ECE 2704
Signals and Systems
Spring 2006

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Lecture #17: Introduction to the Laplace Transform

Overview

- Today we introduce the Laplace Transform which is a generalization of the Fourier Transform
- The Laplace Transform applies to a more general class of functions than the Fourier Transform does and is thus more useful in many cases.
- What to read – Sections 9.1-9.2 in the text

The Generalized Fourier Transform

- Consider the time function

$$g(t) = Au(t)$$

- The Fourier Transform of this function can be written as

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} Au(t)e^{-j2\pi ft}dt \\ &= A\int_{0}^{\infty} e^{-j2\pi ft}dt \end{aligned}$$

- Unfortunately, this integral doesn't converge
- We can, however, find a *Generalized* Fourier Transform as we did before

Convergence

- Now consider the integral

$$A\int_0^{\infty} e^{-\sigma t}e^{-j2\pi ft}dt$$

- This integral will converge provided that $\sigma > 0$
- Thus, let us define a new transform that includes this factor $e^{-\sigma t}$
- The resulting transform is more useful than the Fourier Transform since it will apply not only to functions that have no Fourier Transform, but also to functions that do not have a *generalized* Fourier Transform

The Laplace Transform

- Define a transform of the function *x(t) as*

$$\begin{aligned} L\{g(t)\} &= \int_{-\infty}^{\infty} g(t)e^{-\sigma t}e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} g(t)e^{-\sigma t - j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j2\pi f)t}dt \\ &= \int_{-\infty}^{\infty} g(t)e^{-st}dt \qquad \boxed{\begin{aligned} s &= \sigma + j2\pi f \\ &= \sigma + j\omega \end{aligned}} \\ &= G_L(s) \end{aligned}$$

- This is termed the Laplace Transform of *g(t)* after the French mathematician Pierre-Simon Laplace

Formal Definition

- The bilateral Laplace Transform is thus defined as

$$\begin{aligned} L\{g(t)\} &= G_L(s) \\ &= \int_{-\infty}^{\infty} g(t)e^{-st}dt \end{aligned}$$

which is the Fourier Transform with $j2\pi f$ replaced with $\sigma + j2\pi f = \sigma + j\omega$

- Note that this assumes that the integral converges which will only be true for certain values of σ. (Recall that for *g(t) = Au(t)* we required that $\sigma > 0$ for convergence)

Inverse Laplace Transform

- Consider the function

$$g_\sigma(t) = g(t)e^{-\sigma t}$$

- This function has a Fourier Transform

$$G_\sigma(f) = \int_{-\infty}^{\infty} g_\sigma(t)e^{-j2\pi ft}dt$$

- Recall that the inverse Fourier Transform can be written as

$$g_\sigma(t) = F^{-1}\{G_\sigma(f)\} = \int_{-\infty}^{\infty} G_\sigma(f)e^{j2\pi ft}df$$

Inverse Laplace Transform (cont.)

- Using $s = \sigma + j2\pi f$ and $ds = j2\pi\, df$

$$g_\sigma(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} G_L(s)e^{(s-\sigma)t}ds = \frac{e^{-\sigma t}}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} G_L(s)e^{st}ds$$

- Now substituting for $g_\sigma(t) = e^{-\sigma t} g(t)$

$$g(t)e^{-\sigma t} = \frac{e^{-\sigma t}}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} G_L(s)e^{st}ds$$

$$g(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} G_L(s)e^{st}ds$$

Inverse Laplace Transform – cont.

- Formally we define the Laplace Transform as

$$g(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} G_L(s)e^{st}ds$$

which states that a function can be represented as a linear combination of general complex exponentials (as opposed to complex sinusoids)

System Transfer Function

- Consider a complex exponential

$$x(t) = Ae^{st}$$

- The output of a linear time-invariant system when x(t) is applied at the input is

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) Ae^{s(t-\tau)}\, d\tau \\ &= Ae^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau \end{aligned}$$

System Transfer Function – cont.

- Thus, we have

$$y(t) = \underbrace{Ae^{st}}_{x(t)} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau}_{\text{Laplace Transform of } h(t)}$$

which says that the output of a system in response to a complex exponential is simply the same complex exponential weighted by the Laplace transform of the sytem impulse response.

- Most functions can be represented by a linear combination of complex exponentials.
- Thus, if the Laplace transform is linear (which we will show next class) the output of a system to any excitation can be found by multiplying the Laplace Transform of the input with the Laplace Transform of the impulse response

Region of Convergence

- Consider the function

$$g(t) = Ae^{\alpha t} u(t) \quad \alpha > 0$$

Note α is all real

- This function does not have a Fourier Transform nor a generalized Fourier Transform
- The Laplace Transform can be written as

$$\begin{aligned} G(s) &= \int_{-\infty}^{\infty} g(t) e^{-st}\, dt \\ &= \int_{0}^{\infty} Ae^{\alpha t} e^{-st}\, dt \\ &= \int_{0}^{\infty} Ae^{(\alpha-\sigma)t} e^{-j2\pi ft}\, dt \end{aligned}$$

Region of Convergence – cont.

