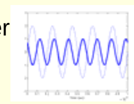
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer

Lecture #2: Mathematical Description of Signals



Introduction

- What to read:
  - Chapter 2 – Sections 2.1-2.8, 2.14
- Purpose of today's lecture
  - To describe some important mathematical functions that we will use in this course
  - To define several important properties associated with signals
  - Provide examples which demonstrate key ideas and functions

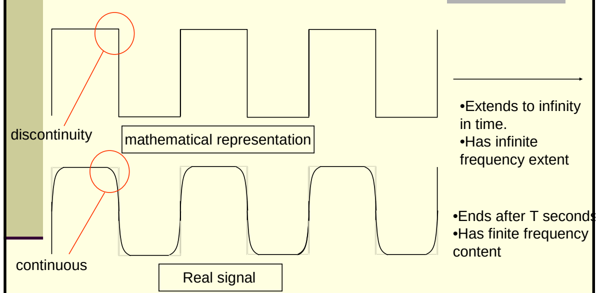
Mathematical Descriptions

- Many systems which are encountered in engineering process signals which represent physical processes that are measured, controlled, or recorded.
- We design and analyze systems by representing these signals using mathematical descriptions which are typically functions of time or space.
- Our representations will not exactly match the real-world signals, but they are sufficiently close to allow extremely accurate prediction of the system behavior.

Physically Realizable Functions

- Have finite time duration (finite energy!)
- Occupy finite frequency spectrum
- Are continuous
- Have finite peak value
- Are real-valued
- All real-world signals will have these properties, although sometimes we use mathematical models which violate these conditions.

Mathematical Representations



Classification of Signals

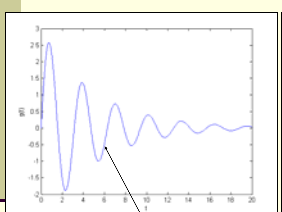
- Although functions can operate on any type of variable, we will be most concerned with functions of *time*
- Signals (or more specifically their mathematical representations) can be categorized according to a few major features
  - Continuous Time vs. Discrete Time
  - Deterministic vs. Random
  - Power vs. Energy
  - Periodic vs. Aperiodic

Continuous Time vs. Discrete Time

- Continuous Time function – a function which is defined on a continuum of points in time
  - We will use the notation *g(t)*
  - Note that a *continuous time* function may be continuous or discontinuous depending on whether or not the derivative is defined at all points – more on this in a moment
- Discrete Time function – a function which is defined only at discrete points in time and undefined at times between those points
  - We will use the notation *g[n]* where *n* is an integer
  - We will not address discrete time signals and systems in this class. This is contained in 3704.

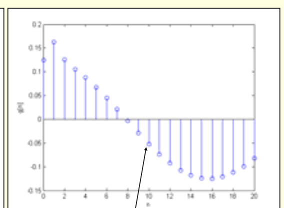
Continuous Time vs. Discrete Time

Continuous Time



Function defined for all values of *t*

Discrete Time



Function only defined at integer values of *n*

Continuous vs. Continuous Time

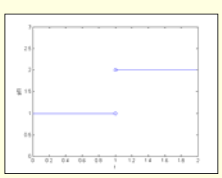
- A *continuous time* function is a function which is defined over a continuum of time values
- A *continuous* function is a function which has no discontinuities
  - Discontinuity defined as $\lim_{\varepsilon \to 0} g(t+\varepsilon) \neq \lim_{\varepsilon \to 0} g(t-\varepsilon)$

Example: The function plotted to the right is *discontinuous* at the value *t = 1*.

$$\lim_{\varepsilon \to 0} g(t+\varepsilon) = 2$$

$$\lim_{\varepsilon \to 0} g(t-\varepsilon) = 1$$

However, the function is defined for all values of *t* so it is *time continuous*.



