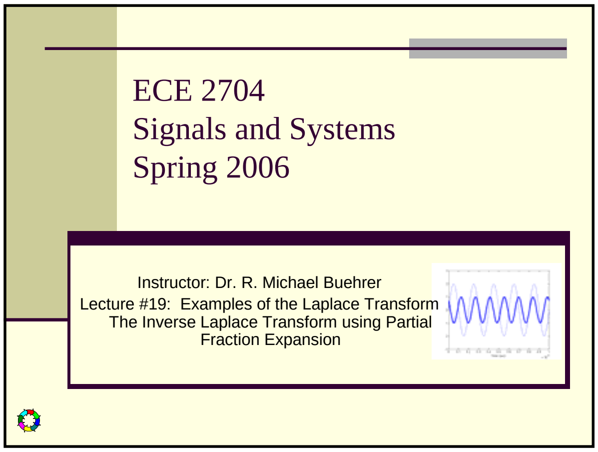
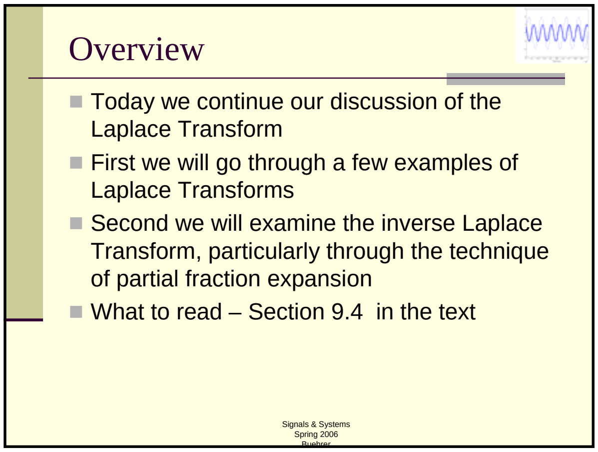


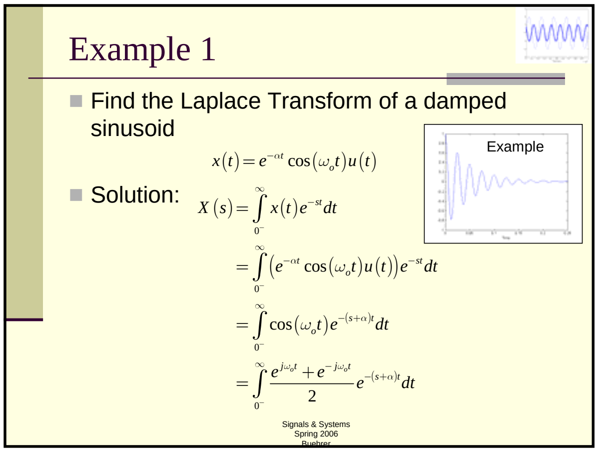
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer
Lecture #19: Examples of the Laplace Transform
The Inverse Laplace Transform using Partial Fraction Expansion



Overview

- Today we continue our discussion of the Laplace Transform
- First we will go through a few examples of Laplace Transforms
- Second we will examine the inverse Laplace Transform, particularly through the technique of partial fraction expansion
- What to read – Section 9.4 in the text



Example 1

- Find the Laplace Transform of a damped sinusoid

$$x(t) = e^{-\alpha t}\cos(\omega_o t)u(t)$$

- Solution:

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$= \int_{0^-}^{\infty} \left(e^{-\alpha t}\cos(\omega_o t)u(t)\right)e^{-st}dt$$

$$= \int_{0^-}^{\infty} \cos(\omega_o t)e^{-(s+\alpha)t}dt$$

$$= \int_{0^-}^{\infty} \frac{e^{j\omega_o t}+e^{-j\omega_o t}}{2}e^{-(s+\alpha)t}dt$$

Example 1 – cont.

