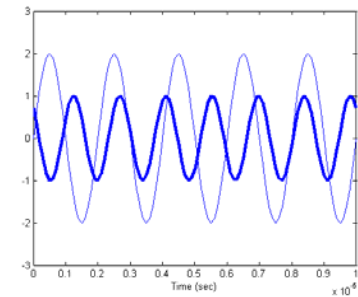


# ECE 2704

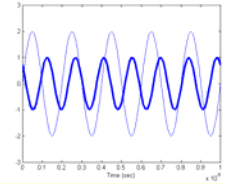
## Signals and Systems

### Spring 2006

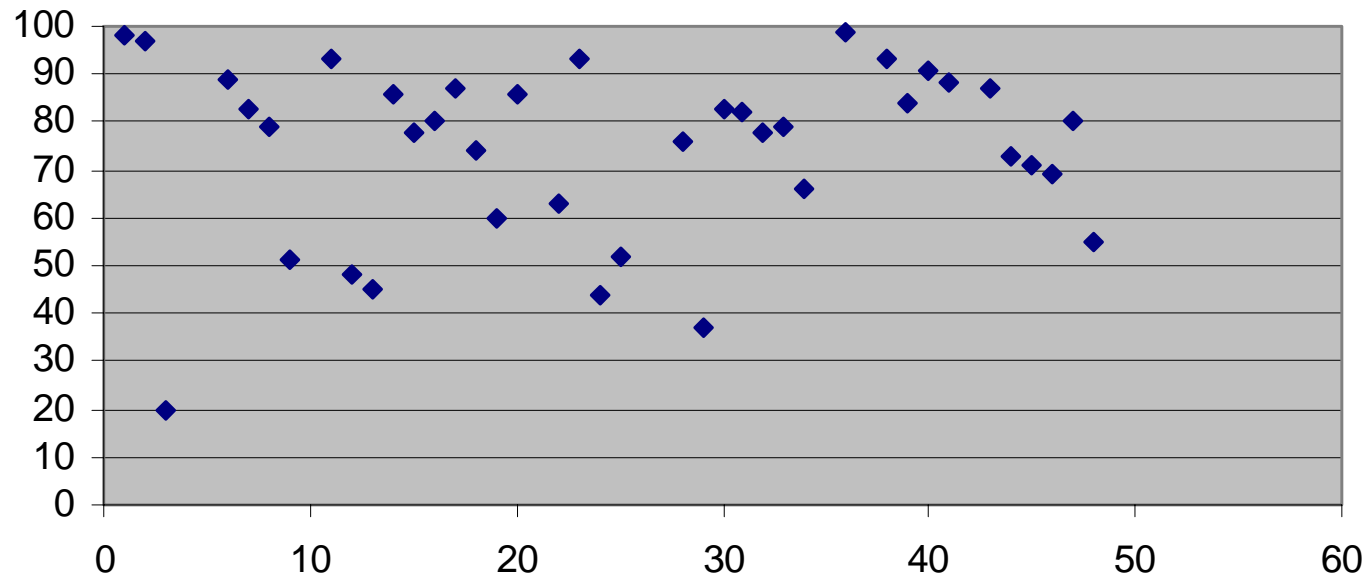
Instructor: Dr. R. Michael Buehrer  
Lecture #20: Additional examples of the  
Inverse LT  
Applications of the LT



# Midterm 3



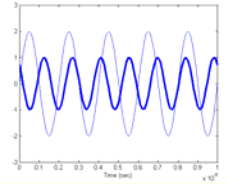
Exam 3



■ Mean = 74  
■ Median 79

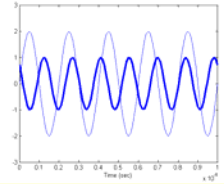
90-100	7
80-89	12
70-79	9
60-69	3
< 60	9

# Overview



- Today we look at a few additional examples of taking the inverse Laplace Transform
- Additionally, we will look at the application of the Laplace Transform for solving differential equations, examining stability and examining feedback systems
- What to read – Section 9.6, 10.1-10.4 in the text