Random vs. Deterministic

- A deterministic function is one for which we can write a mathematical function to predict its future values
- A random function is one for which we cannot predict exactly what values it will take on.
  - Rather we can only say what the probabilities are that it take on certain values
- In this course we will always deal with deterministic functions, however, in future courses you may likely run into random signals which are extremely important for many disciplines

Even and Odd Functions

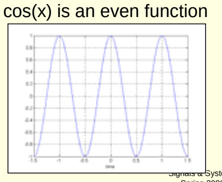
- For an even function

$$g(t) = g(-t)$$

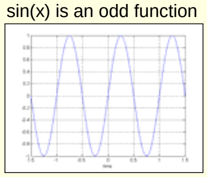
- For an odd function

$$g(t) = -g(-t)$$

- Examples:
  - cos(x) is an even function



sin(x) is an odd function



Periodic Functions

- A periodic function is one which repeats an exact pattern over all time
- In other words, a periodic function is one for which

$$g(t) = g(t + nT)$$

for any integer *n*

  - *T* is termed the period
- If a function is not periodic, we say that it is *aperiodic*
- A periodic function is invariant under the transformation $t \rightarrow t+nT$
  - More on transformations shortly

Sinusoidal Function of Time

$$x(t) = A\cos(2\pi ft + \theta)$$

- *x(t)* is a sinusoidal function of time, *t*
- *A* = amplitude
- *f* = frequency
- θ = phase
- Note that:

$$A\cos(2\pi ft + \theta) = A\cos\left(2\pi f\left[t + \frac{\theta}{2\pi f}\right]\right)$$

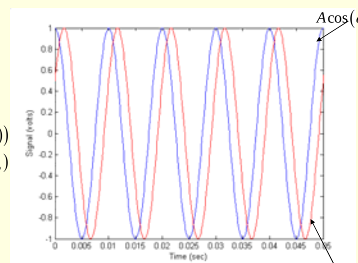
Constant with respect to time

- Thus, phase is simply a normalized time delay/advance

Phase

- Phase represents time delay of a sinusoid

$$x(t) = A\cos(\omega_c t)$$
$$x(t - t_o) = A\cos(\omega_c (t - t_o))$$
$$= A\cos(\omega_c t - \omega_c t_o)$$
$$= A\cos(\omega_c t - \theta)$$



Sinsoids

- The following identity is very useful

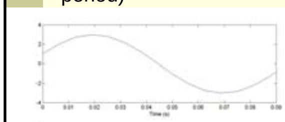
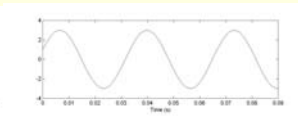
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(2\pi ft + \theta) = \cos(\theta)\cos(2\pi ft) - \sin(\theta)\sin(2\pi ft)$$

- Thus,

$$\cos\left(2\pi ft + \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)\cos(2\pi ft) - \sin\left(\frac{\pi}{2}\right)\sin(2\pi ft)$$
$$= \sin(2\pi ft)$$

Frequency

- For a sinusoidal function the frequency is the inverse of the time it takes to complete one cycle (i.e., the period)

$f_o = 10$ $f_o = 30$

$$3\sin(2\pi f_o t + \pi/9)$$

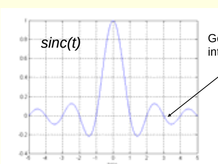
Sinc function

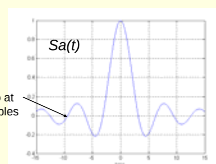
- The sinc function is very important in Fourier Transform analysis and is defined as

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

- A similarly defined function is called the *sampling function* and is defined as

$$Sa(t) = \frac{\sin(t)}{t}$$





Evaluating *sinc(0)*

- The value of *sinc(0)* evaluates to

$$\mathrm{sinc}(0) = \frac{\sin(0)}{0} = \frac{0}{0}$$

which is undefined.

- Using L'Hopital's rule

$$\lim_{t\to 0}\mathrm{sinc}(t) = \lim_{t\to 0}\frac{\sin(\pi t)}{\pi t} = \lim_{t\to 0}\frac{d\{\sin(\pi t)\}/dt}{d\{\pi t\}/dt} = \lim_{t\to 0}\frac{\pi\cos(\pi t)}{\pi} = 1$$

Important Discontinuous Functions

- A very useful set of functions in system analysis have discontinuities or discontinuous derivatives and are related to one another through integrals and derivatives
- Included in this group are
  - Unit step function
  - Unit ramp function
  - Unit Impulse function
  - Signum function
- Useful in creating mathematical descriptions of signals and systems

Unit Step Function

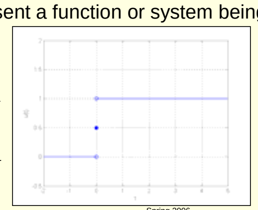
- The *unit step function* is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

