ECE 2704
Signals and Systems
Spring 2006

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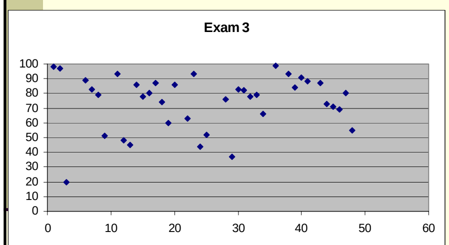
Lecture #20: Additional examples of the Inverse LT
Applications of the LT

Midterm 3



- Mean = 74
- Median 79

90-100	7
80-89	12
70-79	9
60-69	3
< 60	9

Overview

- Today we look at a few additional examples of taking the inverse Laplace Transform
- Additionally, we will look at the application of the Laplace Transform for solving differential equations, examining stability and examining feedback systems
- What to read – Section 9.6,10.1-10.4 in the text

Example

- Find the inverse Laplace Transform of

$$H(s) = \frac{10s^2}{(s+1)(s+3)} e^{-s}$$

- **Solution:**
- The coefficient in front of e^{-s} is an improper fraction (power of *s* in the numerator is not less than the power of *s* in the denominator). Thus, we must first divide it out

$$\begin{array}{r|l} & \quad\quad 10 \\ s^2+4s+3 & 10s^2 \\ & \underline{10s^2 \quad +40s \quad +30} \\ & \quad -40s \quad -30 \end{array}$$

$$H(s) = \left[10 - \frac{40s+30}{(s+1)(s+3)}\right] e^{-s}$$

Example – cont.

- The next step is to expand the proper fraction in partial fractions:

$$H(s) = \left[10 + \frac{A}{(s+1)} + \frac{B}{(s+3)}\right] e^{-s}$$

$$A = (s+1)\left[\frac{40s+30}{(s+1)(s+3)}\right]\Bigg|_{s=-1} = \left[\frac{40s+30}{(s+3)}\right]\Bigg|_{s=-1} = -5$$

$$B = (s+3)\left[\frac{40s+30}{(s+1)(s+3)}\right]\Bigg|_{s=-3} = \left[\frac{40s+30}{(s+1)}\right]\Bigg|_{s=-3} = 45$$

$$H(s) = \left[10 - \frac{5}{(s+1)} + \frac{45}{(s+3)}\right] e^{-s}$$

Example – cont.

- First, recall the time-shift property of the Laplace transform:

$$x(t-t_o) \xleftrightarrow{\mathcal{L}} e^{-st_o} X(s)$$

- Thus, if $H(s) = G(s)e^{-s}$ then $h(t) = g(t-1)$
- Using the two transform pairs

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$$

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1$$

- We can write $g(t) = 10\delta(t) + \left(5e^{-t} - 45e^{-3t}\right)u(t)$

$$h(t) = 10\delta(t-1) + \left(5e^{-(t-1)} - 45e^{-3(t-1)}\right)u(t-1)$$

Example 2

- Find the inverse Laplace Transform of

$$H(s)=\frac{6s+10}{s^2+4s+5}$$

- **Solution:**
- We could factor the denominator into complex roots, but it is easier to recall the transform pairs

$$e^{-\alpha t}\cos(\omega_o t)u(t)\xleftrightarrow{\mathcal{L}}\frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}\qquad \sigma>-\alpha$$

$$e^{-\alpha t}\sin(\omega_o t)u(t)\xleftrightarrow{\mathcal{L}}\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}\qquad \sigma>-\alpha$$

thus we should attempt to factor the denominator into the form $(s+a)^2+b^2$

Example 2 – cont.

- Thus, we have

$$\begin{aligned}H(s)&=\frac{6s+10}{s^2+4s+5}\\&=\frac{6s+10}{(s^2+4s+4)+1}\\&=\frac{6s+10}{(s+2)^2+1}\end{aligned}$$

- To get it into a form that we can use with the previous transform pairs we need either *s+2* in the numerator or 1 in the numerator.

