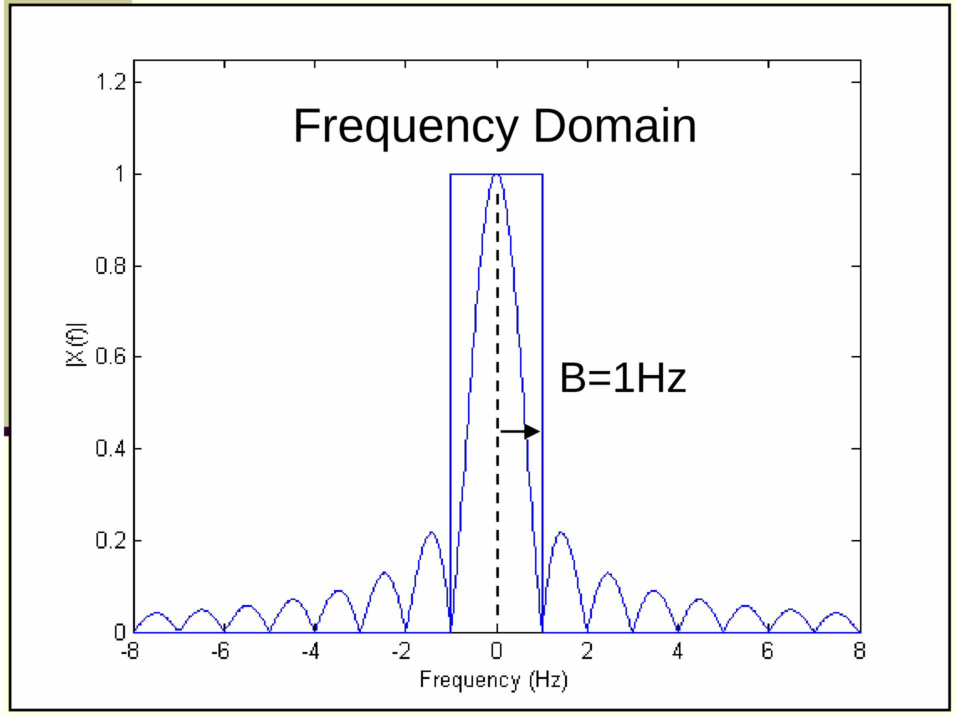


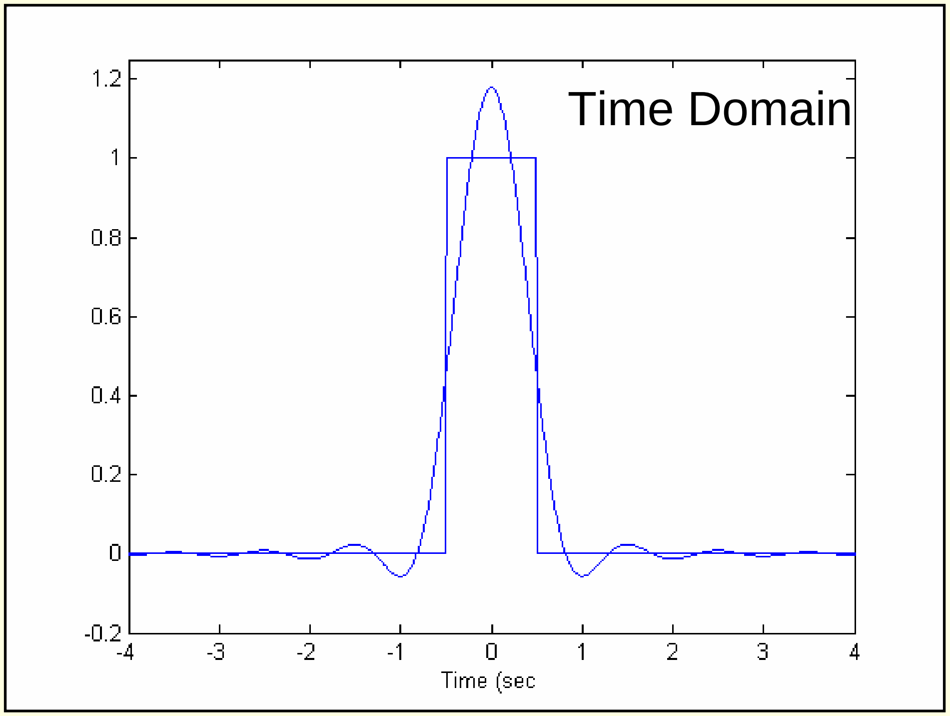


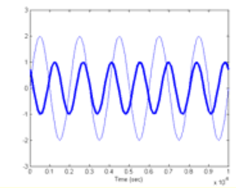


Example – cont.