*-Note that the definition at t=0 is irrelevant as long as it is finite

- This function is very useful and is commonly used to represent a function or system being switched on

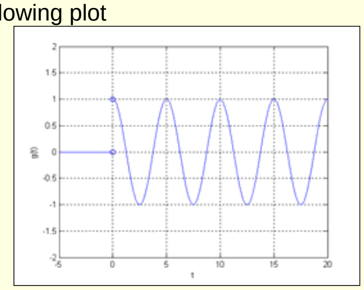
Since the height is "1" we call this the *unit* step function



Note that there is a discontinuity at $t = 0$ which can represents a signal being switched "on"

Example

- Write a mathematical expression for the following plot



Example (cont.)

- The plot
  - is a sinusoid which starts at *t = 0*
  - is equal to one at *t = 0* and is thus a cosine
  - Completes one period at t = 5. Thus, frequency is 1/5.
  - Amplitude is 1
- Answer:

$$g(t) = \cos\left(\frac{2\pi}{5}t\right)u(t)$$

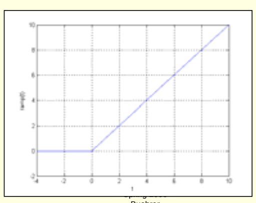
Switches “on” the sinusoid

Unit Ramp Function

- Another useful function is one which turns on at *t = 0* and increases linearly with time
- This is termed the *unit ramp function* and is defined as

$$ramp(t) = \begin{cases} t & t > 0 \\ 0 & t \le 0 \end{cases}$$

Since the slope is “1” we call this the *unit* ramp function



Relationship between the Step and Ramp functions

- It is easy to show that

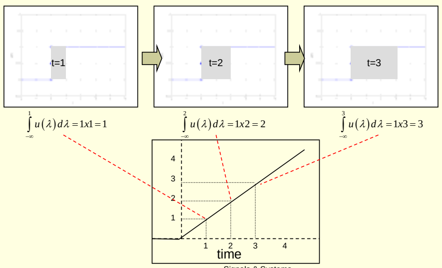
$$ramp(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda$$

- Mathematically

$$\begin{aligned}\int_{-\infty}^{t} u(\lambda)\,d\lambda &= \begin{cases} \int_0^t d\lambda & t > 0 \\ 0 & t \le 0 \end{cases} \\ &= \begin{cases} \lambda\big|_0^t & t > 0 \\ 0 & t \le 0 \end{cases} \\ &= \begin{cases} t & t > 0 \\ 0 & t \le 0 \end{cases} \\ &= ramp(t)\end{aligned}$$

Relationship between the Step and Ramp functions

- Graphically integration is equal to the area under the curve

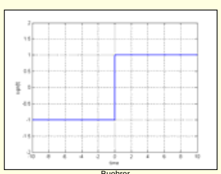


Signum Function

- The signum function is related to the unit step function and is defined as

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

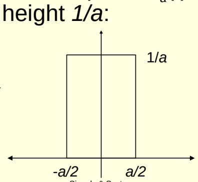
- This is also sometimes called the *sign* function since it essentially produces the sign of its argument



Unit Impulse Function

- One of the most useful, yet strange, functions that we will encounter in this class is the unit impulse function, $\delta(t)$ (sometimes also called a delta function).
- To understand the unit impulse function consider a unit area pulse $\delta_a(t)$ which has width *a* and height *1/a*:

$$\delta_a(t) = \begin{cases} \frac{1}{a} & |t| < \frac{a}{2} \\ 0 & else \end{cases}$$



Area = a*1/a = 1

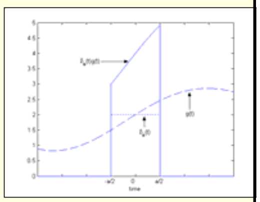
Unit Impulse Function (cont.)

- Now, consider a the integral of the unit pulse times a function g(t):

$$A = \int_{-\infty}^{\infty} \delta_a(t) g(t)\,dt = \frac{1}{a}\int_{-a/2}^{a/2} g(t)\,dt$$

If we let the interval, *a*, get very small:

$$\lim_{a\to 0} A = \lim_{a\to 0}\left\{\int_{-\infty}^{\infty} \delta_a(t) g(t)\,dt\right\} = \lim_{a\to 0}\left\{\frac{1}{a}\int_{-a/2}^{a/2} g(t)\,dt\right\} = g(0)\lim_{a\to 0}\left\{\frac{1}{a}\int_{-a/2}^{a/2} dt\right\} = g(0)\lim_{a\to 0}\frac{1}{a}a = g(0)$$



Thus, in the limit integrating the multiplication of unit pulse with a function results in the value of the function at zero.