Example 2 – cont.

- Continuing:

$$\begin{aligned}H(s)&=\frac{6s+10}{(s+2)^2+1}\\&=\frac{6(s+2)-2}{(s+2)^2+1}\\&=6\frac{s+2}{(s+2)^2+1}-2\frac{1}{(s+2)^2+1}\end{aligned}$$

- Now, comparing to the previous transform pairs we have:

$$h(t)=\left(6e^{-2t}\cos(t)-2e^{-2t}\sin(t)\right)u(t)$$

Solution of Differential Equations

- One of the most powerful uses of the Laplace Transform is in the analysis of linear systems.
- Often linear systems are described by linear differential equations which are easily solved using Laplace Transforms since the time derivative is described by the multiplication by *s* in the Laplace Domain
- This is also true of the Fourier Transform
- However, the unilateral Laplace Transform is more useful since (a) it naturally applies to functions which are causal and (b) it can handle forcing functions which are unbounded by time.

Example 3

- Solve the differential equation

$$\frac{d^2}{dt^2}\left[x(t)\right]+7\frac{d}{dt}\left[x(t)\right]+12x(t)=0$$

for times $t > 0$, given the initial conditions

$$x\left(0^-\right)=2$$

$$\left[\frac{d}{dt}x(t)\right]_{t=0^-}=-4$$

Example 3 – cont.

Solution:

The first step is to Laplace Transform both sides of the equation

$$\frac{d^2}{dt^2}\left[x(t)\right]+7\frac{d}{dt}\left[x(t)\right]+12x(t)=0$$

$$\left\{s^2X(s)-sx\left(0^-\right)-\frac{d}{dt}\left[x(t)\right]_{t=0^-}\right\}+7\left\{sX(s)-x\left(0^-\right)\right\}+12X(s)=0$$

Solving for *X(s):*

$$X(s)=\frac{sx\left(0^-\right)+\frac{d}{dt}\left[x(t)\right]_{t=0^-}+7x\left(0^-\right)}{s^2+7s+12}$$

$$=\frac{2s+10}{s^2+7s+12}$$

Example 3 – cont.

- Continuing:

$$\begin{aligned} X(s) &= \frac{2s+10}{s^2+7s+12} \\ &= \frac{A}{s+4} + \frac{B}{s+3} \\ &= -\frac{2}{s+4} + \frac{4}{s+3} \end{aligned}$$

- Which has a transform

$$h(t) = \left(4e^{-3t} - 2e^{-4t}\right)u(t)$$

Example 3 – cont.

- As a check we can substitute the solution into the original differential equation for times *t>=0*

$$\frac{d^2}{dt^2}[x(t)] + 7\frac{d}{dt}[x(t)] + 12x(t) = 0$$

$$\frac{d^2}{dt^2}\left[\left(4e^{-3t} - 2e^{-4t}\right)\right] + 7\frac{d}{dt}\left[\left(4e^{-3t} - 2e^{-4t}\right)\right] + 12\left(4e^{-3t} - 2e^{-4t}\right) = 0$$

$$\frac{d}{dt}\left[\left(-12e^{-3t} + 8e^{-4t}\right)\right] + 7\left[\left(-12e^{-3t} + 8e^{-4t}\right)\right] + \left(48e^{-3t} - 24e^{-4t}\right) = 0$$

$$\left[\left(36e^{-3t} - 32e^{-4t}\right)\right] + \left[\left(-84e^{-3t} + 56e^{-4t}\right)\right] + \left(48e^{-3t} - 24e^{-4t}\right) = 0$$

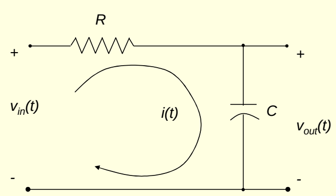
$$0 = 0$$

- Checking the initial conditions:

$$x\left(0^-\right) = 4 - 2 = 2 \quad \left[\frac{d}{dt}x(t)\right]_{t=0^-} = -12 + 8 = -4$$

Example 4

- The lowpass filter shown below is excited by a unit voltage impulse at time $t = t_o$ where $t_o > 0$. Find the response $v_{out}(t)$ without assuming that the initial voltage $v_{out}(t)$ is zero. Also, check the initial output voltage from the result.