- Continuing…

$$X(s)=\int_{0^-}^{\infty}\frac{e^{j\omega_o t}+e^{-j\omega_o t}}{2}e^{-(s+\alpha)t}dt$$

$$=\frac{1}{2}\int_{0^-}^{\infty}e^{(j\omega_o-s-\alpha)t}dt+\frac{1}{2}\int_{0^-}^{\infty}e^{-(j\omega_o+s+\alpha)t}dt$$

$$=\frac{1}{2}\left[\frac{1}{j\omega_o-s-\alpha}e^{(j\omega_o-s-\alpha)t}-\frac{1}{j\omega_o+s+\alpha}e^{-(j\omega_o+s+\alpha)t}\right]\Bigg|_{0^-}^{\infty}$$

$$=\frac{1}{2}\left[-\frac{1}{j\omega_o-s-\alpha}+\frac{1}{j\omega_o+s+\alpha}\right]\quad \sigma>-\alpha$$

Example 1 – cont.

- Continuing…

$$X(s)=\frac{1}{2}\left[-\frac{1}{j\omega_o-s-\alpha}+\frac{1}{j\omega_o+s+\alpha}\right]\quad \sigma>-\alpha$$

$$=\frac{1}{2}\left[\frac{1}{j\omega_o+(s+\alpha)}-\frac{1}{j\omega_o-(s+\alpha)}\right]\quad \sigma>-\alpha$$

$$=\frac{1}{2}\left[\frac{(s+\alpha)-j\omega_o}{(s+\alpha)^2+\omega_o^2}+\frac{(s+\alpha)+j\omega_o}{(s+\alpha)^2+\omega_o^2}\right]\quad \sigma>-\alpha$$

$$=\frac{(s+\alpha)}{(s+\alpha)^2+\omega_o^2}\quad \sigma>-\alpha$$

Example 2

- Fine the Laplace Transform of a damped sinusoid

$$x(t)=e^{-\alpha t}\sin(\omega_o t)u(t)$$

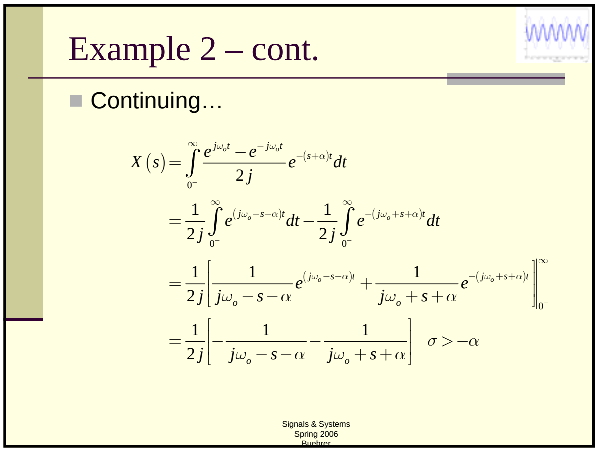
- Solution:

$$X(s)=\int_{0^-}^{\infty}x(t)e^{-st}dt$$

$$=\int_{0^-}^{\infty}\left(e^{-\alpha t}\sin(\omega_o t)u(t)\right)e^{-st}dt$$

$$=\int_{0^-}^{\infty}\sin(\omega_o t)e^{-(s+\alpha)t}dt$$

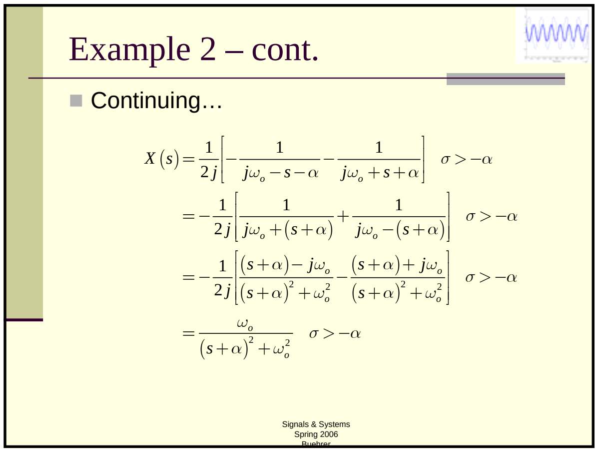
$$=\int_{0^-}^{\infty}\frac{e^{j\omega_o t}-e^{-j\omega_o t}}{2j}e^{-(s+\alpha)t}dt$$



Example 2 – cont.