Unit Impulse Function (cont.)

- Thus, in the limit as *a* → 0, the function $\delta_a(t)$ has the property that it extracts the value of the function at time equal 0 when their product is integrated over any limits which include *t=0*.
- Note that we could arrive at this same property with an entirely different function:

$$\delta_a(t) = \begin{cases} \frac{1}{a}\left(1 - \frac{|t|}{a}\right) & |t| < a \\ 0 & |t| > a \end{cases}$$

$$\lim_{a\to 0} A = \lim_{a\to 0}\left\{\frac{1}{a}\int_{-a}^{a}\left(1-\frac{|t|}{a}\right) g(t)\,dt\right\} = g(0)\lim_{a\to 0}\left\{\frac{2}{a}\int_{0}^{a}\left(1-\frac{t}{a}\right)dt\right\} = g(0)\lim_{a\to 0}\frac{2}{a}\left[t - \frac{t^2}{2a}\right]_0^a = g(0)$$

- The full derivation is in the book. The key to this property is that the function has unit area. The shape of the function is irrelevant.

The Unit Impulse Function: Defined

- The unit impulse is defined as a function which when multiplied by another function *g(t)* (which is finite and continuous at *t=0*) and the product is integrated between limits which include *t=0*, the result is *g(0)*:

$$g(0) = \int_{-\infty}^{\infty} \delta(t) g(t)\,dt$$

- The impulse can thus be defined as

$$\delta(t) = 0 \quad t \neq 0 \qquad \int_{t_1}^{t_2} \delta(t)\,dt = \begin{cases} 1 & t_1 < 0 < t_2 \\ 0 & else \end{cases}$$

Relationship between the Impulse and Step Functions

- We saw previously that

$$ramp(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda$$

- What is the derivative of the unit step?
- For functions with discontinuities, we must use the generalized derivative:

$$\frac{d}{dt}\{g(t)\} = \frac{d}{dt}\{g(t)\}_{t \neq t_o} + \lim_{\varepsilon \to 0}\left[g(t+\varepsilon) - g(t-\varepsilon)\right]\delta(t - t_o)$$

where t_o is the point of the discontinuity

- Let's apply this to the unit step function

Derivative of the Unit Step

- Taking the derivative:

$$\frac{d}{dt}\{u(t)\} = \frac{d}{dt}\{u(t)\}_{t \neq 0} + \lim_{\varepsilon \to 0}\left[u(t+\varepsilon) - u(t-\varepsilon)\right]\delta(t)$$

- The derivative for t< 0 is zero. The derivative for t > 0 is also zero.
- Thus we have

$$\frac{d}{dt}\{u(t)\} = 0 + [1-0]\delta(t)$$
$$= \delta(t)$$

- The unit impulse function is the generalized derivative of the unit step function.
- Further

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda$$

Properties of the impulse function

- The strength of an impulse is equal to the area of the impulse.
- The unit impulse has area or strength of one.
- Consider an impulse of strength *k* written as *k* δ(t):

$$\int_{-\infty}^{\infty} k\delta(\lambda)\, g(\lambda)\, d\lambda = k\, g(0)$$

- Equivalence property:

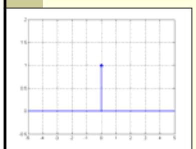
$$g(t)\, k\, \delta(t) = k\, g(0)\, \delta(t)$$

- Sampling property

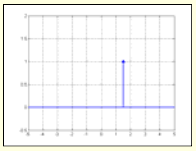
$$\int_{-\infty}^{\infty} \delta(t - t_o)\, g(t)\, dt = g(t_o)$$

Graphical Representation

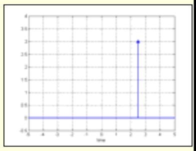
- We typically represent the impulse as an arrow where the height corresponds to the strength or area of the impulse



δ(t)



δ(t-1.5)



