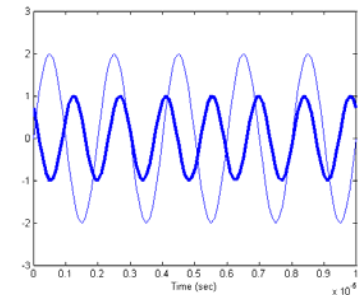


# ECE 2704

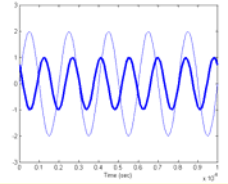
## Signals and Systems

### Spring 2006

Instructor: Dr. R. Michael Buehrer  
Lecture #19: Examples of the Laplace Transform  
The Inverse Laplace Transform using Partial  
Fraction Expansion

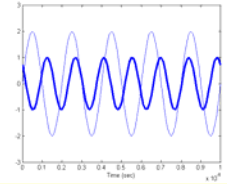


# Overview



- Today we continue our discussion of the Laplace Transform
- First we will go through a few examples of Laplace Transforms
- Second we will examine the inverse Laplace Transform, particularly through the technique of partial fraction expansion
- What to read – Section 9.4 in the text

# Example 1



- Find the Laplace Transform of a damped sinusoid

$$x(t) = e^{-\alpha t} \cos(\omega_o t) u(t)$$

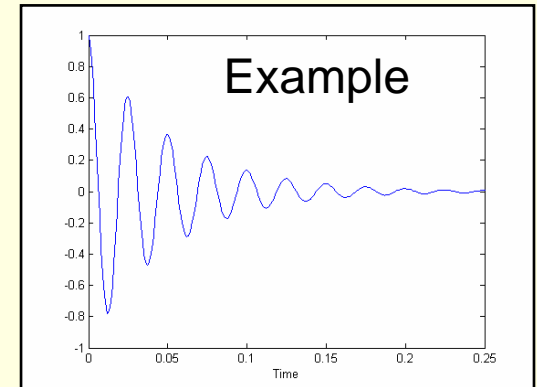
- Solution:

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

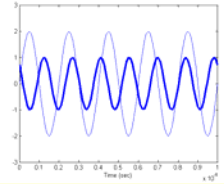
$$= \int_{0^-}^{\infty} (e^{-\alpha t} \cos(\omega_o t) u(t)) e^{-st} dt$$

$$= \int_{0^-}^{\infty} \cos(\omega_o t) e^{-(s+\alpha)t} dt$$

$$= \int_{0^-}^{\infty} \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} e^{-(s+\alpha)t} dt$$



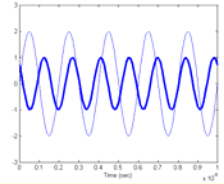
# Example 1 – cont.



## ■ Continuing...

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} e^{-(s+\alpha)t} dt \\ &= \frac{1}{2} \int_{0^-}^{\infty} e^{(j\omega_o - s - \alpha)t} dt + \frac{1}{2} \int_{0^-}^{\infty} e^{-(j\omega_o + s + \alpha)t} dt \\ &= \frac{1}{2} \left[ \frac{1}{j\omega_o - s - \alpha} e^{(j\omega_o - s - \alpha)t} - \frac{1}{j\omega_o + s + \alpha} e^{-(j\omega_o + s + \alpha)t} \right] \Bigg|_{0^-}^{\infty} \\ &= \frac{1}{2} \left[ -\frac{1}{j\omega_o - s - \alpha} + \frac{1}{j\omega_o + s + \alpha} \right] \quad \sigma > -\alpha \end{aligned}$$

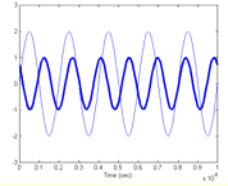
# Example 1 – cont.