# Example



- Find the inverse Laplace Transform of

$$H(s) = \frac{10s^2}{(s+1)(s+3)} e^{-s}$$

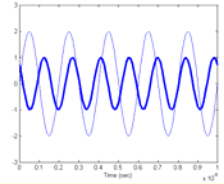
- **Solution:**

- The coefficient in front of  $e^{-s}$  is an improper fraction (power of  $s$  in the numerator is not less than the power of  $s$  in the denominator). Thus, we must first divide it out

$$\begin{array}{r} 10 \\ s^2 + 4s + 3 \overline{) 10s^2} \\ \underline{10s^2 \quad +40s \quad +30} \\ -40s \quad -30 \end{array}$$

$$H(s) = \left[ 10 - \frac{40s + 30}{(s+1)(s+3)} \right] e^{-s}$$

# Example – cont.



- The next step is to expand the proper fraction in partial fractions:

$$H(s) = \left[ 10 + \frac{A}{(s+1)} + \frac{B}{(s+3)} \right] e^{-s}$$

$$A = (s+1) \left[ \frac{40s+30}{(s+1)(s+3)} \right] \Bigg|_{s=-1}$$

$$= \left[ \frac{40s+30}{(s+3)} \right]_{s=-1}$$

$$= -5$$

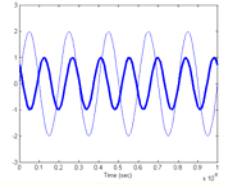
$$B = (s+3) \left[ \frac{40s+30}{(s+1)(s+3)} \right] \Bigg|_{s=-3}$$

$$= \left[ \frac{40s+30}{(s+1)} \right]_{s=-3}$$

$$= 45$$

$$H(s) = \left[ 10 - \frac{5}{(s+1)} + \frac{45}{(s+3)} \right] e^{-s}$$

# Example – cont.



- First, recall the time-shift property of the Laplace transform:

$$x(t - t_o) \xleftrightarrow{\mathcal{L}} e^{-st_o} X(s)$$

- Thus, if  $H(s) = G(s)e^{-s}$  then  $h(t) = g(t-1)$
- Using the two transform pairs

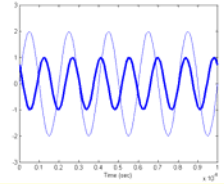
$$e^{-at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s + a}$$

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1$$

- We can write  $g(t) = 10\delta(t) + (5e^{-t} - 45e^{-3t})u(t)$

$$h(t) = 10\delta(t-1) + (5e^{-(t-1)} - 45e^{-3(t-1)})u(t-1)$$

# Example 2



- Find the inverse Laplace Transform of

$$H(s) = \frac{6s + 10}{s^2 + 4s + 5}$$

- **Solution:**

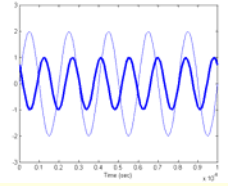
- We could factor the denominator into complex roots, but it is easier to recall the transform pairs

$$e^{-\alpha t} \cos(\omega_o t) u(t) \xleftrightarrow{\mathcal{L}} \frac{s + \alpha}{(s + \alpha)^2 + \omega_o^2} \quad \sigma > -\alpha$$

$$e^{-\alpha t} \sin(\omega_o t) u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_o}{(s + \alpha)^2 + \omega_o^2} \quad \sigma > -\alpha$$

thus we should attempt to factor the denominator into the form  $(s+a)^2 + b^2$

## Example 2 – cont.

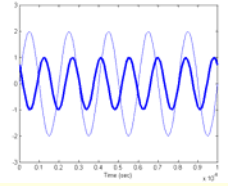


- Thus, we have

$$\begin{aligned} H(s) &= \frac{6s + 10}{s^2 + 4s + 5} \\ &= \frac{6s + 10}{(s^2 + 4s + 4) + 1} \\ &= \frac{6s + 10}{(s + 2)^2 + 1} \end{aligned}$$

- To get it into a form that we can use with the previous transform pairs we need either  $s+2$  in the numerator or 1 in the numerator.