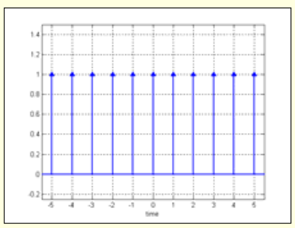
3 δ(t-2.5)

Unit Comb

- The unit comb is a sequence of uniformly spaced unit impulses (sometimes also called an *impulse train*)

$$comb(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$ where *n* is an integer

Since the strength of each impulse is "1" and the spacing of the impulses is unity, we call this the *unit* comb function



Singularity Functions

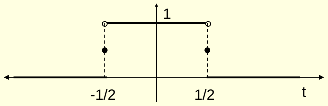
- The unit impulse, unit step, and unit ramp are part of a larger family of functions termed *singularity functions* written as $u_k(t)$ where *k* represents the number of times the unit impulse is differentiated
- A negative value of *k* represents an integral

$$u_0(t) = \delta(t)$$
$$u_{-1}(t) = u(t)$$
$$u_{-2}(t) = ramp(t)$$

Other Functions

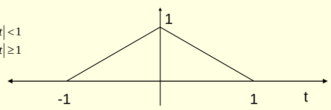
- Unit Rectangle Function [Rectangular Pulse]

$$rect(t)=\begin{cases}1 & |t|<\frac{1}{2}\\ \frac{1}{2} & |t|=\frac{1}{2}\\ 0 & |t|>\frac{1}{2}\end{cases}$$



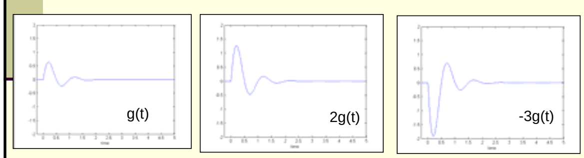
- Unit Triangle

$$tri(t)=\begin{cases}1-|t| & |t|<1\\ 0 & |t|\geq 1\end{cases}$$



Transformations

- Amplitude Scaling $g(t)\rightarrow Ag(t)$
  - Multiplies every value of the time function by the scaling factor
  - Negative scaling values change the sign of the function values
- Ex: g(t) = exp(-2t)sin(2πt)u(t)



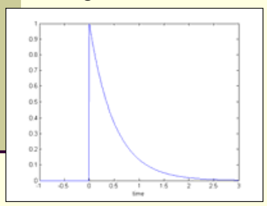
Transformations (cont.)

- Time Shifting $g(t)\rightarrow g(t-t_o)$
  - *t* is replaced by *t*-t_o at every instance
- Example:
  - g(t) = exp(-2t)u(t)

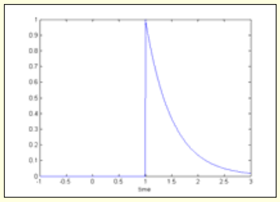
t	g(t)	g(t-1)
-0.99	0	0
-0.49	0	0
0.01	1	0
0.51	0.37	0
1.01	0.14	1
1.51	0.05	0.37
2.01	0.02	0.14
2.51	0.007	0.05

Time Shifting

- Time shifting by a positive number ($t_o>0$) corresponds to shifting the function to the right



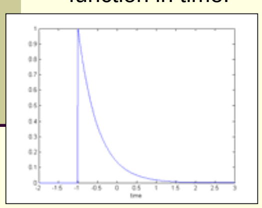
g(t)



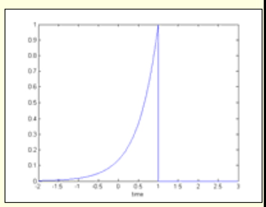
g(t-1)

Time Shifting

- Time shifting by a negative number ($t_o < 0$) corresponds to shifting to the left
- Multiplying time by negative one flips the function in time.



g(t+1)



g(1-t) = g(-[t-1])

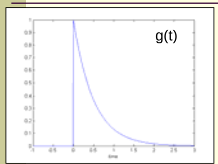
Time Scaling

- Time scaling is the transformation

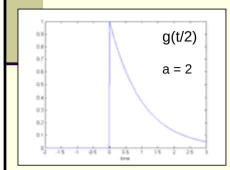
$$g(t) \rightarrow g\left(\frac{t}{a}\right)$$