- The integral

$$G(s)=\int_0^{\infty} Ae^{(\alpha-\sigma)t}e^{-j2\pi ft}dt$$

will converge provided that $\sigma > \alpha$

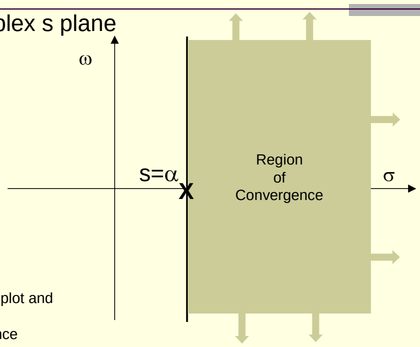
- When $\sigma > \alpha$ we get

$$G(s)=\int_0^{\infty} Ae^{-(s-\alpha)t}dt$$
$$=\frac{A}{s-\alpha} \quad \sigma=\mathrm{Re}\{s\}>\alpha$$

- In the complex plane (where σ is plotted on the x-axis and $\omega=2\pi f$ is plotted on the y-axis) the value $\sigma = \alpha$ is a *pole* of the transform G(s) which is defined as a point where the transform goes to infinity.

Region of Convergence – cont.

Complex s plane



Pole-zero plot and region of convergence

Example 1

- Find the Laplace Transform of the function

$$g(t)=e^{-t}u(t)+e^{-2t}u(t)$$

and plot the poles and the region of convergence

$$G(s)=\int_{-\infty}^{\infty} g(t)e^{-st}dt$$
$$=\int_{-\infty}^{\infty}\left(e^{-t}u(t)+e^{-2t}u(t)\right)e^{-st}dt$$
$$=\int_0^{\infty} e^{-t}e^{-st}dt+\int_0^{\infty} e^{-2t}e^{-st}dt$$
$$=\int_0^{\infty} e^{-(s+1)t}dt+\int_0^{\infty} e^{-(s+2)t}dt$$

Example 1 – cont.

- Continuing…

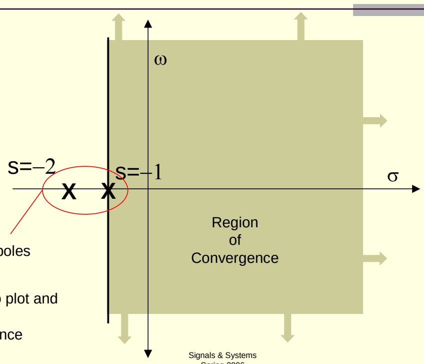
$$G(s) = \int_0^\infty e^{-(s+1)t}dt + \int_0^\infty e^{-(s+2)t}dt$$
$$= \frac{1}{s+1} + \frac{1}{s+2} \quad \sigma > -1$$

- Note that the region of convergence (ROC) for the first part of the transform is $R_1 = \sigma > -1$ while the ROC for the second part of the transform is $R_2 = \sigma > -2$. Thus, the overall ROC is

$$R = R_1 \cap R_2$$
$$= \sigma > -1$$

- The poles of this transform are $p_1 = -1$, $p_2 = -2$

Example 1 – cont.



Pole-zero plot and region of convergence

Example 2

- Find the Laplace Transform of the function

$$g(t) = e^{-t}u(t) + e^{2t}u(-t)$$

and plot the poles and the region of convergence

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} \left(e^{-t}u(t) + e^{2t}u(-t)\right)e^{-st}dt$$
$$= \int_0^\infty e^{-t}e^{-st}dt + \int_{-\infty}^0 e^{2t}e^{-st}dt$$
$$= \int_0^\infty e^{-(s+1)t}dt + \int_{-\infty}^0 e^{-(s-2)t}dt$$

Example 2 – cont.

- Continuing…

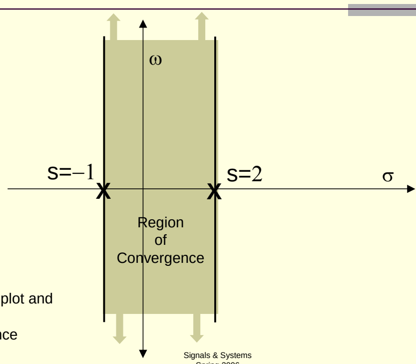
$$G(s) = \int_0^{\infty} e^{-(s+1)t}dt + \int_{-\infty}^{0} e^{-(s-2)t}dt$$
$$= \frac{1}{s+1} - \frac{1}{s-2} \quad \sigma > -1, \sigma < 2$$

- Note that the region of convergence (ROC) for the first part of the transform is $R_1 = \sigma > -1$ while the ROC for the second part of the transform is $R_2 = \sigma < 2$. Thus, the overall ROC is

$$R = R_1 \cap R_2$$
$$= -1 < \sigma < 2$$

- The poles of this transform are $p_1 = -1$, $p_2 = 2$

Example 2 – cont.