- Now consider a a filter with a bandwidth of 1Hz
- The bandwidth restriction now changes the shape considerably

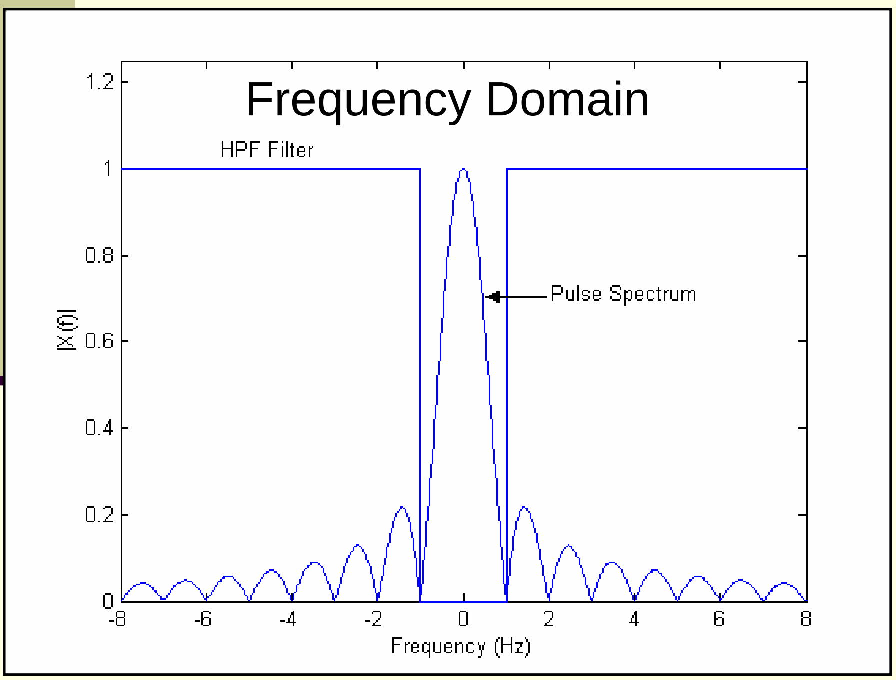


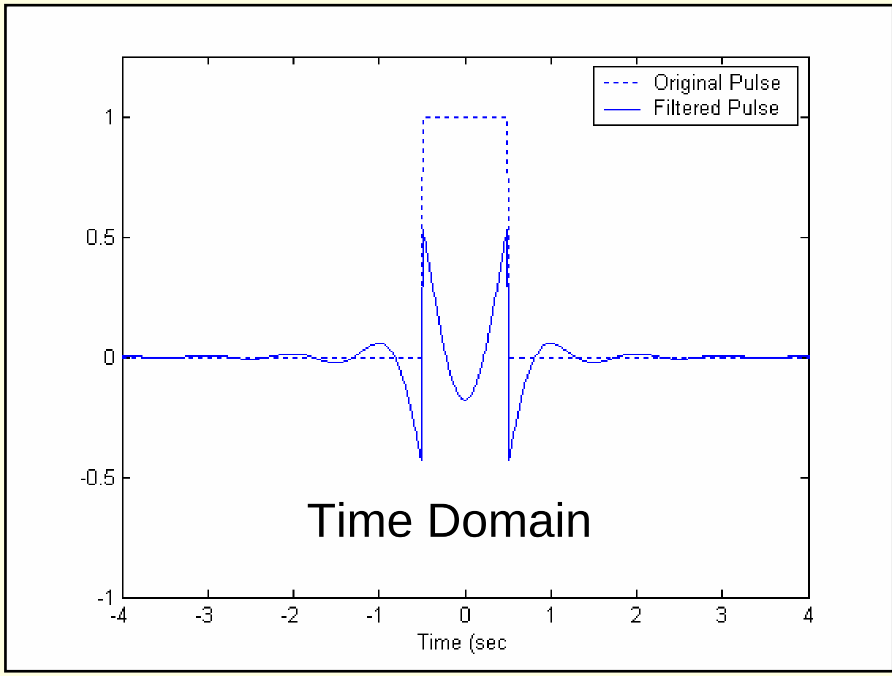




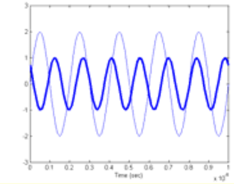
Example 2

- Consider a time-domain square pulse of width 1 second which is passed through a HP filter with a bandstop bandwidth of 1Hz
- The loss of the low-frequency energy causes a large change in the pulse. Note that the pulse now has zero average value.





