ECE 2704
Signals and Systems
Spring 2006

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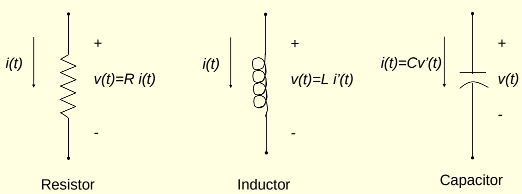
Lecture #16: Applications of the Fourier Transform – Part II: Practical Filters and Bode Diagrams

Overview

- Today we continue our discussion of applications of the Continuous Time Fourier Transform (CTFT)
- We specifically consider
  - Practical Filters
  - Bode plots
  - Log-magnitude plots
  - Communication example
- What to read – Sections 6.4-6.5,6.9 in the text

Resistors, capacitors, inductors

- The voltage across an element in a circuit can be written as



Frequency Domain

- Using our knowledge of Fourier Transforms, we can write the relationship between voltage and current for each:

Resistor	Inductor	Capacitor
$\frac{V(f)}{I(f)} = R$	$\frac{V(f)}{I(f)} = j2\pi fL$	$\frac{V(f)}{I(f)} = \frac{1}{j2\pi fC}$

- Using radian frequency $\omega = 2\pi f$

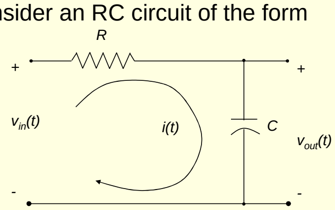
$$\frac{V(j\omega)}{I(j\omega)} = R \qquad \frac{V(j\omega)}{I(j\omega)} = j\omega L \qquad \frac{V(j\omega)}{I(j\omega)} = \frac{1}{j\omega C}$$

- This gives rise to the concept of *impedence $Z(\omega)$*

$$Z(j\omega) = \frac{V(j\omega)}{I(j\omega)} \qquad Z(f) = \frac{V(f)}{I(f)}$$

RC Lowpass Filter

- Consider an RC circuit of the form



- Using Kirchoff's Voltage law

$$v_{in}(t) = i(t)R + v_{out}(t)$$
$$= v'_{out}(t)RC + v_{out}(t)$$

RC Filter (cont.)

- We could solve this using differential equations as we did in the past. However, using Fourier Transforms is more straightforward.

$$v_{in}(t) = v'_{out}(t)RC + v_{out}(t)$$

- Taking the Fourier Transform of both sides

$$V_{in}(f) = j2\pi f V_{out}(f)RC + V_{out}(f)$$

- Now we can directly solve for the system transfer function

$$H(f) = \frac{V_{out}(f)}{V_{in}(f)}$$
$$= \frac{1}{1 + j2\pi fRC}$$

RC Filter – cont.

- The system transfer function can be plotted in terms of its magnitude response and phase response

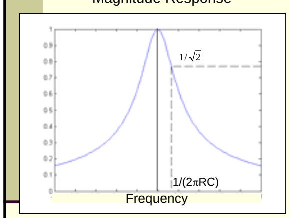
$$H(f) = \frac{1}{1 + j2\pi fRC}$$

$$|H(f)| = \frac{1}{\sqrt{1 + (2\pi fRC)^2}}$$

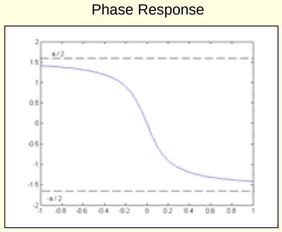
$$\angle H(f) = 0 - \tan^{-1}(2\pi fRC) = -\tan^{-1}(2\pi fRC)$$

RC Filter – Magnitude and Phase

Magnitude Response



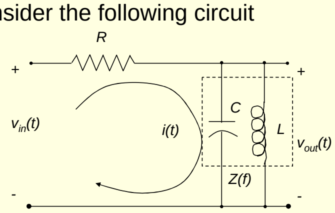
Phase Response



Clearly this is a low pass filter with a 3dB bandwidth of 1/(2πRC)
However, note that it is defined as lowpass strictly because of the input and output definitions

RLC Bandpass Filter

- Consider the following circuit



- If we define the combined capacitor and inductor to have an impedence *Z(f)*, we can solve the problem just as in the previous case
- We will use the frequency domain directly

RLC Circuit (cont.)

