ECE 2704
Signals and Systems
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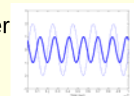
Instructor: Dr. R. Michael Buehrer
Lecture #18: Properties of the Laplace Transform



Overview

- Today we continue our discussion of the Laplace Transform
- The current lecture focuses on the properties of the Laplace Transform
  - These properties are very similar to the properties that we studied earlier for the Fourier Series and Fourier Transform
  - Understanding the properties and a handful of basic Laplace Transforms allows us to determine most of the Laplace Transforms of interest
- Note that we consider the *unilateral Laplace Transform*
- What to read – Section 9.3 in the text

Preliminaries

- For the following lecture we assume that

$$x(t) = 0 \quad t < 0$$
$$y(t) = 0 \quad t < 0$$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

$$\mathcal{L}\{y(t)\} = Y(s) = \int_{0^-}^{\infty} y(t) e^{-st} dt$$

Linearity

- If $z(t) = \alpha x(t) + \beta y(t)$
- Then

$$\begin{aligned} Z(s) &= \int_{0^-}^{\infty} z(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} \{\alpha x(t) + \beta y(t)\} e^{-st} dt \\ &= \alpha \int_{0^-}^{\infty} x(t) e^{-st} dt + \beta \int_{0^-}^{\infty} y(t) e^{-st} dt \\ &= \alpha X(s) + \beta Y(s) \end{aligned}$$

- In other words

$$\boxed{\alpha x(t) + \beta y(t) \overset{\mathcal{L}}{\longleftrightarrow} \alpha X(s) + \beta Y(s)}$$

Time Shifting

- Let $z(t) = x(t - t_o)$
- Then

$$\begin{aligned} Z(s) &= \int_{0^-}^{\infty} z(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} x(t - t_o) e^{-st} dt \\ &= \int_{0^-}^{\infty} x(\tau) e^{-s(\tau + t_o)} d\tau \\ &= e^{-st_o} \int_{0^-}^{\infty} x(\tau) e^{-s\tau} d\tau \\ &= e^{-st_o} X(s) \end{aligned}$$

let $\tau = t - t_o$

Note that this property only applies for $t_o > 0$. Why?

$$\boxed{x(t - t_o) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_o} X(s)}$$

Example

- We know from before that

$$u(t) \longleftrightarrow \frac{1}{s}$$

- Additionally, we have seen

$$g(t - t_0) \longleftrightarrow G(s) e^{-st_0}$$

- If we want to find the unilateral Laplace transform of *u*(t) – *u*(t-4)

$$u(t) - u(t-4) \longleftrightarrow \frac{1}{s} - \frac{1}{s} e^{-s(4)} = \frac{1 - e^{-4s}}{s}$$

Complex Frequency Shifting

- Let $z(t) = e^{s_o t} x(t)$
- Then

$$\begin{aligned} Z(s) &= \int_{0^-}^{\infty} z(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} e^{s_o t} x(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} x(t) e^{-(s-s_o)t} dt \\ &= X(s - s_o) \end{aligned}$$

$$e^{s_o t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_o)$$

Time Scaling

- Let $z(t) = x(at)$
- Then the Laplace Transform is

$$\begin{aligned} Z(s) &= \int_{0^-}^{\infty} z(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} x(at) e^{-st} dt \\ &= \frac{1}{a} \int_{0^-}^{\infty} x(\lambda) e^{-(s/a)\lambda} d\lambda \\ &= \frac{1}{a} X\left(\frac{s}{a}\right) \quad a > 0 \end{aligned}$$

Let λ=at

Note that this property only applies for $a > 0$. Why?

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right) \quad a > 0$$

Frequency Scaling

- From the last slide we have

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right) \quad a > 0$$

- Now let $b = 1/a$

$$x\left(\frac{t}{b}\right) \overset{L}{\longleftrightarrow} bX(bs) \quad b > 0$$

$$\frac{1}{b} x\left(\frac{t}{b}\right) \overset{L}{\longleftrightarrow} X(bs) \quad b > 0$$

Note that this property only applies for $a > 0$. Why?

$$\frac{1}{a} x\left(\frac{t}{a}\right) \overset{\mathcal{L}}{\longleftrightarrow} X(as) \quad a > 0$$

Convolution in Time

- Let $$z(t) = x(t) * y(t)$$
- Then
$$Z(s) = \int_{0^-}^{\infty} z(t) e^{-st} dt$$
$$= \int_{0^-}^{\infty} \{x(t) * y(t)\} e^{-st} dt$$
$$= \int_{0^-}^{\infty} \left\{ \int_{0^-}^{\infty} x(\tau) y(t-\tau) d\tau \right\} e^{-st} dt$$
- Changing the order of integration:
$$= \int_{0^-}^{\infty} x(\tau) \left\{ \int_{0^-}^{\infty} y(t-\tau) e^{-st} dt \right\} d\tau$$

Convolution (cont.)