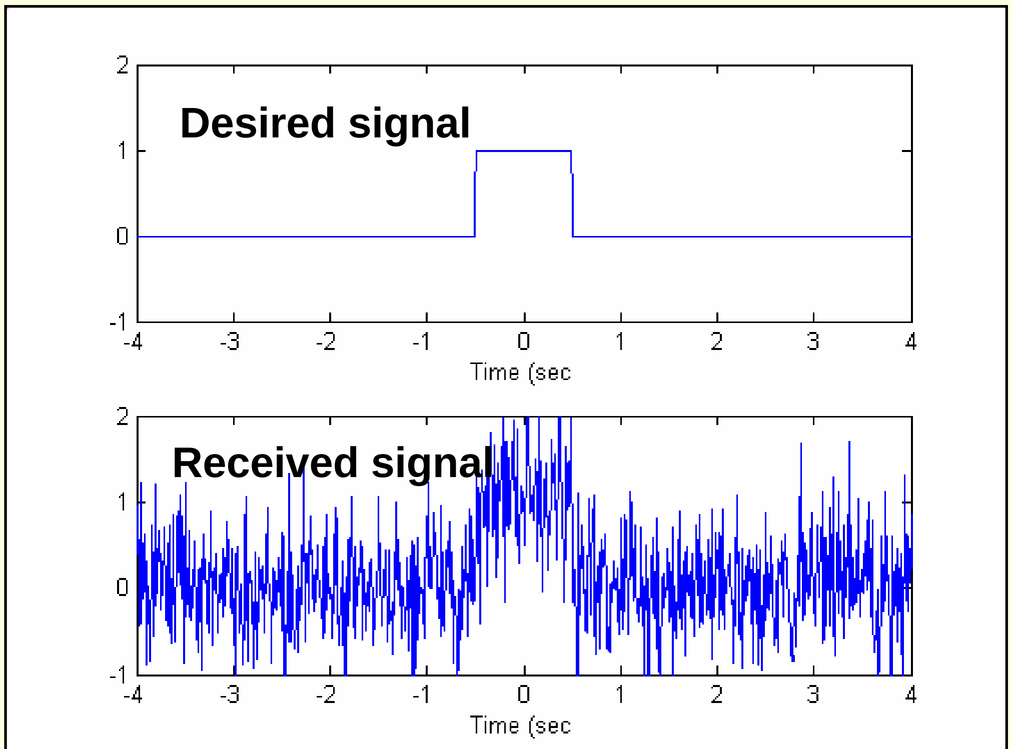
Example 3 - Noise



- A major use of filters is the elimination of noise.
- Noise typically has a much larger bandwidth than the signal of interest.
  - Filtering the received signal with a bandpass or lowpass filter can reduce the amount of noise
- We typically consider *additive noise* where the received signal *r(t)* is equal to the desired signal *x(t)* plus noise *n(t)*
  - $r(t) = x(t) + n(t)$