- Summing voltages around the circuit:

$$\begin{aligned}
V_{in}(f) &= RI(f) + V_{out}(f) \\
&= R\frac{V_{out}(f)}{Z(f)} + V_{out}(f) \\
&= V_{out}(f)\left\{1 + \frac{R}{Z(f)}\right\} \\
&= V_{out}(f)\left\{1 + \frac{R}{\frac{j2\pi fL}{1-(2\pi f)^2 LC}}\right\} \\
&= V_{out}(f)\left\{1 + \frac{R-(2\pi f)^2 RLC}{j2\pi fL}\right\}
\end{aligned}$$

Find equivalent impedence

$$\begin{aligned}
Z(f) &= \frac{Z_C(f)Z_L(f)}{Z_C(f)+Z_L(f)} \\
&= \frac{j2\pi fL\frac{1}{j2\pi fC}}{j2\pi fL + \frac{1}{j2\pi fC}} \\
&= \frac{j2\pi fL}{1-(2\pi f)^2 LC}
\end{aligned}$$

RLC Filter – cont.

- Continuing…

$$V_{in}(f) = V_{out}(f)\left\{1 + \frac{R-(2\pi f)^2 RLC}{j2\pi fL}\right\}$$

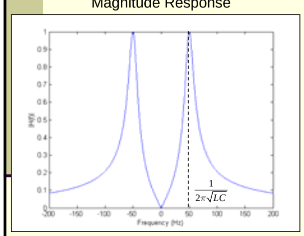
$$\begin{aligned}
H(f) &= \frac{V_{out}(f)}{V_{in}(f)} \\
&= \frac{j2\pi fL}{\left(R-(2\pi f)^2 RLC\right) + j2\pi fL}
\end{aligned}$$

- Now at very low frequencies (near zero) the response is nearly zero
- At very high frequencies, the response also goes to zero as the denominator dominates
- Thus, this circuit is a *bandpass filter*

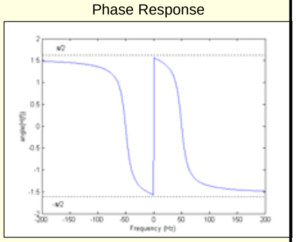
RLC Filter - Example

- R = 10,000 L = 10 C = 10^{-6}

Magnitude Response



Phase Response

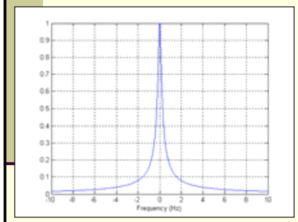


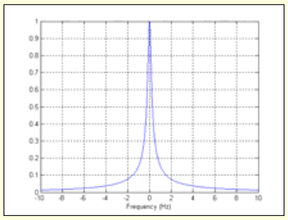
The filter gain is maximum at the frequencies near $\pm\frac{1}{2\pi\sqrt{LC}}$

Log-magnitude Frequency-Response

- Consider the two system responses

$H(f)=\frac{1}{1+j2\pi f}$ $H_2(f)=\frac{30}{30-4\pi^2 f^2+j62\pi f}$



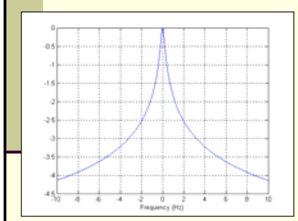


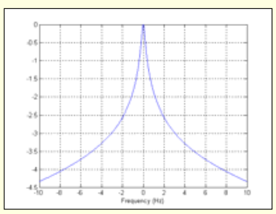
- The two responses appear identical

Log-magnitude Frequency-Response

- More subtle differences are observable when plotted on a log scale $\ln|H(f)|$

$H(f)=\frac{1}{1+j2\pi f}$ $H_2(f)=\frac{30}{30-4\pi^2 f^2+j62\pi f}$

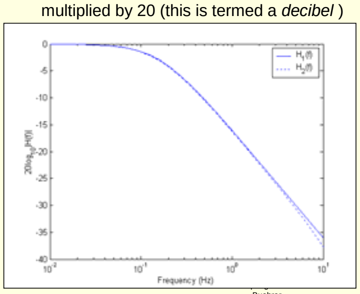




Bode Diagrams

- A Bode diagram is a plot of the transfer function that is logarithmic in both dimensions
- Further, the base of the logarithm is 10 rather than *e* and multiplied by 20 (this is termed a *decibel*)