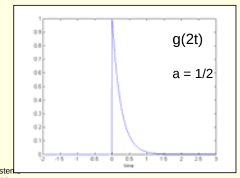
t	g(t)	g(t/2)
-0.99	0	0
-0.49	0	0
0.01	0.99	0.99
0.51	0.36	0.6
1.01	0.13	0.36
1.51	0.05	0.22
2.01	0.02	0.13
2.51	0.007	0.08

Time Scaling



- If a > 1 → Scaling slows the function in time
- If a < 1 → Scaling speeds the function in time





Multiple Transformations

- The transformation

$$g(t)\to Ag\left(\frac{t-t_o}{a}\right)$$

is equivalent to multiple transformations

$$g(t)\to \underbrace{Ag(t)}_{\text{amplitude scaling}} \to \underbrace{Ag\left(\frac{t}{a}\right)}_{\text{time scaling}} \to \underbrace{Ag\left(\frac{t-t_o}{a}\right)}_{\text{time shift}}$$

- Note that order can be important

$$g(t)\to \underbrace{g\left(\frac{t}{a}\right)}_{\text{time scaling}} \to \underbrace{g\left(\frac{t-t_o}{a}\right)}_{\text{time shift}} \neq g(t)\to \underbrace{g(t-t_o)}_{\text{time shift}} \to \underbrace{g\left(\frac{t}{a}-t_o\right)}_{\text{time scaling}}$$

- Key is to remember that when scaling we replace *t* by *t/a* and when time shifting we replace *t* by $t-t_o$

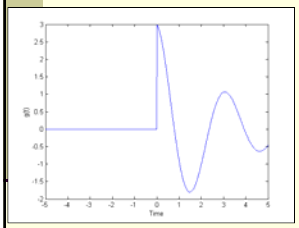
Example

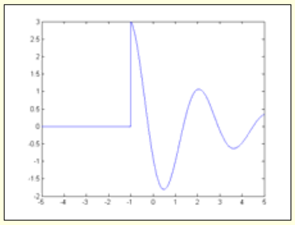
- Plot the function
  - g(t) = 3cos(2t)exp(-t/3)u(t)

  for -5 < t < 5
- Repeat for the following transformations
  - g(t+1)
  - g(2t)
  - g(3-t)
  - -2*g((t-2)/5)

Solution

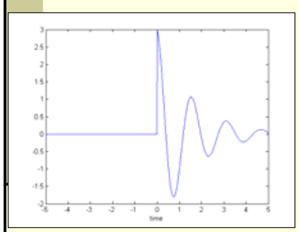
g(t) = 3cos(2t)exp(-t/3)u(t)

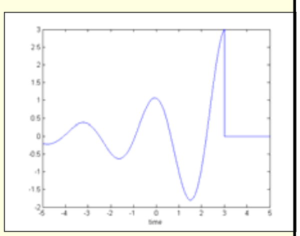

g(t)


g(t+1)

Solution (cont.)

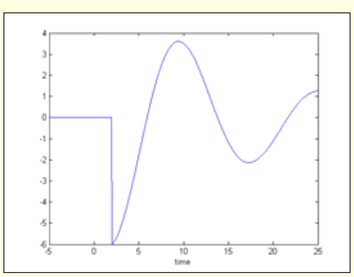
g(t) = 3cos(2t)exp(-t/3)u(t)


g(2t)


g(3-t)

Solution (cont.)

- -2*g((t-2)/5)



Energy and Power

- The **Energy** of a signal *g(t)* is defined as**:**

$$E = \int_{-\infty}^{\infty} g^2(t)\,dt$$

- A signal *g(t)* is classified as an <u>Energy Signal</u> if

$$0 < E < \infty$$

- The **Power** of a signal *g(t)* is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)\,dt$$

- A signal *g(t)* is a <u>Power Signal</u> if $0 < P < \infty$

Energy and Power (cont.)

- Note: For periodic signals, power can be computed by integrating over one period
- Questions:
  - If a signal is a power signal how much energy does it have?
  - If a signal is an energy signal how much power does it have?
  - Can a signal be both an energy signal and a power signal?

Conclusions

- In this lecture we have discussed functions which we use to mathematically describe signals
  - We discussed several important functions that will be useful in describing signals and systems including: the unit step function, the unit ramp, the unit impulse, the unit comb, and others.
- We also discussed several important properties of time functions that will be useful in describing and analyzing signals and systems
- There are several homework problems given that will help you understand these signals/properties and commit them to memory