## Example 2 – cont.



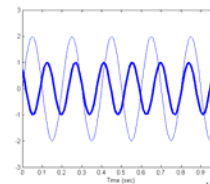
■ Continuing:

$$\begin{aligned} H(s) &= \frac{6s + 10}{(s + 2)^2 + 1} \\ &= \frac{6(s + 2) - 2}{(s + 2)^2 + 1} \\ &= 6 \frac{s + 2}{(s + 2)^2 + 1} - 2 \frac{1}{(s + 2)^2 + 1} \end{aligned}$$

- Now, comparing to the previous transform pairs we have:

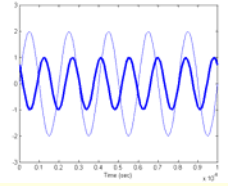
$$h(t) = \left( 6e^{-2t} \cos(t) - 2e^{-2t} \sin(t) \right) u(t)$$

# Solution of Differential Equations



- One of the most powerful uses of the Laplace Transform is in the analysis of linear systems.
- Often linear systems are described by linear differential equations which are easily solved using Laplace Transforms since the time derivative is described by the multiplication by  $s$  in the Laplace Domain
- This is also true of the Fourier Transform
- However, the unilateral Laplace Transform is more useful since (a) it naturally applies to functions which are causal and (b) it can handle forcing functions which are unbounded by time.

# Example 3



- Solve the differential equation

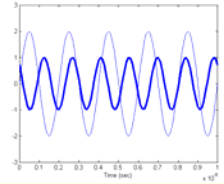
$$\frac{d^2}{dt^2}[x(t)] + 7 \frac{d}{dt}[x(t)] + 12x(t) = 0$$

for times  $t > 0$ , given the initial conditions

$$x(0^-) = 2$$

$$\left[ \frac{d}{dt} x(t) \right]_{t=0^-} = -4$$

# Example 3 – cont.



## Solution:

The first step is to Laplace Transform both sides of the equation

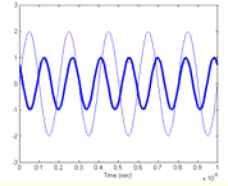
$$\frac{d^2}{dt^2}[x(t)] + 7\frac{d}{dt}[x(t)] + 12x(t) = 0$$

$$\left\{ s^2 X(s) - sx(0^-) - \frac{d}{dt}[x(t)]_{t=0^-} \right\} + 7 \left\{ sX(s) - x(0^-) \right\} + 12X(s) = 0$$

Solving for  $X(s)$ :

$$\begin{aligned} X(s) &= \frac{sx(0^-) + \frac{d}{dt}[x(t)]_{t=0^-} + 7x(0^-)}{s^2 + 7s + 12} \\ &= \frac{2s + 10}{s^2 + 7s + 12} \end{aligned}$$

# Example 3 – cont.



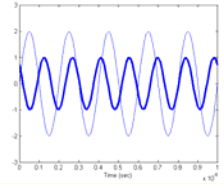
- Continuing:

$$\begin{aligned} X(s) &= \frac{2s + 10}{s^2 + 7s + 12} \\ &= \frac{A}{s + 4} + \frac{B}{s + 3} \\ &= -\frac{2}{s + 4} + \frac{4}{s + 3} \end{aligned}$$

- Which has a transform

$$h(t) = (4e^{-3t} - 2e^{-4t})u(t)$$

# Example 3 – cont.



- As a check we can substitute the solution into the original differential equation for times  $t \geq 0$

$$\frac{d^2}{dt^2}[x(t)] + 7\frac{d}{dt}[x(t)] + 12x(t) = 0$$

$$\frac{d^2}{dt^2}[(4e^{-3t} - 2e^{-4t})] + 7\frac{d}{dt}[(4e^{-3t} - 2e^{-4t})] + 12(4e^{-3t} - 2e^{-4t}) = 0$$

$$\frac{d}{dt}[(-12e^{-3t} + 8e^{-4t})] + 7[(-12e^{-3t} + 8e^{-4t})] + (48e^{-3t} - 24e^{-4t}) = 0$$