$$20\log_{10}\left(\left|H(f)\right|\right)$$



Single Pole Filter

- Consider a filter with system transfer function

$$H(f) = \frac{1}{1 + j2\pi f / p_1}$$

- At very low and high frequencies we have

$$H(f) \approx 1 \quad 2\pi f << p_1$$

$$H(f) \approx \frac{p_1}{j2\pi f} \quad 2\pi f >> p_1$$

- Thus, at low frequencies, the filter response is unity and at high frequencies the filter response is inversely proportional with $2\pi f$

Single Pole Filter – cont.

- Consider two high frequencies f_1 and $f_2 = 10f_1$

$$20\log_{10}|H(f_1)| \approx 20\log_{10}\left(\frac{p_1}{2\pi f_1}\right)$$

$$20\log_{10}|H(f_2)| \approx 20\log_{10}\left(\frac{p_1}{2\pi f_2}\right)$$

$$= 20\log_{10}\left(\frac{p_1}{2\pi 10 f_1}\right)$$

$$= 20\log_{10}\left(\frac{1}{10}\right) + 20\log_{10}\left(\frac{p_1}{2\pi f_1}\right)$$

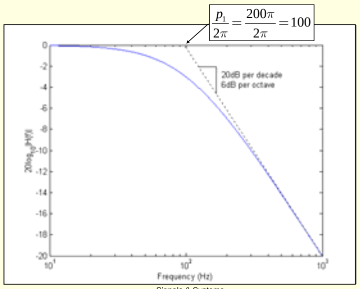
$$= 20\log_{10}\left(\frac{p_1}{2\pi f_1}\right) - 20dB$$

- A change in frequency of a factor of ten is termed a *decade.*
- A change in frequency by a factor of two is termed an *octave*
- Thus, a single pole filter response decays at 20dB per decade or 6dB per octave

Single Pole Filter – cont.

- Example $H(f) = \frac{1}{1 + j2\pi f / (200\pi)}$

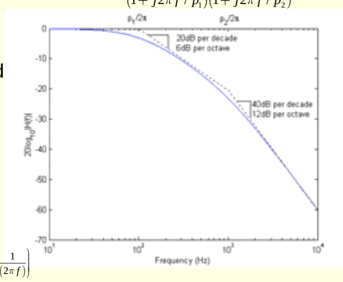
$$\frac{p_1}{2\pi} = \frac{200\pi}{2\pi} = 100$$



Two-pole Filter

- Second pole increases the slope of the filter decay after pole has been reached
- $p_1 = 200\pi$
- $p_2 = 2000\pi$

$$H(f) = \frac{1}{(1 + j2\pi f / p_1)(1 + j2\pi f / p_2)}$$



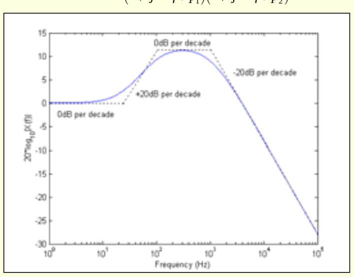
$$|H(f)| \approx \frac{p_1 p_2}{(2\pi f)^2} \quad 2\pi f >> p_1$$

$$20\log_{10}|H(f)| \approx 20\log_{10}\left(\frac{p_1 p_2}{(2\pi f)^2}\right) = 20\log_{10}(p_1 p_2) - 40\log_{10}\left(\frac{1}{(2\pi f)}\right)$$