- Continuing…

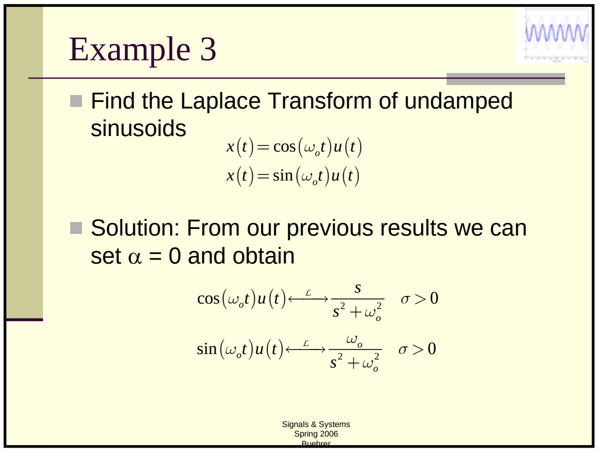
$$X(s)=\int_{0^-}^{\infty}\frac{e^{j\omega_o t}-e^{-j\omega_o t}}{2j}e^{-(s+\alpha)t}dt$$
$$=\frac{1}{2j}\int_{0^-}^{\infty}e^{(j\omega_o-s-\alpha)t}dt-\frac{1}{2j}\int_{0^-}^{\infty}e^{-(j\omega_o+s+\alpha)t}dt$$
$$=\frac{1}{2j}\left[\frac{1}{j\omega_o-s-\alpha}e^{(j\omega_o-s-\alpha)t}+\frac{1}{j\omega_o+s+\alpha}e^{-(j\omega_o+s+\alpha)t}\right]\Bigg|_{0^-}^{\infty}$$
$$=\frac{1}{2j}\left[-\frac{1}{j\omega_o-s-\alpha}-\frac{1}{j\omega_o+s+\alpha}\right]\quad\sigma>-\alpha$$



Example 2 – cont.

- Continuing…

$$X(s)=\frac{1}{2j}\left[-\frac{1}{j\omega_o-s-\alpha}-\frac{1}{j\omega_o+s+\alpha}\right]\quad\sigma>-\alpha$$
$$=-\frac{1}{2j}\left[\frac{1}{j\omega_o+(s+\alpha)}+\frac{1}{j\omega_o-(s+\alpha)}\right]\quad\sigma>-\alpha$$
$$=-\frac{1}{2j}\left[\frac{(s+\alpha)-j\omega_o}{(s+\alpha)^2+\omega_o^2}-\frac{(s+\alpha)+j\omega_o}{(s+\alpha)^2+\omega_o^2}\right]\quad\sigma>-\alpha$$
$$=\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}\quad\sigma>-\alpha$$



Example 3

- Find the Laplace Transform of undamped sinusoids

$$x(t)=\cos(\omega_o t)u(t)$$
$$x(t)=\sin(\omega_o t)u(t)$$

- Solution: From our previous results we can set $\alpha = 0$ and obtain

$$\cos(\omega_o t)u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{s}{s^2+\omega_o^2}\quad\sigma>0$$
$$\sin(\omega_o t)u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{\omega_o}{s^2+\omega_o^2}\quad\sigma>0$$

Example 4

- Find the Laplace Transform of the function

$$x(t)=tu(t)$$

- Solution:

$$X(s)=\int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$=\int_{0^-}^{\infty} tu(t)e^{-st}dt$$

$$=\int_{0^-}^{\infty} te^{-st}dt$$

$$=\left.\left\{\frac{e^{-st}}{s^2}[st-1]\right\}\right|_{0^-}^{\infty}$$

Example 4 – cont.

- Continuing …

$$X(s)=\left.\left\{\frac{e^{-st}}{s^2}[st-1]\right\}\right|_{0^-}^{\infty}$$

$$=0-\frac{-1}{s^2}$$

$$=\frac{1}{s^2}$$

$$\boxed{tu(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s^2}}$$

Example 5

- Find the Laplace Transform of

$$x(t)=te^{-at}u(t)$$

- Solution

$$X(s)=\int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$=\int_{0^-}^{\infty} te^{-at}u(t)e^{-st}dt$$

$$=\int_{0^-}^{\infty} te^{-(s+a)t}dt$$

$$=\left.\left\{\frac{e^{-(s+a)t}}{(s+a)^2}[-(s+a)t-1]\right\}\right|_{0^-}^{\infty}$$