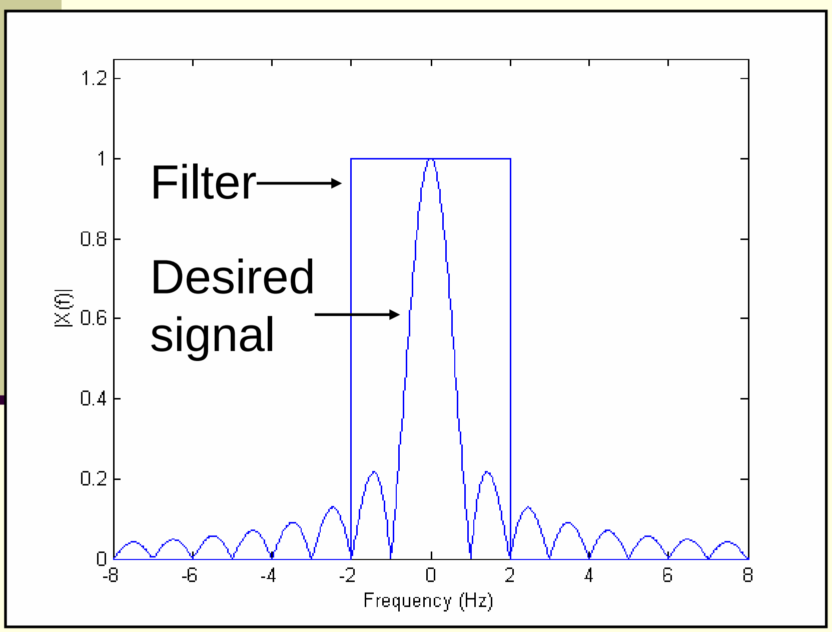
Example 3 (cont.)

- Consider a square pulse with duration one second that is received with the addition of noise
- The ratio of the received desired signal power to the noise power, signal-to-noise ratio or SNR, is 2

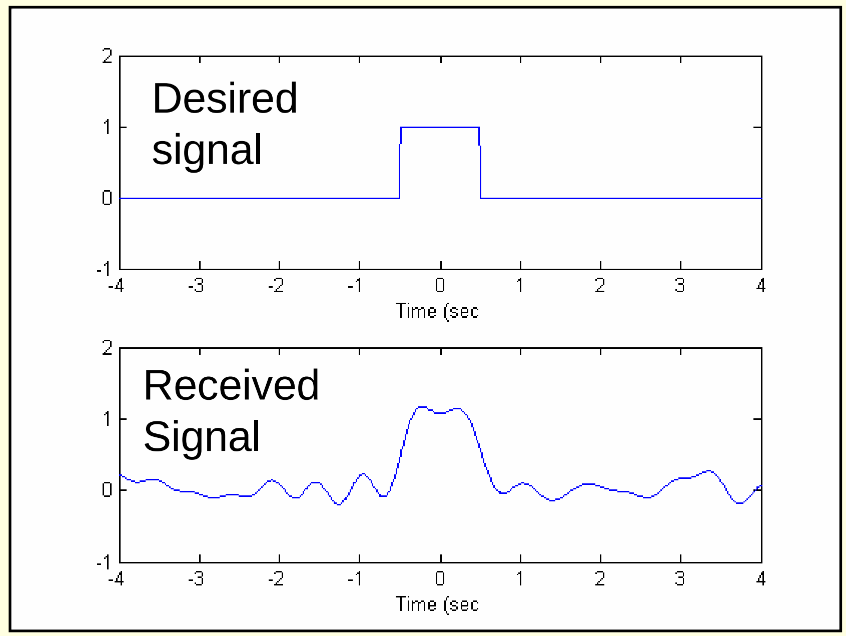


Example 3 (cont.)

- If we filter the signal with an ideal LPF with bandwidth B = 2Hz, we know that we will introduce some distortion to the desired signal, but we can also eliminate much of the noise