Pole-Zero Filter

- Zero provides positive slope
- First pole cancels the zero
- Second pole provide negative slope
- $z_1 = 50\pi$
- $p_1 = 200\pi$
- $p_2 = 2000\pi$

$$H(f) = \frac{(1 + j2\pi f / z_1)}{(1 + j2\pi f / p_1)(1 + j2\pi f / p_2)}$$



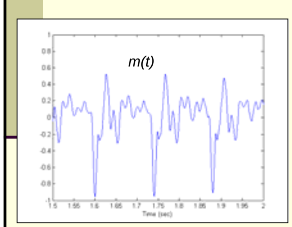
$$|H(f)| \approx \frac{p_1 p_2}{z_1} \frac{2\pi f}{(2\pi f)^2} \quad 2\pi f >> p_1$$

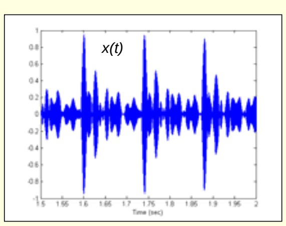
$$= \frac{p_1 p_2}{z_1 2\pi f}$$

Communications Example

- Amplitude Modulation – Double Sideband Suppressed Carrier (DSBSC)

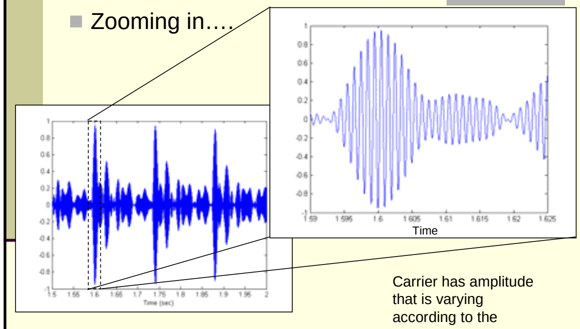
$$x(t) = m(t)\cos(2\pi f_c t)$$





AM – cont.

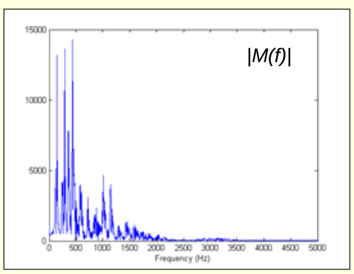
- Zooming in….



Carrier has amplitude that is varying according to the message

AM – cont.

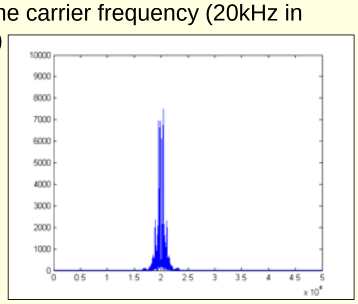
- Message Signal Spectrum
- Voice signal has most of its energy below 4kHz



AM - cont.

- After modulation the spectrum of the signal is centered at the carrier frequency (20kHz in this example)

$$X(f)=\frac{1}{2}\{M(f-f_c)+M(f+f_c)\}$$

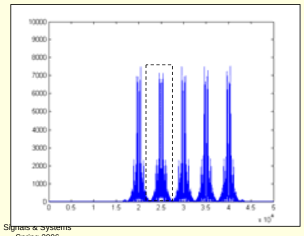


AM – cont.

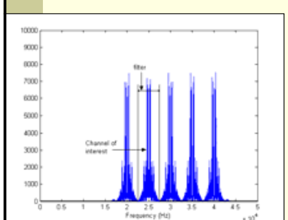
- An AM radio must select one signal from several received signals.
- We can visualize what the receiver must do by examining the frequency domain:

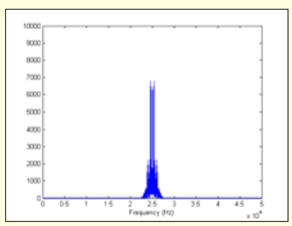
If we desire to listen to channel 2, the reciever must use a bandpass filter to isolate the second signal from the others

This would be extremely hard to visualize in the time domain



AM – cont.





Summary

- In this lecture we have examined further application of the Fourier Transform
- Specifically we have looked at the practical lowpass and bandpass filters using simple RLC circuits.
- We also showed how these circuits are analyzed through the use of Bode plots and component diagrams
- We finished with a common but simple communications example – AM modulation