$\boxed{\sigma>a}$

$$=\frac{1}{(s+a)^2}$$

Laplace Transform Pairs

- From the last few lectures we have developed the following Laplace Transform Pairs:

$$e^{-\alpha t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \quad \sigma > -\alpha$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \sigma > 0$$

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1$$

$$e^{-\alpha t}\cos(\omega_o t)u(t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2} \quad \sigma > -\alpha$$

$$e^{-\alpha t}\sin(\omega_o t)u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_o}{(s+\alpha)^2+\omega_o^2} \quad \sigma > -\alpha$$

$$\cos(\omega_o t)u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+\omega_o^2} \quad \sigma > 0$$

$$\sin(\omega_o t)u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_o}{s^2+\omega_o^2} \quad \sigma > 0$$

$$te^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2} \quad \sigma > a$$

Inverse Laplace Transform

- The most convenient way to find the inverse Laplace transform of an arbitrary function is to attempt to put the function into a sum of terms that exist in the preceding table and then find the inverse LT via linearity and inspection
- Example: Find the inverse Laplace Transform of

$$X(s) = \frac{10s}{(s+1)(s+3)}$$

- Using the table on the previous page doesn't help us here. However, we can rewrite the expression as

$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

Inverse Laplace Transforms – cont.

- The unknown terms in the expression

$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

can be found as

$$X(s) = \frac{15}{(s+3)} - \frac{5}{(s+1)}$$

- Using our table of Laplace Transform pairs we have

$$x(t) = 15e^{-3t}u(t) - 5e^{-t}u(t)$$

General Inverse Laplace Transforms

- The general procedure can be used to determine the inverse Laplace Transform of

$$X(s)=\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{s^D+a_{D-1}s^{D-1}+a_{D-2}s^{D-2}\ldots+a_1 s+a_0}$$

- It is in principle possible to factor the denominator such that

$$X(s)=\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{(s-p_1)(s-p_2)(s-p_3)\ldots(s-p_D)}$$

where p_i are the poles of *X(s)*.

- If there are no repeated poles and $D > N$ we write

$$X(s)=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\ldots\frac{K_D}{s-p_D}$$

Determining the Coefficients

$$\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{(s-p_1)(s-p_2)(s-p_3)\ldots(s-p_D)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\ldots\frac{K_D}{s-p_D}$$

- The coefficients K_i can be found by multiplying both expressions $(s\text{-}p_i)$ and setting $s=p_i$

$$\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{(s-p_1)(s-p_2)(s-p_3)\ldots(s-p_D)}(s-p_1)=\frac{K_1}{s-p_1}(s-p_1)+\frac{K_2}{s-p_2}(s-p_1)+\ldots\frac{K_D}{s-p_D}(s-p_1)$$

$$\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{(s-p_2)(s-p_3)\ldots(s-p_D)}=K_1+\frac{K_2}{s-p_2}(s-p_1)+\ldots\frac{K_D}{s-p_D}(s-p_1)$$

$$K_1=\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{(s-p_2)(s-p_3)\ldots(s-p_D)}-\frac{K_2}{s-p_2}(s-p_1)-\ldots\frac{K_D}{s-p_D}(s-p_1)$$

$$K_1=\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{(s-p_2)(s-p_3)\ldots(s-p_D)}-\frac{K_2}{s-p_2}(s-p_1)-\ldots\frac{K_D}{s-p_D}(s-p_1)\bigg|_{s=p_1}$$

$$=\frac{b_N s^N+b_{N-1}s^{N-1}+b_{N-2}s^{N-2}\ldots+b_1 s+b_0}{(s-p_2)(s-p_3)\ldots(s-p_D)}\bigg|_{s=p_1}$$

Example

- Find the inverse transform of

$$X(s)=\frac{6}{(s+1)(s+3)}$$

$$=\frac{K_1}{(s+1)}+\frac{K_2}{(s+3)}$$

- We can find K_1 as

$$K_1=(s+1)X(s)\big|_{s=-1}$$

$$=\frac{6}{(s+3)}\bigg|_{s=-1}$$

$$=3$$

- and K_2 as

$$K_2=(s+3)X(s)\big|_{s=-3}$$

$$=\frac{6}{(s+1)}\bigg|_{s=-3}$$

$$=-2$$

Example – cont.

