ECE 2704
Signals and Systems
Spring 2006

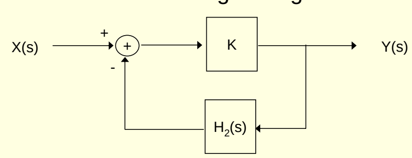
Instructor: Dr. R. Michael Buehrer
Lecture #21: Applications of the LT

Overview

- Today we look at a the application of the Laplace Transform for examining feedback systems. Further we examine system responses to common inputs and using the pole-zero diagrams for calculating the frequency response
- What to read – Sections 10.5, 10.7, 10.8 in the text

Inversion

- A common use of feedback it inversion
- Consider the following arrangement



- The overall transfer function is

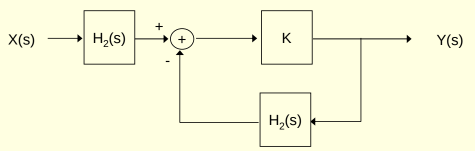
$$\frac{Y(s)}{X(s)} = \frac{K}{1 + KH_2(s)}$$

Inversion – cont.

- For large values of K than for at least some range of values for s we have

$$KH_2(s) >> 1$$

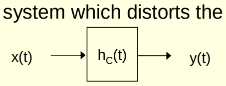
- Thus if we have $\frac{Y(s)}{X(s)} \approx \frac{1}{H_2(s)}$



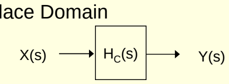
the overall gain is unity. What is the usefulness of this?

Equalization

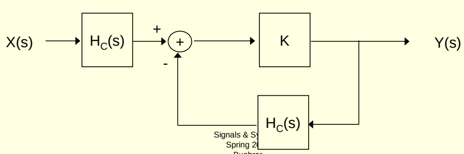
- Consider a system which distorts the signal of interest



- In the Laplace Domain

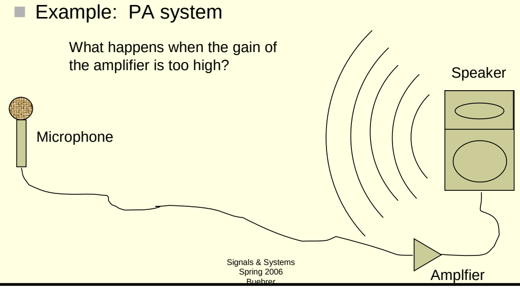


- To invert the impact of the channel:



Positive Feedback

- Just as negative feedback can introduce stability into an unstable system, positive feedback can introduce instability into a stable system.
- Example: PA system



Positive Feedback – cont.

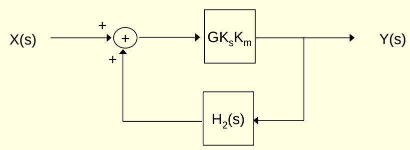
- Let's model this system.
- Let
  - the input signal be $x_{in}(t)$
  - the sound coming out of the speaker be $x_{out}(t)$
  - G = gain of the amplifier, K_m = gain of the microphone and K_s be the gain of the speaker
  - Since the sound from the speaker can travel back to the microphone (at speed v) we have a positive feedback system where the feedback signal is

$$\frac{1}{d^2} x_s\left(t - \frac{d}{v}\right)$$

- The Laplace transform of this is $\underbrace{\frac{1}{d^2} e^{-\frac{d}{v}s}}_{H_2(s)} X_s(s)$

Positive Feedback – cont.

- Thus, we can write a system model as



- The transfer function can be written as

$$\frac{Y(s)}{X(s)} = \frac{GK_s K_m}{1 - GK_s K_m H_2(s)}$$

Positive Feedback – cont.

- Substituting for the feedback transfer function:

$$\frac{Y(s)}{X(s)} = \frac{GK_s K_m}{1 - GK_s K_m H_2(s)}$$

$$= \frac{GK_s K_m}{1 - GK_s K_m \frac{1}{d^2} e^{-(d/v)s}}$$

- We can determine the stability of the system by examining the poles of the transfer function.
- We desire the poles to be in the left-half plane for system stability. The poles are the values of s for which the denominator is zero

$$1 - GK_s K_m \frac{1}{d^2} e^{-(d/v)s} = 0$$

Positive Feedback – cont.