$$Z(s) = \int_{0^-}^{\infty} x(\tau) \left\{ \int_{0^-}^{\infty} y(t-\tau) e^{-st} dt \right\} d\tau$$

- Since y(t) = 0 for t < 0 we have
$$Z(s) = \int_{0^-}^{\infty} x(\tau) \left\{ \int_{\tau^-}^{\infty} y(t-\tau) e^{-st} dt \right\} d\tau$$
- Now, letting $\lambda = t - \tau$ and $d\lambda = dt$

$$Z(s) = \int_{0^-}^{\infty} x(\tau) \left\{ \int_{0^-}^{\infty} y(\lambda) e^{-s(\lambda+\tau)} d\lambda \right\} d\tau$$
$$= \int_{0^-}^{\infty} x(\tau) e^{-s\tau} \underbrace{\left\{ \int_{0^-}^{\infty} y(\lambda) e^{-s\lambda} d\lambda \right\}}_{Y(s)} d\tau$$

Convolution (cont.)

- Continuing
$$Z(s) = \int_{0^-}^{\infty} x(\tau) e^{-s\tau} \underbrace{\left\{ \int_{0^-}^{\infty} y(\lambda) e^{-s\lambda} d\lambda \right\}}_{Y(s)} d\tau$$
$$= Y(s) \int_{0^-}^{\infty} x(\tau) e^{-s\tau} d\tau$$
$$= Y(s) X(s)$$

$$\boxed{x(t) * y(t) \xleftrightarrow{\mathcal{L}} X(s) Y(s)}$$

Multiplication in Time

- Now let $z(t) = x(t)\,y(t)$

$$
\begin{aligned}
Z(s) &= \int_{0^-}^{\infty} z(t) e^{-st} dt \\
&= \int_{0^-}^{\infty} x(t) y(t) e^{-st} dt \\
&= \int_{0^-}^{\infty} x(t) \left\{ \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} Y(\lambda) e^{\lambda t} d\lambda \right\} e^{-st} dt \\
&= \int_{\sigma-j\infty}^{\sigma+j\infty} Y(\lambda) \underbrace{\left\{ \int_{0^-}^{\infty} x(t) e^{-(s-\lambda)t} dt \right\}}_{X(s-\lambda)} d\lambda
\end{aligned}
$$

- Changing the order of integration

Note we must choose s such that X(s) and Y(s) exist.

Multiplication (cont.)

- Continuing …

$$
\begin{aligned}
Z(s) &= \int_{\sigma-j\infty}^{\sigma+j\infty} Y(\lambda) \underbrace{\left\{ \int_{0^-}^{\infty} x(t) e^{-(s-\lambda)t} dt \right\}}_{X(s-\lambda)} d\lambda \\
&= \int_{\sigma-j\infty}^{\sigma+j\infty} Y(\lambda) X(s-\lambda) d\lambda
\end{aligned}
$$

$$
x(t)\,y(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_{\sigma-j\infty}^{\sigma+j\infty} Y(\lambda) X(s-\lambda) d\lambda
$$

- Which is almost an aperiodic convolution but not exactly

Time Differentiation

- From the definition of the Laplace Transform we have

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

- Now let us evaluate this integral by parts specifically let

$$
\begin{aligned}
u &= x(t) & dv &= e^{-st} dt \\
du &= \frac{d}{dt}\{x(t)\} dt & v &= -\frac{1}{s} e^{-st}
\end{aligned}
$$

- Then, $\displaystyle \int_{0^-}^{\infty} x(t) e^{-st} dt = x(t)\left(-\frac{1}{s} e^{-st}\right)\Bigg|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} \frac{d}{dt}\{x(t)\} e^{-st} dt$

Time Differentiation – cont.

- Continuing..

$$\int_{0^-}^{\infty} x(t)e^{-st}dt = x(t)\left(-\frac{1}{s}e^{-st}\right)\Big|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty}\frac{d}{dt}\{x(t)\}e^{-st}dt$$

$$X(s) = \frac{1}{s}x(0^-) + \frac{1}{s}\int_{0^-}^{\infty}\frac{d}{dt}\{x(t)\}e^{-st}dt$$

$$\int_{0^-}^{\infty}\frac{d}{dt}\{x(t)\}e^{-st}dt = sX(s) + x(0^-)$$

$$L\left\{\frac{d}{dt}\{x(t)\}\right\} = sX(s) + x(0^-)$$

$$\boxed{\frac{d}{dt}\{x(t)\} \overset{\mathcal{L}}{\longleftrightarrow} sX(s) - x(0^-)}$$

Time Differentiation – cont.