- Substituting for K_1 and K_2

$$X(s)=\frac{K_1}{(s+1)}+\frac{K_2}{(s+3)}$$
$$=\frac{3}{(s+1)}-\frac{2}{(s+3)}$$

- Now the inverse transform can be found using our table as

$$x(t)=3e^{-t}u(t)-2e^{-3t}u(t)$$

Dealing with Multiple Poles

- Consider the Laplace Transform

$$H(s)=\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}$$

- We would like to use partial fraction expansion to find the inverse Laplace Transform. How do we deal with the fact that there are two poles at $s=p_1$?

Attempt 1 Let's use the same approach as before

$$H(s)=\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}=\frac{A}{(s-p_1)}+\frac{B}{(s-p_1)}+\frac{C}{(s-p_2)}$$

However, due to the common denominator we can combined the two terms

$$\frac{A}{(s-p_1)}+\frac{B}{(s-p_1)}+\frac{C}{(s-p_2)}=\frac{A+B}{(s-p_1)}+\frac{C}{(s-p_2)}$$
$$=\frac{D}{(s-p_1)}+\frac{C}{(s-p_2)}$$

Dealing with Multiple Poles – cont.

- But $\frac{D}{(s-p_1)}+\frac{C}{(s-p_2)}=\frac{D(s-p_2)+C(s-p_1)}{(s-p_1)(s-p_2)}\neq\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}$

Attempt 2

$$H(s)=\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}=\frac{A}{(s-p_1)^2}+\frac{B}{(s-p_2)}$$

However, if we attempt to solve for A and B

$$\frac{A}{(s-p_1)^2}+\frac{B}{(s-p_2)}=\frac{A(s-p_2)+B(s-p_1)^2}{(s-p_1)^2(s-p_2)}$$
$$=\frac{Bs^2+(A-2Bp_1)s+(-Ap_2+Bp_1^2)}{(s-p_1)^2(s-p_2)}$$

This gives us 3 equations and 2 unknowns and there is no unique solution. In general doing this results in D equations and $D-1$ unknowns.

Dealing with mulitple poles – cont.

Attempt 3

$$H(s)=\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}=\frac{A}{(s-p_1)^2}+\frac{B}{(s-p_1)}+\frac{C}{(s-p_2)}$$

This will result in D equations and D unknowns which can be solved for a unique solution. We can find A and C using the same technique as before. Specifically, to find *A* let's multiply through by *(s-p_1)*^{2} :

$$\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}(s-p_1)^2=\frac{A(s-p_1)^2}{(s-p_1)^2}+\frac{B(s-p_1)^2}{(s-p_1)}+\frac{C(s-p_1)^2}{(s-p_2)}$$

$$\frac{b_2s^2+b_1s+b_0}{(s-p_2)}=A+B(s-p_1)+\frac{C(s-p_1)^2}{(s-p_2)}$$

Now, we let s = p_1

$$\left.\frac{b_2s^2+b_1s+b_0}{(s-p_2)}\right|_{s=p_1}=A+0+0$$

$$A=\left.\frac{b_2s^2+b_1s+b_0}{(s-p_2)}\right|_{s=p_1}$$

Dealing with multiple poles – cont.

- Doing the same for *C*:

$$C=\left.\left[(s-p_2)H(s)\right]\right|_{s=p_2}$$

$$=\left.\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2}\right|_{s=p_2}$$

- However, when we attempt to do the same procedure for finding *B*

$$\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}(s-p_1)=\frac{A(s-p_1)}{(s-p_1)^2}+\frac{B(s-p_1)}{(s-p_1)}+\frac{C(s-p_1)}{(s-p_2)}$$

$$\frac{b_2s^2+b_1s+b_0}{(s-p_1)(s-p_2)}=\frac{A}{(s-p_1)}+B+\frac{C(s-p_1)}{(s-p_2)}$$

- Now if we let *s=p_1* we see that the equations have a divide by zero problem.

Dealing with multiple poles – cont.