Example 4 – cont.

- **Solution**: Using Kirchoff's Voltage law we can write
$$v_{in}(t) = i(t)R + v_{out}(t)$$
$$= v'_{out}(t)RC + v_{out}(t)$$
- Now, taking the Laplace Transform:
$$v_{in}(t) = v'_{out}(t)RC + v_{out}(t)$$
$$V_{in}(s) = \left[sV_{out}(s) - v_{out}(0^-)\right]RC + V_{out}(s)$$
- Re-arranging the equation for *$V_{out}(s)$*:
$$V_{out}(s) = \frac{V_{in}(s) + RCv_{out}(0^-)}{sRC + 1}$$

Example 4 – cont.

- For an impulse excitation, we have
$$V_{in}(s) = 1$$
- However, for an arbitrary time delay t_o:
$$V_{in}(s) = e^{-t_o s}$$
- Thus, we have at the output
$$V_{out}(s) = \frac{e^{-t_o s} + RCv_{out}(0^-)}{sRC + 1}$$
- To find the inverse Laplace Transform let us rewrite the equation as
$$V_{out}(s) = e^{-t_o s}\frac{1}{sRC+1} + RCv_{out}(0^-)\frac{1}{sRC+1}$$
$$= \frac{e^{-t_o s}}{RC}\frac{1}{s+1/RC} + v_{out}(0^-)\frac{1}{s+1/RC}$$

Example 4 – cont.

- Continuing…
$$V_{out}(s) = \frac{e^{-t_o s}}{RC}\frac{1}{s+1/RC} + v_{out}(0^-)\frac{1}{s+1/RC}$$
- Now taking the inverse Laplace Transform
$$v_{out}(t) = \frac{1}{RC}e^{-(t-t_o)/RC}u(t-t_o) + v_{out}(0^-)e^{-(t-t_o)/RC}u(t)$$
- Thus, there are two parts to the response, an exponential decay due to the initial voltage $v_{out}(0^-)$ and the response to the impulse at t_o.

Example 4 – cont.

- What is the initial value of the output voltage?
- **Solution**: The initial value theorem says that

$$x(0^+) = \lim_{s\to\infty}[sX(s)]$$

- From the solution we have

$$V_{out}(s) = \frac{e^{-t_o s}}{RC}\frac{1}{s+1/RC} + v_{out}(0^-)\frac{1}{s+1/RC}$$

- Thus

$$\begin{aligned} x(0^+) &= \lim_{s\to\infty}[sV_{out}(s)] \\ &= \lim_{s\to\infty}\left[s\left(\frac{e^{-t_o s}}{RC}\frac{1}{s+1/RC} + v_{out}(0^-)\frac{1}{s+1/RC}\right)\right] \\ &= \lim_{s\to\infty}\left[\left(\frac{e^{-t_o s}}{RC}\frac{s}{s+1/RC} + v_{out}(0^-)\frac{s}{s+1/RC}\right)\right] \\ &= v_{out}(0^-) \end{aligned}$$

Example 4 – cont.

- However, what if the impulse is applied at $t_o=0$?

$$\begin{aligned} x(0^+) &= \lim_{s\to\infty}[sV_{out}(s)] \\ &= \lim_{s\to\infty}\left[s\left(\frac{e^{-t_o s}}{RC}\frac{1}{s+1/RC} + v_{out}(0^-)\frac{1}{s+1/RC}\right)\right] \\ &= \lim_{s\to\infty}\left[\left(\frac{1}{RC}\frac{s}{s+1/RC} + v_{out}(0^-)\frac{s}{s+1/RC}\right)\right] \\ &= \frac{1}{RC} + v_{out}(0^-) \end{aligned}$$

- Which captures the fact that the impulse applies an initial voltage when it is applied at zero.