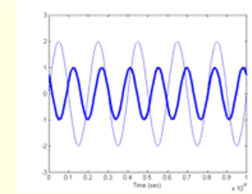


Frequency Domain

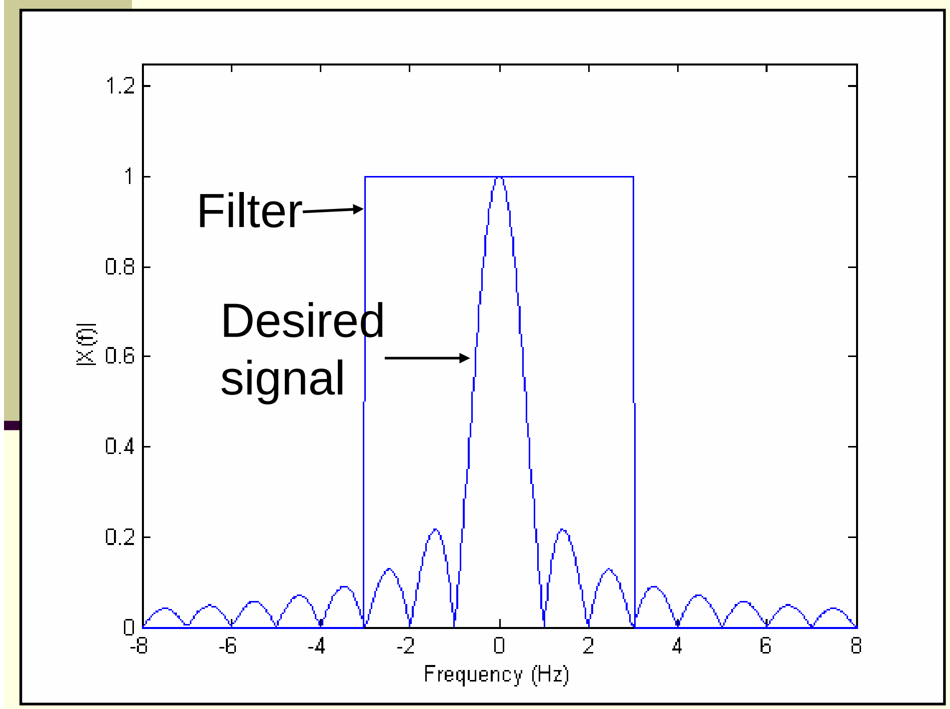


Time Domain

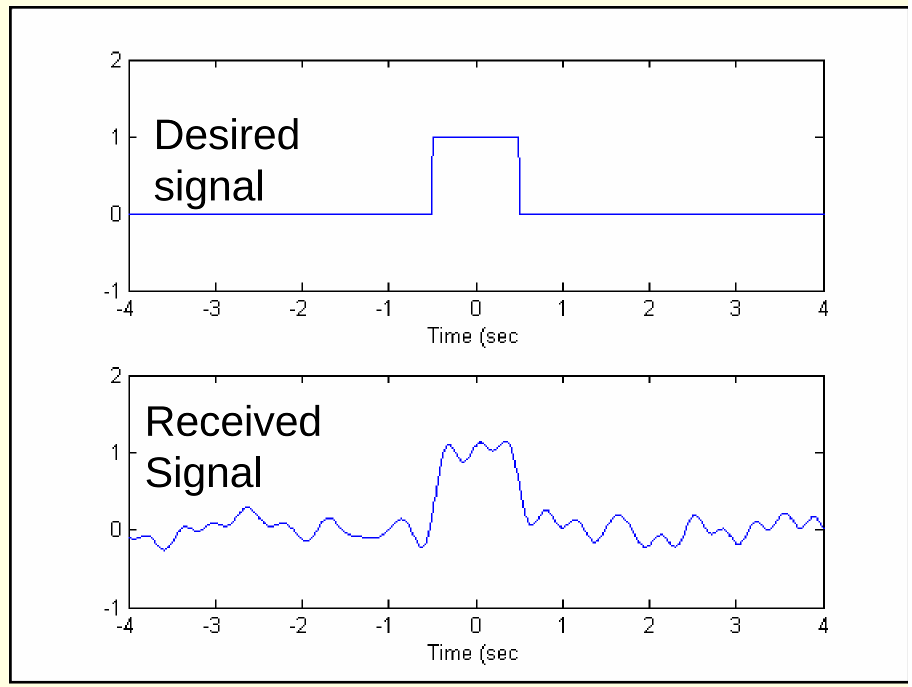
Example 3 (cont.)



- We know that increasing the bandwidth to B = 3Hz will reduce the amount of distortion to the original signal
- However, it also lets more noise in