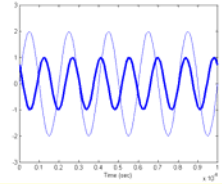
$$[(36e^{-3t} - 32e^{-4t})] + [(-84e^{-3t} + 56e^{-4t})] + (48e^{-3t} - 24e^{-4t}) = 0$$

$$0 = 0$$

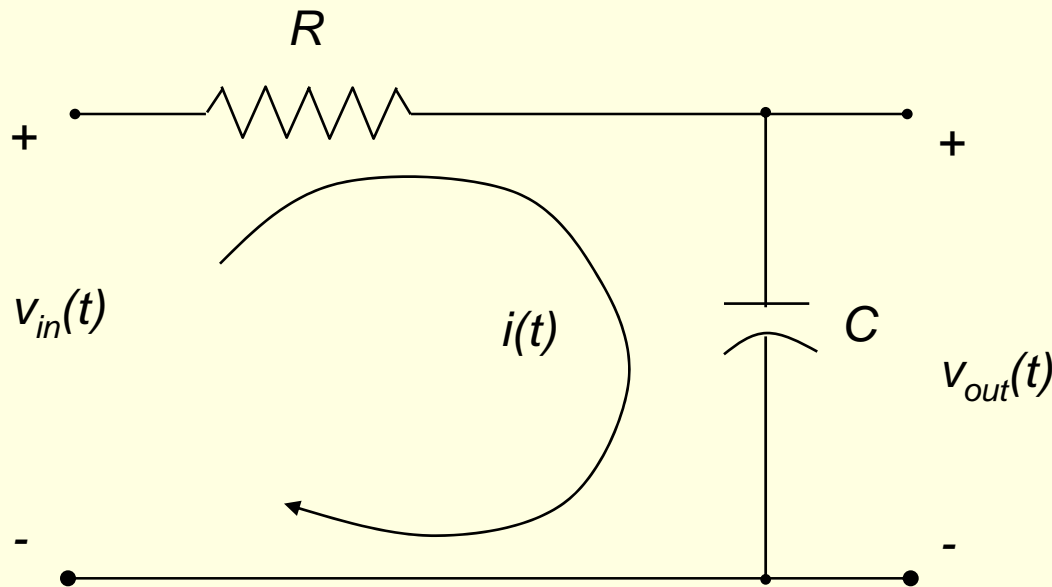
- Checking the initial conditions:

$$x(0^-) = 4 - 2 = 2 \quad \left[ \frac{d}{dt} x(t) \right]_{t=0^-} = -12 + 8 = -4$$

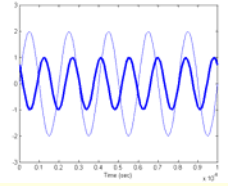
# Example 4



- The lowpass filter shown below is excited by a unit voltage impulse at time  $t = t_0$  where  $t_0 > 0$ . Find the response  $v_{out}(t)$  without assuming that the initial voltage  $v_{out}(t)$  is zero. Also, check the initial output voltage from the result.



# Example 4 – cont.



- **Solution:** Using Kirchoff's Voltage law we can write

$$\begin{aligned}v_{in}(t) &= i(t)R + v_{out}(t) \\ &= v'_{out}(t)RC + v_{out}(t)\end{aligned}$$

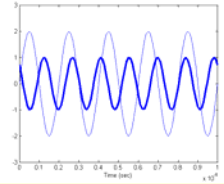
- Now, taking the Laplace Transform:

$$\begin{aligned}v_{in}(t) &= v'_{out}(t)RC + v_{out}(t) \\ V_{in}(s) &= [sV_{out}(s) - v_{out}(0^-)]RC + V_{out}(s)\end{aligned}$$