- Examining the roots of $1 - GK_sK_m \frac{1}{d^2} e^{-(d/v)s} = 0$

$$GK_sK_m \frac{1}{d^2} e^{-(d/v)s} = 1$$

$$e^{-(d/v)s} = \frac{d^2}{GK_sK_m}$$

$$-\left(\frac{d}{v}\right)s = \ln\left(\frac{d^2}{GK_sK_m}\right)$$

$$s = -\left(\frac{v}{d}\right)\ln\left(\frac{d^2}{GK_sK_m}\right)$$

Positive Feedback – cont.

- The root is all real

$$s = -\left(\frac{v}{d}\right)\ln\left(\frac{d^2}{GK_sK_m}\right)$$

- However, any value of *s* of the form

$$s = -\left(\frac{v}{d}\right)\left[\ln\left(\frac{d^2}{GK_sK_m}\right) + jn2\pi\right]$$

for integer values of *n* would be a root of the equation. Thus, the system has an infinite number of poles.

What does stability require?

Positive Feedback – cont.

- The poles must lie in the left-half plane for stability. This means that Re(s) < 0. That is:

$$-\left(\frac{v}{d}\right)\ln\left(\frac{d^2}{GK_sK_m}\right) < 0$$

$$\ln\left(\frac{d^2}{GK_sK_m}\right) > 0$$

$$d^2 > GK_sK_m$$

- Or in other words the loss in the feedback path (d^2) must be greater than the gain through the amplifer/microphone and speaker. (This limits how high we can increase the gain)
- This should make sense. If the signal arriving at the microphone is larger than (or equal) to the signal coming in, an infinite gain will ultimately result.
- In real systems the gain is finite, but we still hear a very large tone.

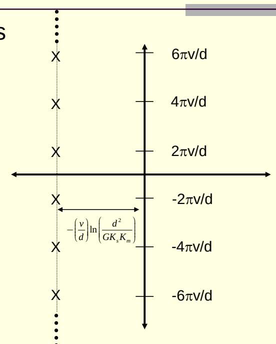
System poles

- Plotting the poles

If the poles are on the axis, the system will oscillate with frequencies ω=2πnv/d

What happens as we turn up the gain of the amplifier?

What happens as we move the speaker closer to the microphone?



System Response to Unit Step

- The response *y(t)* of a system to any input *x(t)* can be found from the inverse Laplace transform of

$$Y(s) = X(s)H(s)$$

where *H(s)* is the system transfer function

- For a unit step input $X(s) = \frac{1}{s}$
- The transfer function can be written as the ratio of two polynomials in s N(s)/D(s). The general output is

$$Y(s) = \frac{1}{s}H(s)$$
$$= \frac{N(s)}{sD(s)}$$

Response to a unit step – cont.

- The output can then be factored as

$$Y(s) = \frac{N(s)}{sD(s)}$$
$$= \frac{K}{s} + \frac{N_1(s)}{D(s)}$$

- where *K = H(0)*.
- If the system is stable, the poles of *H(s)* will be in the left-half plane. Thus, the term $N_1(s)/D(s)$ will have poles in the left-half plane and will decay to zero as *t* approaches infinity. This term is called the *transient response.*

Unit step response – cont.

- Let $$H(s) = \frac{A}{1 - s/p}$$
- This system has a single pole as s=p. If p<0, the system is stable.
- The response to a unit step is

$$\begin{aligned} Y(s) &= \frac{A}{s(1 - s/p)} \\ &= \frac{K}{s} + \frac{B}{1 - s/p} \\ &= \frac{K}{s} - \frac{C}{s - p} \end{aligned}$$

Step response – cont.