## ■ Continuing...

$$\begin{aligned} X(s) &= \frac{1}{2} \left[ -\frac{1}{j\omega_o - s - \alpha} + \frac{1}{j\omega_o + s + \alpha} \right] \quad \sigma > -\alpha \\ &= \frac{1}{2} \left[ \frac{1}{j\omega_o + (s + \alpha)} - \frac{1}{j\omega_o - (s + \alpha)} \right] \quad \sigma > -\alpha \\ &= \frac{1}{2} \left[ \frac{(s + \alpha) - j\omega_o}{(s + \alpha)^2 + \omega_o^2} + \frac{(s + \alpha) + j\omega_o}{(s + \alpha)^2 + \omega_o^2} \right] \quad \sigma > -\alpha \\ &= \frac{(s + \alpha)}{(s + \alpha)^2 + \omega_o^2} \quad \sigma > -\alpha \end{aligned}$$

# Example 2



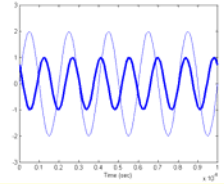
- Find the Laplace Transform of a damped sinusoid

$$x(t) = e^{-\alpha t} \sin(\omega_o t) u(t)$$

- Solution:

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} x(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} (e^{-\alpha t} \sin(\omega_o t) u(t)) e^{-st} dt \\ &= \int_{0^-}^{\infty} \sin(\omega_o t) e^{-(s+\alpha)t} dt \\ &= \int_{0^-}^{\infty} \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j} e^{-(s+\alpha)t} dt \end{aligned}$$

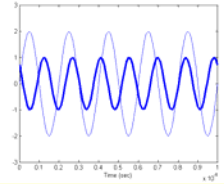
# Example 2 – cont.



## ■ Continuing...

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j} e^{-(s+\alpha)t} dt \\ &= \frac{1}{2j} \int_{0^-}^{\infty} e^{(j\omega_o - s - \alpha)t} dt - \frac{1}{2j} \int_{0^-}^{\infty} e^{-(j\omega_o + s + \alpha)t} dt \\ &= \frac{1}{2j} \left[ \frac{1}{j\omega_o - s - \alpha} e^{(j\omega_o - s - \alpha)t} + \frac{1}{j\omega_o + s + \alpha} e^{-(j\omega_o + s + \alpha)t} \right] \Bigg|_{0^-}^{\infty} \\ &= \frac{1}{2j} \left[ -\frac{1}{j\omega_o - s - \alpha} - \frac{1}{j\omega_o + s + \alpha} \right] \quad \sigma > -\alpha \end{aligned}$$

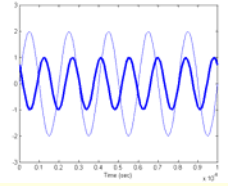
# Example 2 – cont.



## ■ Continuing...

$$\begin{aligned} X(s) &= \frac{1}{2j} \left[ \frac{1}{j\omega_o - s - \alpha} - \frac{1}{j\omega_o + s + \alpha} \right] \quad \sigma > -\alpha \\ &= -\frac{1}{2j} \left[ \frac{1}{j\omega_o + (s + \alpha)} + \frac{1}{j\omega_o - (s + \alpha)} \right] \quad \sigma > -\alpha \\ &= -\frac{1}{2j} \left[ \frac{(s + \alpha) - j\omega_o}{(s + \alpha)^2 + \omega_o^2} - \frac{(s + \alpha) + j\omega_o}{(s + \alpha)^2 + \omega_o^2} \right] \quad \sigma > -\alpha \\ &= \frac{\omega_o}{(s + \alpha)^2 + \omega_o^2} \quad \sigma > -\alpha \end{aligned}$$

# Example 3



- Find the Laplace Transform of undamped sinusoids

$$x(t) = \cos(\omega_o t)u(t)$$

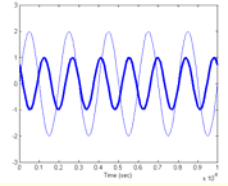
$$x(t) = \sin(\omega_o t)u(t)$$

- Solution: From our previous results we can set  $\alpha = 0$  and obtain

$$\cos(\omega_o t)u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + \omega_o^2} \quad \sigma > 0$$

$$\sin(\omega_o t)u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_o}{s^2 + \omega_o^2} \quad \sigma > 0$$

# Example 4

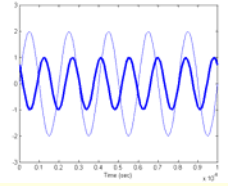


- Find the Laplace Transform of the function

$$x(t) = tu(t)$$

- Solution: 
$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} x(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} tu(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} te^{-st} dt \\ &= \left\{ \frac{e^{-st}}{s^2} [st - 1] \right\} \Big|_{0^-}^{\infty} \end{aligned}$$