- Re-arranging the equation for  $V_{out}(s)$ :

$$V_{out}(s) = \frac{V_{in}(s) + RCv_{out}(0^-)}{sRC + 1}$$

# Example 4 – cont.



- For an impulse excitation, we have

$$V_{in}(s) = 1$$

- However, for an arbitrary time delay  $t_o$ :

$$V_{in}(s) = e^{-t_o s}$$

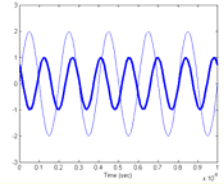
- Thus, we have at the output

$$V_{out}(s) = \frac{e^{-t_o s} + RCv_{out}(0^-)}{sRC + 1}$$

- To find the inverse Laplace Transform let us rewrite the equation as

$$\begin{aligned} V_{out}(s) &= e^{-t_o s} \frac{1}{sRC + 1} + RCv_{out}(0^-) \frac{1}{sRC + 1} \\ &= \frac{e^{-t_o s}}{RC} \frac{1}{s + 1/RC} + v_{out}(0^-) \frac{1}{s + 1/RC} \end{aligned}$$

# Example 4 – cont.



- Continuing...

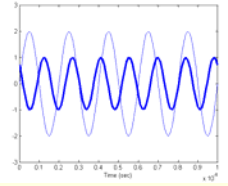
$$V_{out}(s) = \frac{e^{-t_0 s}}{RC} \frac{1}{s + 1/RC} + v_{out}(0^-) \frac{1}{s + 1/RC}$$

- Now taking the inverse Laplace Transform

$$v_{out}(t) = \frac{1}{RC} e^{-(t-t_0)/RC} u(t-t_0) + v_{out}(0^-) e^{-(t-t_0)/RC} u(t)$$

- Thus, there are two parts to the response, an exponential decay due to the initial voltage  $v_{out}(0^-)$  and the response to the impulse at  $t_0$ .

# Example 4 – cont.



- What is the initial value of the output voltage?
- **Solution:** The initial value theorem says that

$$x(0^+) = \lim_{s \rightarrow \infty} [sX(s)]$$

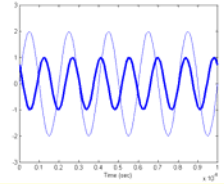
- From the solution we have

$$V_{out}(s) = \frac{e^{-t_0 s}}{RC} \frac{1}{s + 1/RC} + v_{out}(0^-) \frac{1}{s + 1/RC}$$

- Thus

$$\begin{aligned} x(0^+) &= \lim_{s \rightarrow \infty} [sV_{out}(s)] \\ &= \lim_{s \rightarrow \infty} \left[ s \left( \frac{e^{-t_0 s}}{RC} \frac{1}{s + 1/RC} + v_{out}(0^-) \frac{1}{s + 1/RC} \right) \right] \\ &= \lim_{s \rightarrow \infty} \left[ \left( \frac{e^{-t_0 s}}{RC} \frac{s}{s + 1/RC} + v_{out}(0^-) \frac{s}{s + 1/RC} \right) \right] \\ &= v_{out}(0^-) \end{aligned}$$

# Example 4 – cont.

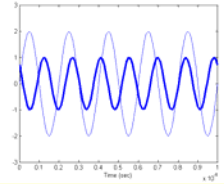


- However, what if the impulse is applied at  $t_0=0$ ?

$$\begin{aligned}x(0^+) &= \lim_{s \rightarrow \infty} [sV_{out}(s)] \\&= \lim_{s \rightarrow \infty} \left[ s \left( \frac{e^{-t_0 s}}{RC} \frac{1}{s + 1/RC} + v_{out}(0^-) \frac{1}{s + 1/RC} \right) \right] \\&= \lim_{s \rightarrow \infty} \left[ \left( \frac{1}{RC} \frac{s}{s + 1/RC} + v_{out}(0^-) \frac{s}{s + 1/RC} \right) \right] \\&= \frac{1}{RC} + v_{out}(0^-)\end{aligned}$$

- Which captures the fact that the impulse applies an initial voltage when it is applied at zero.