Pole-zero plot and region of convergence

Example 2 – cont.

- From these two examples we can write two Laplace transform pairs

$$e^{-\alpha t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+\alpha} \quad \sigma > -\alpha$$
$$-e^{-\alpha t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+\alpha} \quad \sigma < -\alpha$$

- Note that despite the fact that the functions are different, they have the same Laplace Transform.
- However, they have *different* regions of convergence. In fact, the ROC's are mutually exclusive for the same value of α. This means that we cannot find a Laplace transform for the sum of these two functions if the values of α are equal.

ROC and the Fourier Transform

- Note that the region of convergence tells us which values of *s* allow the integral

$$\int_{-\infty}^{\infty} g(t)e^{-st}dt$$

to converge

- Thus, if the ROC includes the $j\omega = j2\pi f$ axis, the integral

$$\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$

converges and thus the Fourier Transform exists.

Convergence

- Consider the constant function

$$g(t) = A$$

- The Laplace Transform is

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$
$$= A\int_{-\infty}^{\infty} e^{-st}dt$$
$$= A\int_{-\infty}^{0} e^{-st}dt + A\int_{0}^{\infty} e^{-st}dt$$

- Note that the first integral will converge if $\sigma < 0$ while the second integral will converge if $\sigma > 0$
- Thus, a Laplace Transform cannot be found

Convergence - cont.

- We saw that the function g(t) = A is not Laplace Transformable. However, the function *g(t) = Au(t)* is.

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$
$$= A\int_{0}^{\infty} e^{-st}dt$$
$$= \frac{A}{s} \quad \sigma > 0$$

- In general functions which are non-zero for a semi-infinite range of time (i.e., causal functions) and increase in time no faster than an exponential will have a Laplace Transform.

The Unilateral Laplace Transform

- The previous fact concerning causal functions gives rise to a second definition of the Laplace Transform termed the *unilateral Laplace Transform:*

$$G(s) = \int_{0^-}^{\infty} g(t) e^{-st} dt$$

- Due to the change in the lower limit, any function which grows no faster than an exponential in positive time has a unilateral Laplace Transform
- This is also termed the *one-sided Laplace Transform*
- The previous definition

$$G(s) = \int_{0^-}^{\infty} g(t) e^{-st} dt$$

is often termed the bilateral or two-sided Laplace Transform

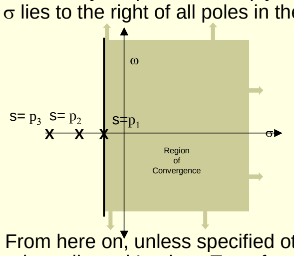
Unilateral Laplace Transform – cont.

- The unilateral Laplace Transform is restrictive in the sense that it excludes the negative-time behavior of a function
  - However this is not a problem practically since practical functions are causal.
- This means that any two functions which have the same behavior after time t=0 will have the same unilateral Laplace transform (ex: *g(t) = A* and *g(t) = Au(t)*)
- Note: The integration starts at 0^- to ensure capturing any impulse at $t = 0$.
- The inverse unilateral Laplace Transform is the same as the bilateral:

$$g(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} G_L(s) e^{(s-\sigma)t} ds$$

Region of Convergence

- The region of convergence for the unilateral Laplace Transform is very simple – it is simply the region for which σ lies to the right of all poles in the complex s-plane



- Note: From here on, unless specified otherwise, we will use the unilateral Laplace Transform

Example 3

- Find the unilateral Laplace Transform of

$$g(t) = e^{-\alpha t}u(t)$$

$$\begin{aligned} G(s) &= \int_{0^-}^{\infty} g(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} e^{-\alpha t}u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} e^{-(s+\alpha)t}dt \\ &= \frac{1}{s+\alpha} \quad \sigma > -\alpha \end{aligned}$$

- which is the same as the bilateral LT. Why?

Example 4

- Find the unilateral Laplace Transform of

$$g(t) = u(t)$$

$$\begin{aligned} G(s) &= \int_{0^-}^{\infty} g(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} e^{-st}dt \\ &= \frac{1}{s} \quad \sigma > 0 \end{aligned}$$

- which is the same as the bilateral LT. Why?

Example 5

- Find the unilateral Laplace Transform of

$$g(t) = \delta(t)$$

$$\begin{aligned} G(s) &= \int_{0^-}^{\infty} g(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} \delta(t)e^{-st}dt \\ &= 1 \end{aligned}$$

- Note that the ROC is the entire s plane

Summary

- In this lecture we have introduced the unilateral and bilateral Laplace Transforms
- The Laplace Transform is more general than the Fourier Transform and thus is more useful in many applications
- Although most of the development was for the bilateral Laplace Transform, we will focus on the unilateral transform
  - We develop the bilateral transform due to its similarity to the Fourier Transform
  - Note that the unilateral and bilateral transforms will yield the same transforms for functions defined only for t >= 0