# Example 4 – cont.

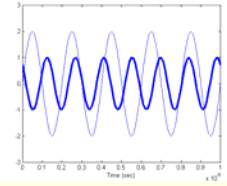


## ■ Continuing ...

$$\begin{aligned} X(s) &= \left. \left\{ \frac{e^{-st}}{s^2} [st - 1] \right\} \right|_{0^-}^{\infty} \\ &= 0 - \frac{-1}{s^2} \\ &= \frac{1}{s^2} \end{aligned}$$

$$\boxed{tu(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}}$$

# Example 5



- Find the Laplace Transform of

$$x(t) = te^{-at}u(t)$$

- Solution

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st} dt$$

$$= \int_{0^-}^{\infty} te^{-at}u(t)e^{-st} dt$$

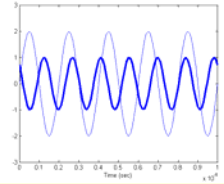
$$= \int_{0^-}^{\infty} te^{-(s+a)t} dt$$

$$= \left. \left\{ \frac{e^{-(s+a)t}}{(s+a)^2} [-(s+a)t - 1] \right\} \right|_{0^-}^{\infty}$$

$$\boxed{\sigma > a}$$

$$= \frac{1}{(s+a)^2}$$

# Laplace Transform Pairs



- From the last few lectures we have developed the following Laplace Transform Pairs:

$$e^{-\alpha t} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s + \alpha} \quad \sigma > -\alpha$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \sigma > 0$$

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1$$

$$e^{-\alpha t} \cos(\omega_o t) u(t) \xleftrightarrow{\mathcal{L}} \frac{s + \alpha}{(s + \alpha)^2 + \omega_o^2} \quad \sigma > -\alpha$$

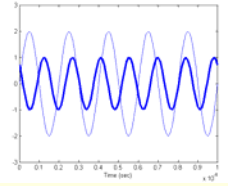
$$e^{-\alpha t} \sin(\omega_o t) u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_o}{(s + \alpha)^2 + \omega_o^2} \quad \sigma > -\alpha$$

$$\cos(\omega_o t) u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + \omega_o^2} \quad \sigma > 0$$

$$\sin(\omega_o t) u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_o}{s^2 + \omega_o^2} \quad \sigma > 0$$

$$te^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s + a)^2} \quad \sigma > a$$

# Inverse Laplace Transform



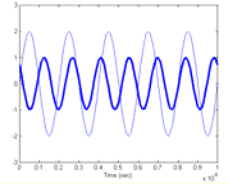
- The most convenient way to find the inverse Laplace transform of an arbitrary function is to attempt to put the function into a sum of terms that exist in the preceding table and then find the inverse LT via linearity and inspection
- Example: Find the inverse Laplace Transform of

$$X(s) = \frac{10s}{(s+1)(s+3)}$$

- Using the table on the previous page doesn't help us here. However, we can rewrite the expression as

$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

# Inverse Laplace Transforms – cont.



- The unknown terms in the expression

$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

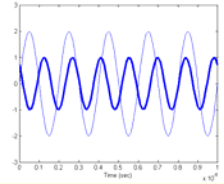
can be found as

$$X(s) = \frac{15}{(s+3)} - \frac{5}{(s+1)}$$

- Using our table of Laplace Transform pairs we have

$$x(t) = 15e^{-3t}u(t) - 5e^{-t}u(t)$$

# General Inverse Laplace Transforms



- The general procedure can be used to determine the inverse Laplace Transform of

$$X(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{s^D + a_{D-1} s^{D-1} + a_{D-2} s^{D-2} \dots + a_1 s + a_0}$$

- It is in principle possible to factor the denominator such that

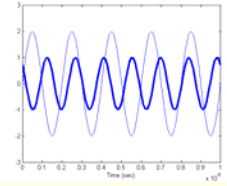
$$X(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{(s - p_1)(s - p_2)(s - p_3) \dots (s - p_D)}$$

where  $p_i$  are the poles of  $X(s)$ .