Using the Laplace Transform to Examine System Stability

- Consider a linear system which can be described by differential equations of the form

$$\sum_{k=0}^{D} a_k \frac{d^k}{dt^k}\{y(t)\} = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k}\{x(t)\}$$

- Such a system can be represented by a transfer function (in the Laplace domain)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{D} a_k s^k}{\sum_{k=0}^{N} b_k s^k}$$

Stability – cont.

- The denominator can be factored as

$$H(s)=\frac{Y(s)}{X(s)}=\frac{\sum_{k=0}^{D} a_k s^k}{\prod_{k=0}^{N}(s-p_k)}$$

- If none of the poles are repeated we can write the transfer function as

$$H(s)=\frac{Y(s)}{X(s)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\ldots\frac{K_D}{s-p_D}$$

- The impulse response is then

$$h(t)=\left(K_1e^{p_1t}+K_2e^{p_2t}+\ldots K_De^{p_Dt}\right)u(t)$$

Stability – cont.

- For a system to be *stable,* its impulse response must be *absolutely integrable.*
- Recall that in the transfer function

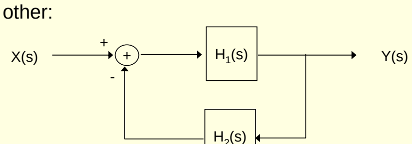
$$H(s)=\frac{Y(s)}{X(s)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\ldots\frac{K_D}{s-p_D}$$

p_i are the poles of the transfer function.

- In order for the impulse response to be absolutely integrable, the real part of each p_i must be *negative.*
- Therefore, the poles must lie in the Left Hand Plane (LHP) which is the plane to the left (not including) the jω axis.

Systems with Feedback

- Consider two systems which have transfer functions *$H_1(s)$* and *$H_2(s)$*. Let us arrange these two systems such that the output of one is fed into the input of the other:



- This is referred to as a negative feedback connection of systems since the output is negated and fed back into the input

Transfer Function

- The transfer function of the overall system can be found in terms of the transfer functions of the individual systems. First we define the output of the summation as

$$E(s) = X(s) - H_2(s)Y(s)$$

- Further we can write

$$Y(s) = H_1(s)E(s)$$

- Combining the two equations we have

$$\frac{Y(s)}{H_1(s)} = X(s) - H_2(s)Y(s)$$

- Solving for Y(s)/X(s) we have

$$Y(s)\left(1 + H_1(s)H_2(s)\right) = H_1(s)X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Often called the "Loop Transfer Function"

Open Loop

- Notice that if the feedback goes to zero the overall transfer function

$$\frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

collapses to the forward path gain

$$\frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)*0}$$

$$= H_1(s)$$

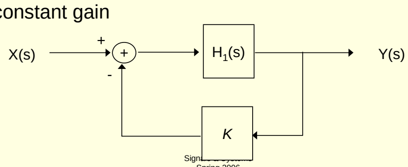
This is termed *open-loop*

Stabilizing a System with Feeback

- Consider a system with transfer function

$$H_1(s) = \frac{1}{s-2}$$

- As we have stated before, since this system has poles in the right hand plane (s=2) the system is unstable. (Note that the response of the system to an impulse is a growing exponential.)
- However, consider a simple feedback system with a constant gain



Stabilizing a system – cont.

- The new transfer function is now

$$H(s)=\frac{Y(s)}{X(s)}=\frac{H_1(s)}{1+H_1(s)H_2(s)}$$
$$=\frac{\frac{1}{s-2}}{1+K\frac{1}{s-2}}$$
$$=\frac{1}{s-2+K}$$

- Thus, provided that *K > 2,* we have stabilized the system

Summary

- In this lecture we have examined a couple more examples of the inverse Laplace Transform and discussed a couple of applications of the Laplace transform including
  - Solving differential equations
  - Examining stability
  - Examining feedback systems
- We will continue this discussion in the next lecture