# Using the Laplace Transform to Examine System Stability



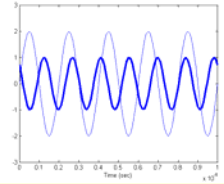
- Consider a linear system which can be described by differential equations of the form

$$\sum_{k=0}^D a_k \frac{d^k}{dt^k} \{y(t)\} = \sum_{k=0}^N b_k \frac{d^k}{dt^k} \{x(t)\}$$

- Such a system can be represented by a transfer function (in the Laplace domain)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^D a_k s^k}{\sum_{k=0}^N b_k s^k}$$

# Stability – cont.



- The denominator can be factored as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^D a_k s^k}{\prod_{k=0}^N (s - p_k)}$$

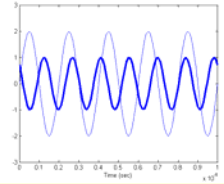
- If none of the poles are repeated we can write the transfer function as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \dots + \frac{K_D}{s - p_D}$$

- The impulse response is then

$$h(t) = (K_1 e^{p_1 t} + K_2 e^{p_2 t} + \dots + K_D e^{p_D t}) u(t)$$

# Stability – cont.



- For a system to be *stable*, its impulse response must be *absolutely integrable*.

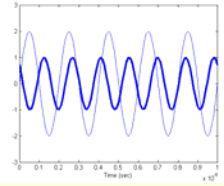
- Recall that in the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \dots + \frac{K_D}{s - p_D}$$

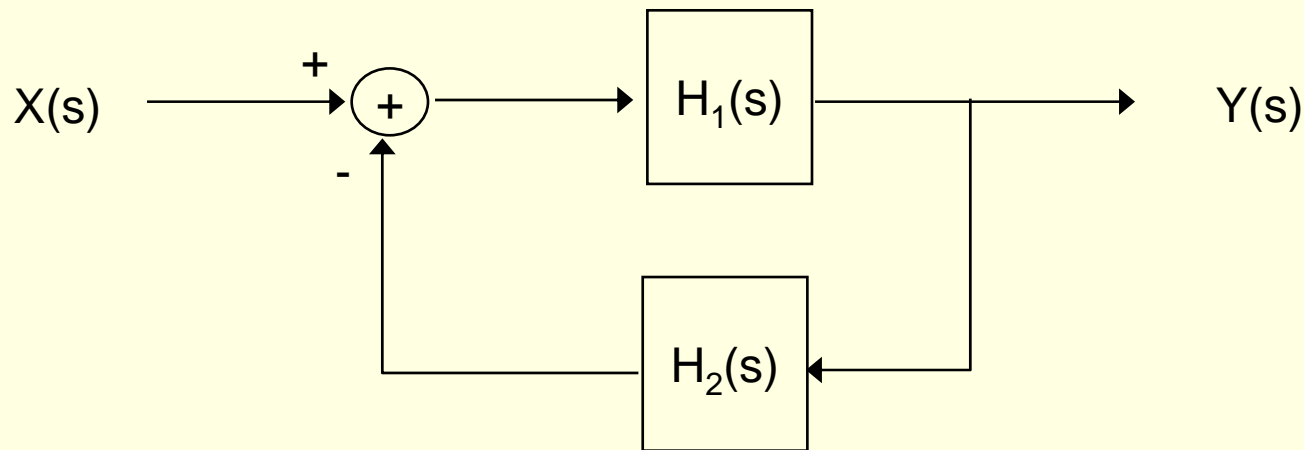
$p_i$  are the poles of the transfer function.

- In order for the impulse response to be absolutely integrable, the real part of each  $p_i$  must be *negative*.
- Therefore, the poles must lie in the Left Hand Plane (LHP) which is the plane to the left (not including) the  $j\omega$  axis.

# Systems with Feedback

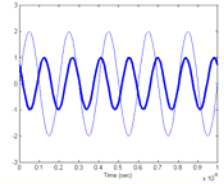


- Consider two systems which have transfer functions  $H_1(s)$  and  $H_2(s)$ . Let us arrange these two systems such that the output of one is fed into the input of the other:



- This is referred to as a negative feedback connection of systems since the output is negated and fed back into the input