- If there are no repeated poles and  $D > N$  we write

$$X(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \dots + \frac{K_D}{s - p_D}$$

# Determining the Coefficients



$$\frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{(s - p_1)(s - p_2)(s - p_3) \dots (s - p_D)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \dots + \frac{K_D}{s - p_D}$$

- The coefficients  $K_i$  can be found by multiplying both expressions  $(s-p_i)$  and setting  $s=p_i$

$$\frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{(s - p_1)(s - p_2)(s - p_3) \dots (s - p_D)} (s - p_1) = \frac{K_1}{s - p_1} (s - p_1) + \frac{K_2}{s - p_2} (s - p_1) + \dots + \frac{K_D}{s - p_D} (s - p_1)$$

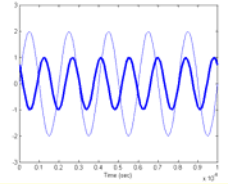
$$\frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{(s - p_2)(s - p_3) \dots (s - p_D)} = K_1 + \frac{K_2}{s - p_2} (s - p_1) + \dots + \frac{K_D}{s - p_D} (s - p_1)$$

$$K_1 = \frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{(s - p_2)(s - p_3) \dots (s - p_D)} - \frac{K_2}{s - p_2} (s - p_1) - \dots - \frac{K_D}{s - p_D} (s - p_1)$$

$$K_1 = \left. \frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{(s - p_2)(s - p_3) \dots (s - p_D)} - \frac{K_2}{s - p_2} (s - p_1) - \dots - \frac{K_D}{s - p_D} (s - p_1) \right|_{s=p_1}$$

$$= \left. \frac{b_N s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} \dots + b_1 s + b_0}{(s - p_2)(s - p_3) \dots (s - p_D)} \right|_{s=p_1}$$

# Example



- Find the inverse transform of

$$\begin{aligned} X(s) &= \frac{6}{(s+1)(s+3)} \\ &= \frac{K_1}{(s+1)} + \frac{K_2}{(s+3)} \end{aligned}$$

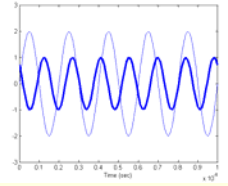
- We can find  $K_1$  as

$$\begin{aligned} K_1 &= (s+1)X(s)\Big|_{s=-1} \\ &= \frac{6}{(s+3)}\Big|_{s=-1} \\ &= 3 \end{aligned}$$

- and  $K_2$  as

$$\begin{aligned} K_2 &= (s+3)X(s)\Big|_{s=-3} \\ &= \frac{6}{(s+1)}\Big|_{s=-3} \\ &= -2 \end{aligned}$$

# Example – cont.



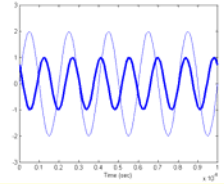
- Substituting for  $K_1$  and  $K_2$

$$\begin{aligned} X(s) &= \frac{K_1}{(s+1)} + \frac{K_2}{(s+3)} \\ &= \frac{3}{(s+1)} - \frac{2}{(s+3)} \end{aligned}$$

- Now the inverse transform can be found using our table as

$$x(t) = 3e^{-t}u(t) - 2e^{-3t}u(t)$$

# Dealing with Multiple Poles



- Consider the Laplace Transform

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)^2 (s - p_2)}$$

- We would like to use partial fraction expansion to find the inverse Laplace Transform. How do we deal with the fact that there are two poles at  $s=p_1$ ?

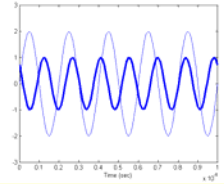
Attempt 1 Let's use the same approach as before

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)^2 (s - p_2)} = \frac{A}{(s - p_1)} + \frac{B}{(s - p_1)} + \frac{C}{(s - p_2)}$$

However, due to the common denominator we can combined the two terms

$$\begin{aligned} \frac{A}{(s - p_1)} + \frac{B}{(s - p_1)} + \frac{C}{(s - p_2)} &= \frac{A + B}{(s - p_1)} + \frac{C}{(s - p_2)} \\ &= \frac{D}{(s - p_1)} + \frac{C}{(s - p_2)} \end{aligned}$$

# Dealing with Multiple Poles – cont.



■ But 
$$\frac{D}{(s-p_1)} + \frac{C}{(s-p_2)} = \frac{D(s-p_2) + C(s-p_1)}{(s-p_1)(s-p_2)} \neq \frac{b_2s^2 + b_1s + b_0}{(s-p_1)^2(s-p_2)}$$