- The second derivative of *x(t)* is

$$\frac{d^2}{dt^2}\{x(t)\} = \frac{d}{dt}\left\{\frac{d}{dt}\{x(t)\}\right\}$$

- The Laplace Transform is then

$$\begin{aligned}\mathcal{L}\left\{\frac{d^2}{dt^2}\{x(t)\}\right\} &= s\mathcal{L}\left\{\frac{d}{dt}\{x(t)\}\right\} - \frac{d}{dt}\{x(t)\}\Big|_{t=0^-}\\ &= s\left\{sX(s) - x(0^-)\right\} - \frac{d}{dt}\{x(t)\}\Big|_{t=0^-}\\ &= s^2X(s) - sx(0^-) - \frac{d}{dt}\{x(t)\}\Big|_{t=0^-}\end{aligned}$$

$$\boxed{\frac{d^2}{dt^2}\{x(t)\} \overset{\mathcal{L}}{\longleftrightarrow} s^2X(s) - sx(0^-) - \frac{d}{dt}\{x(t)\}\Big|_{t=0^-}}$$

Complex Frequency Differentiation

- From the definition of the Laplace Transform

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

- Differentiating with respect to *s*

$$\begin{aligned}\frac{d}{ds}\{X(s)\} &= \frac{d}{ds}\left\{\int_{0^-}^{\infty} x(t)e^{-st}dt\right\}\\ &= \int_{0^-}^{\infty}\frac{d}{ds}\{x(t)e^{-st}\}dt\\ &= \int_{0^-}^{\infty}(-t)x(t)e^{-st}dt\\ &= \mathcal{L}\{-tx(t)\}\end{aligned}$$

$$\boxed{-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}\{X(s)\}}$$

Integration

- Integration of a function *x(t)* can be written as

$$\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$$

- From the convolution property and the Laplace Transform of the unit step we have

$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \frac{X(s)}{s}}$$

Initial Value Theorem

- Using the time differentiation property of the Laplace Transform

$$\int_{0^-}^{\infty} \frac{d}{dt}\{x(t)\} e^{-st} dt = sX(s) - x(0^-)$$

- Taking the limit as *s* approaches infinity

$$\lim_{s\to\infty}\left[\int_{0^-}^{\infty} \frac{d}{dt}\{x(t)\} e^{-st} dt\right] = \lim_{s\to\infty}\left[sX(s) - x(0^-)\right]$$

$$\int_{0^-}^{\infty} \lim_{s\to\infty}\left[\frac{d}{dt}\{x(t)\} e^{-st}\right] dt = \lim_{s\to\infty}\left[sX(s) - x(0^-)\right]$$

Initial Value Theorem – cont.

- Case 1: x(t) is continuous at *t = 0*

$$0 = \lim_{s\to\infty}\left[sX(s) - x(0^-)\right]$$

$$x(0^-) = \lim_{s\to\infty}\left[sX(s)\right]$$

- Case 2: x(t) is discontinuous at *t=0*

$$\lim_{s\to\infty}\left[\int_{0^-}^{\infty} \frac{d}{dt}\{x(t)\} e^{-st} dt\right] = \lim_{s\to\infty}\left[\int_{0^-}^{0^+} \left[g(0^+) - g(0^-)\right]\delta(t) e^{-st} dt\right] + \lim_{s\to\infty}\left[\int_{0^+}^{\infty} \frac{d}{dt}\{x(t)\} e^{-st} dt\right]$$

$$= \lim_{s\to\infty}\left[\int_{0^-}^{0^+} \left[g(0^+) - g(0^-)\right]\delta(t) e^{-st} dt\right]$$

$$\lim_{s\to\infty}\left[\int_{0^-}^{0^+} \left[x(0^+) - x(0^-)\right]\delta(t) e^{-st} dt\right] = \lim_{s\to\infty}\left[sX(s) - x(0^-)\right]$$

$$x(0^+) - x(0^-) = \lim_{s\to\infty}\left[sX(s) - x(0^-)\right]$$

$$x(0^+) = \lim_{s\to\infty}\left[sX(s)\right]$$

Final Value Theorem

- Again using the differentiation property

$$\lim_{s\to 0}\left[\int_{0^-}^{\infty}\frac{d}{dt}\{x(t)\}e^{-st}dt\right]=\lim_{s\to 0}\left[sX(s)-x(0^-)\right]$$

$$\int_{0^-}^{\infty}\lim_{s\to 0}\left[\frac{d}{dt}\{x(t)\}e^{-st}\right]dt=\lim_{s\to 0}\left[sX(s)-x(0^-)\right]$$

$$\int_{0^-}^{\infty}\frac{d}{dt}\{x(t)\}dt=\lim_{s\to 0}\left[sX(s)-x(0^-)\right]$$

$$\lim_{t\to\infty}\left[x(t)-x(0^-)\right]=\lim_{s\to 0}\left[sX(s)-x(0^-)\right]$$

$$\lim_{t\to\infty}\left[x(t)\right]=\lim_{s\to 0}\left[sX(s)\right]$$

Summary

- In this lecture we have examined several properties of the Laplace Transform
- The properties are very similar to those for the Fourier Series and Fourier Transform
  - Some notable and important differences
- We will find these properties very useful in determining the Laplace Transform of arbitrary signals.
  - Using a simple table of Laplace Transforms and LT properties, we can determine the LT of most signals of interest.
  - Note that we focused on the *unilateral* Laplace Transform