- We can avoid this by instead multiplying by (s-p_1)^2:

$$\frac{b_2s^2+b_1s+b_0}{(s-p_1)^2(s-p_2)}(s-p_1)^2=\frac{A(s-p_1)^2}{(s-p_1)^2}+\frac{B(s-p_1)^2}{(s-p_1)}+\frac{C(s-p_1)^2}{(s-p_2)}$$

$$\frac{b_2s^2+b_1s+b_0}{(s-p_2)}=A+B(s-p_1)+\frac{C(s-p_1)^2}{(s-p_2)}$$

- Now, if we take the derivative of both sides with respect to *s:*

$$\frac{d}{ds}\left\{\frac{b_2s^2+b_1s+b_0}{(s-p_2)}\right\}=B+\frac{d}{ds}\left\{\frac{C(s-p_1)^2}{(s-p_2)}\right\}$$

$$=B+C\left\{\frac{(s-p_2)2(s-p_1)-(s-p_1)^2}{(s-p_2)^2}\right\}$$

- Finally, we set s = p_1:

$$B=\left.\left[\frac{d}{ds}\left\{\frac{b_2s^2+b_1s+b_0}{(s-p_2)}\right\}\right]\right|_{s=p_1}$$

Dealing with multiple poles - cont.

- Thus, in general we can find the solution when there are two poles by including the term

$$\frac{A}{(s-p_1)^2}+\frac{B}{(s-p_1)}$$

in the partial fraction expansion. We can find *B* by taking the derivative of both sides with respect to *s* and setting $s=p_1$.

- Once we have solved for the coefficients we have

$$H(s)=\frac{A}{(s-p_1)^2}+\frac{B}{(s-p_1)}+\frac{C}{(s-p_2)}$$

- We can then directly find the inverse Laplace Transform as

$$h(t)=\left(Ate^{p_1 t}+Be^{p_1 t}+Ce^{p_2 t}\right)u(t)$$

Example

- Find the inverse Laplace Transform of

$$H(s)=\frac{s+5}{s^2(s+2)}$$

- Solution:

There is a repeated root at *s=0.* Thus we can write the Laplace Transform as

$$H(s)=\frac{s+5}{s^2(s+2)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{(s+2)}$$

where

$$A=\left[s^2H(s)\right]\Big|_{s=0}=\left[\frac{s+5}{s^2(s+2)}s^2\right]_{s=0}$$

$$=\left[\frac{s+5}{(s+2)}\right]_{s=0}=\frac{5}{2}$$

Example – cont.

- The third coefficient can be found as

$$C=\left[(s+2)H(s)\right]\Big|_{s=-2}=\left[\frac{s+5}{s^2(s+2)}(s+2)\right]_{s=-2}$$

$$=\left[\frac{s+5}{s^2}\right]_{s=-2}=\frac{3}{4}$$

- To find the second coefficient

$$A=\left[\frac{d}{ds}\left[s^2H(s)\right]\right]_{s=0}=\left[\frac{d}{ds}\left[\frac{s+5}{s^2(s+2)}s^2\right]\right]_{s=0}$$

$$=\left[\frac{d}{ds}\left[\frac{s+5}{(s+2)}\right]\right]_{s=0}=\frac{(s+2)-(s+5)}{(s+2)^2}\Bigg|_{s=0}=-\frac{3}{4}$$

- Thus we have

$$H(s)=\frac{5}{2}\frac{1}{s^2}-\frac{3}{4}\frac{1}{s}+\frac{3}{4}\frac{1}{(s+2)}$$

Example – cont.

- The inverse transform of

$$H(s)=\frac{5}{2}\frac{1}{s^2}-\frac{3}{4}\frac{1}{s}+\frac{3}{4}\frac{1}{(s+2)}$$

can be found using our tables as

$$h(t)=\frac{5}{2}tu(t)-\frac{3}{4}u(t)+\frac{3}{4}e^{-2t}u(t)$$

Summary

- In this lecture we have gone through several examples of the Laplace Transform
- Additionally, we introduced the most common technique for determining the inverse Laplace Transform using the technique of *partial fraction expansion*.
- Finally we also looked at a few examples of the inverse Laplace Transform using this technique.