Attempt 2

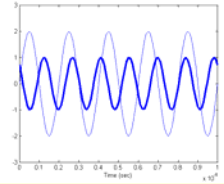
$$H(s) = \frac{b_2s^2 + b_1s + b_0}{(s-p_1)^2(s-p_2)} = \frac{A}{(s-p_1)^2} + \frac{B}{(s-p_2)}$$

However, if we attempt to solve for  $A$  and  $B$

$$\begin{aligned} \frac{A}{(s-p_1)^2} + \frac{B}{(s-p_2)} &= \frac{A(s-p_2) + B(s-p_1)^2}{(s-p_1)^2(s-p_2)} \\ &= \frac{Bs^2 + (A - 2Bp_1)s + (-Ap_2 + Bp_1^2)}{(s-p_1)^2(s-p_2)} \end{aligned}$$

This gives us 3 equations and 2 unknowns and there is no unique solution. In general doing this results in  $D$  equations and  $D-1$  unknowns.

# Dealing with multiple poles – cont.



## Attempt 3

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)^2 (s - p_2)} = \frac{A}{(s - p_1)^2} + \frac{B}{(s - p_1)} + \frac{C}{(s - p_2)}$$

This will result in D equations and D unknowns which can be solved for a unique solution. We can find A and C using the same technique as before. Specifically, to find A let's multiply through by  $(s - p_1)^2$ :

$$\frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)^2 (s - p_2)} (s - p_1)^2 = \frac{A(s - p_1)^2}{(s - p_1)^2} + \frac{B(s - p_1)^2}{(s - p_1)} + \frac{C(s - p_1)^2}{(s - p_2)}$$

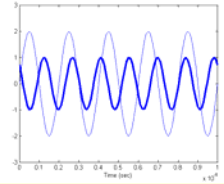
$$\frac{b_2 s^2 + b_1 s + b_0}{(s - p_2)} = A + B(s - p_1) + \frac{C(s - p_1)^2}{(s - p_2)}$$

Now, we let  $s = p_1$

$$\left. \frac{b_2 s^2 + b_1 s + b_0}{(s - p_2)} \right|_{s=p_1} = A + 0 + 0$$

$$A = \left. \frac{b_2 s^2 + b_1 s + b_0}{(s - p_2)} \right|_{s=p_1}$$

# Dealing with multiple poles – cont.



- Doing the same for  $C$ :

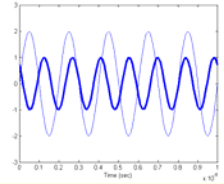
$$C = \left. \left[ (s - p_2) H(s) \right] \right|_{s=p_2}$$
$$= \left. \frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)^2} \right|_{s=p_2}$$

- However, when we attempt to do the same procedure for finding  $B$

$$\frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)^2 (s - p_2)} (s - p_1) = \frac{A(s - p_1)}{(s - p_1)^2} + \frac{B(s - p_1)}{(s - p_1)} + \frac{C(s - p_1)}{(s - p_2)}$$
$$\frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)(s - p_2)} = \frac{A}{(s - p_1)} + B + \frac{C(s - p_1)}{(s - p_2)}$$

- Now if we let  $s=p_1$  we see that the equations have a divide by zero problem.

# Dealing with multiple poles – cont.



- We can avoid this by instead multiplying by  $(s-p_1)^2$ :

$$\frac{b_2s^2 + b_1s + b_0}{(s-p_1)^2(s-p_2)}(s-p_1)^2 = \frac{A(s-p_1)^2}{(s-p_1)^2} + \frac{B(s-p_1)^2}{(s-p_1)} + \frac{C(s-p_1)^2}{(s-p_2)}$$

$$\frac{b_2s^2 + b_1s + b_0}{(s-p_2)} = A + B(s-p_1) + \frac{C(s-p_1)^2}{(s-p_2)}$$

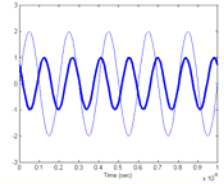
- Now, if we take the derivative of both sides with respect to  $s$ :

$$\begin{aligned} \frac{d}{ds} \left\{ \frac{b_2s^2 + b_1s + b_0}{(s-p_2)} \right\} &= B + \frac{d}{ds} \left\{ \frac{C(s-p_1)^2}{(s-p_2)} \right\} \\ &= B + C \left\{ \frac{(s-p_2)2(s-p_1) - (s-p_1)^2}{(s-p_2)^2} \right\} \end{aligned}$$