# Transfer Function



- The transfer function of the overall system can be found in terms of the transfer functions of the individual systems. First we define the output of the summation as

$$E(s) = X(s) - H_2(s)Y(s)$$

- Further we can write

$$Y(s) = H_1(s)E(s)$$

- Combining the two equations we have

$$\frac{Y(s)}{H_1(s)} = X(s) - H_2(s)Y(s)$$

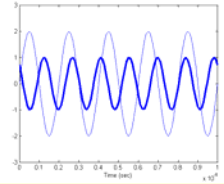
- Solving for  $Y(s)/X(s)$  we have

$$Y(s)(1 + H_1(s)H_2(s)) = H_1(s)X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Often called  
the “Loop  
Transfer  
Function”

# Open Loop



- Notice that if the feedback goes to zero the overall transfer function

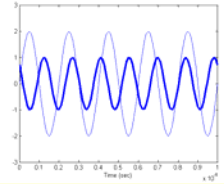
$$\frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

collapses to the forward path gain

$$\begin{aligned}\frac{Y(s)}{X(s)} &= \frac{H_1(s)}{1 + H_1(s)*0} \\ &= H_1(s)\end{aligned}$$

This is termed *open-loop*

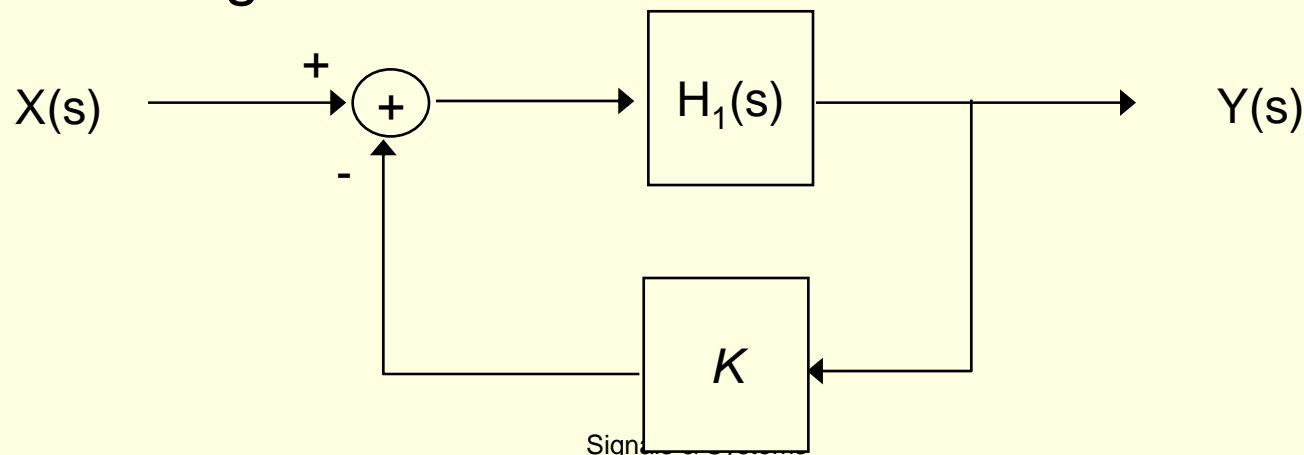
# Stabilizing a System with Feedback



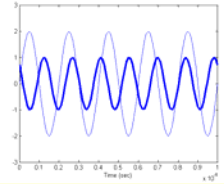
- Consider a system with transfer function

$$H_1(s) = \frac{1}{s-2}$$

- As we have stated before, since this system has poles in the right hand plane ( $s=2$ ) the system is unstable. (Note that the response of the system to an impulse is a growing exponential.)
- However, consider a simple feedback system with a constant gain



# Stabilizing a system – cont.

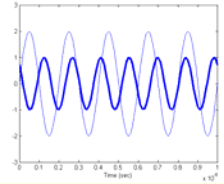


- The new transfer function is now

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \\ &= \frac{1}{s-2} \\ &= \frac{1}{1 + K \frac{1}{s-2}} \\ &= \frac{1}{s-2+K} \end{aligned}$$

- Thus, provided that  $K > 2$ , we have stabilized the system

# Summary



- In this lecture we have examined a couple more examples of the inverse Laplace Transform and discussed a couple of applications of the Laplace transform including
  - Solving differential equations
  - Examining stability
  - Examining feedback systems
- We will continue this discussion in the next lecture