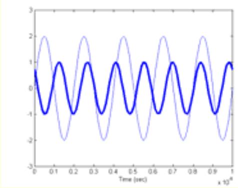


Frequency Domain

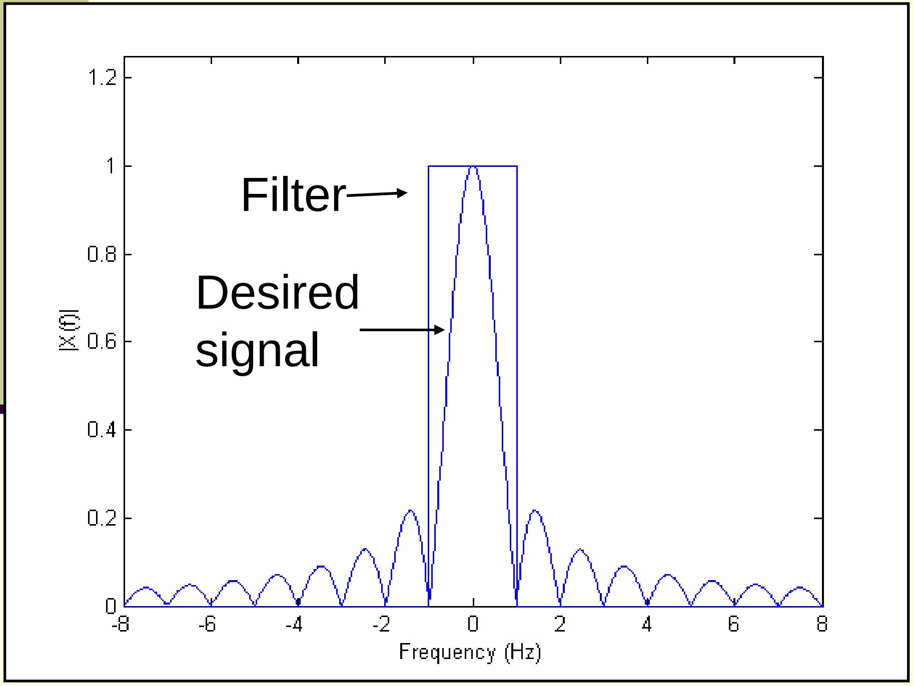


Time Domain

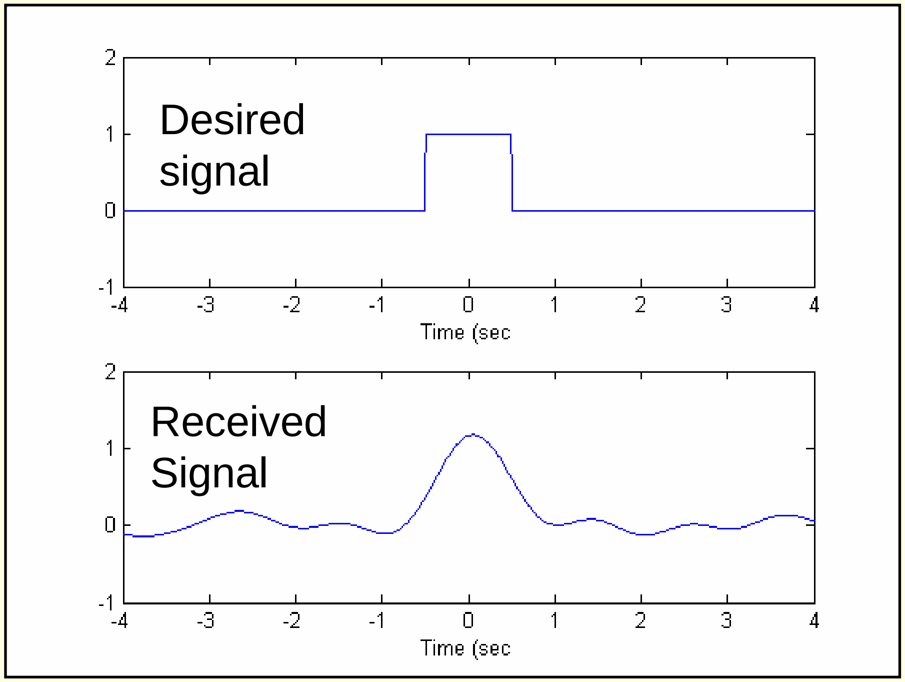
Example 3 (cont.)



- If we descrease the bandwidth to B = 1Hz will reduce the amount of noise in the received signal
- However, it introduces more distortion to the desired signal

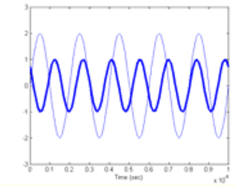


Frequency Domain



Time Domain

Summary



- In this lecture we have examined the application of the Fourier Transform to system analysis, specifically through the system frequency response and through the application of filters
- We defined several ideal filters:
  - LPF, HPF, BPF
- Ideal filters require a non-causal impulse response
  - Real filters must use a delay or non-ideal response
  - Next class we will discuss practical filters