- Finally, we set  $s = p_1$ :

$$B = \left. \left[ \frac{d}{ds} \left\{ \frac{b_2s^2 + b_1s + b_0}{(s-p_2)} \right\} \right] \right|_{s=p_1}$$

# Dealing with multiple poles - cont.



- Thus, in general we can find the solution when there are two poles by including the term

$$\frac{A}{(s-p_1)^2} + \frac{B}{(s-p_1)}$$

in the partial fraction expansion. We can find  $B$  by taking the derivative of both sides with respect to  $s$  and setting  $s=p_1$ .

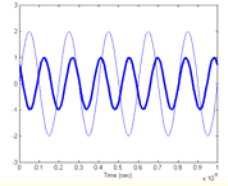
- Once we have solved for the coefficients we have

$$H(s) = \frac{A}{(s-p_1)^2} + \frac{B}{(s-p_1)} + \frac{C}{(s-p_2)}$$

- We can then directly find the inverse Laplace Transform as

$$h(t) = \left( Ate^{p_1t} + Be^{p_1t} + Ce^{p_2t} \right) u(t)$$

# Example



- Find the inverse Laplace Transform of

$$H(s) = \frac{s+5}{s^2(s+2)}$$

- Solution:

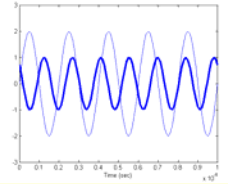
There is a repeated root at  $s=0$ . Thus we can write the Laplace Transform as

$$H(s) = \frac{s+5}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+2)}$$

where

$$\begin{aligned} A &= \left[ s^2 H(s) \right]_{s=0} = \left[ \frac{s+5}{s^2(s+2)} s^2 \right]_{s=0} \\ &= \left[ \frac{s+5}{(s+2)} \right]_{s=0} = \frac{5}{2} \end{aligned}$$

# Example – cont.



- The third coefficient can be found as

$$\begin{aligned} C &= \left[ (s+2)H(s) \right]_{s=-2} = \left[ \frac{s+5}{s^2(s+2)}(s+2) \right]_{s=-2} \\ &= \left[ \frac{s+5}{s^2} \right]_{s=-2} = \frac{3}{4} \end{aligned}$$

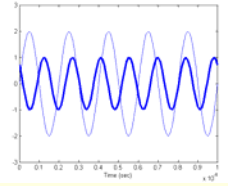
- To find the second coefficient

$$\begin{aligned} A &= \left[ \frac{d}{ds} [s^2 H(s)] \right]_{s=0} = \left[ \frac{d}{ds} \left[ \frac{s+5}{s^2(s+2)} s^2 \right] \right]_{s=0} \\ &= \left[ \frac{d}{ds} \left[ \frac{s+5}{(s+2)} \right] \right]_{s=0} = \frac{(s+2) - (s+5)}{(s+2)^2} \Big|_{s=0} = -\frac{3}{4} \end{aligned}$$

- Thus we have

$$H(s) = \frac{5}{2} \frac{1}{s^2} - \frac{3}{4} \frac{1}{s} + \frac{3}{4} \frac{1}{(s+2)}$$

# Example – cont.



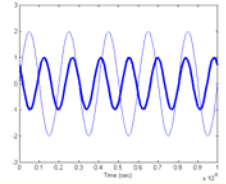
- The inverse transform of

$$H(s) = \frac{5}{2} \frac{1}{s^2} - \frac{3}{4} \frac{1}{s} + \frac{3}{4} \frac{1}{(s+2)}$$

can be found using our tables as

$$h(t) = \frac{5}{2} t u(t) - \frac{3}{4} u(t) + \frac{3}{4} e^{-2t} u(t)$$

# Summary



- In this lecture we have gone through several examples of the Laplace Transform
- Additionally, we introduced the most common technique for determining the inverse Laplace Transform using the technique of *partial fraction expansion*.
- Finally we also looked at a few examples of the inverse Laplace Transform using this technique.