- Continuing…

$$\begin{aligned} Y(s) &= \frac{K}{s} - \frac{C}{s - p} \\ &= \frac{A}{s} - \frac{A}{s - p} \end{aligned}$$

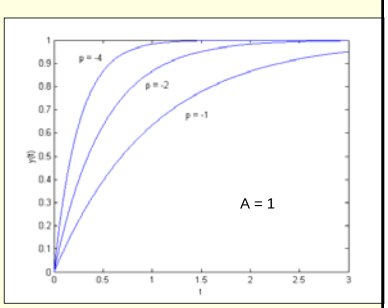
- Taking the inverse Laplace Transform

$$y(t) = A\left(1 - e^{pt}\right)u(t)$$

- If *p* is positive, the system is unstable and the response to a unit step increases without bound.
- If *p* is negative, the system is stable and the step response approaches a constant *A*
- Thus, the output contains a transient response that dies out and a constant

Step response – cont.

- The larger the magnitude of p, the faster the convergence to A.
- We define 1/p as the time constant of the system.



Response to a sinusoid

- Now, let the input to a system with transfer function H(s) by $x(t) = \cos(\omega_o t)u(t)$
- Again we can write the transfer function as

$$H(s) = \frac{N(s)}{D(s)}$$

- The output can be written as

$$Y(s) = \frac{N(s)}{D(s)} \frac{s}{s^2 + \omega_o^2}$$
$$= \frac{N_1(s)}{D(s)} + \frac{A}{s + j\omega_o} + \frac{B}{s - j\omega_o}$$

where it can be readily shown that

$$A = \frac{H(-j\omega_o)}{2}, B = \frac{H(j\omega_o)}{2}$$

Response to a sinusoid – cont.

- Since $H(-j\omega_o) = H^*(j\omega_o)$ and recombining the last two fractions we have

$$Y(s) = \frac{N_1(s)}{D(s)} + \frac{1}{2} \frac{H^*(j\omega_o)(s - j\omega_o) + H(j\omega_o)(s + j\omega_o)}{s^2 + \omega_o^2}$$
$$= \frac{N_1(s)}{D(s)} + \mathrm{Re}\{H(j\omega_o)\} \frac{s}{s^2 + \omega_o^2} - \mathrm{Im}\{H(j\omega_o)\} \frac{\omega_o}{s^2 + \omega_o^2}$$

- Taking the inverse Laplace Transform we have

$$y(t) = \mathcal{L}^{-1}\left\{\frac{N_1(s)}{D(s)}\right\} + \mathrm{Re}\{H(j\omega_o)\}\cos(\omega_o t)u(t) - \mathrm{Im}\{H(j\omega_o)\}\sin(\omega_o t)u(t)$$

Response to a sinusoid – cont.

- Rewriting the response

$$y(t) = \mathcal{L}^{-1}\left\{\frac{N_1(s)}{D(s)}\right\} + \mathrm{Re}\{H(j\omega_o)\}\cos(\omega_o t)u(t) - \mathrm{Im}\{H(j\omega_o)\}\sin(\omega_o t)u(t)$$

- If the system is stable, the first term will be composed of terms which will decay to zero as time goes to infinity. This is the *transient response.*
- The second and third terms can be re-written as

$$\mathrm{Re}\{H(j\omega_o)\}\cos(\omega_o t)u(t) - \mathrm{Im}\{H(j\omega_o)\}\sin(\omega_o t)u(t) =$$
$$|H(j\omega_o)|\cos(\omega_o t + \angle H(j\omega_o))u(t)$$

- Thus, the response to a sinusoid is a transient plus a scaled and phase-shifted sinusoid.

Example

- Find the response of the system

$$H(s)=\frac{10}{s+10}$$

to a unit amplitude cosine of frequency 2Hz which starts at *t=0.*

SOLUTION:

$$Y(s)=\frac{10}{s+10}\frac{s}{s^2+(4\pi)^2}$$

$$=\frac{K}{s+10}+\mathrm{Re}\{H(j4\pi)\}\frac{s}{s^2+\omega_o^2}-\mathrm{Im}\{H(j4\pi)\}\frac{4\pi}{s^2+\omega_o^2}$$

Example – cont.

- Solving for *K*:

$$K=\left[\frac{10}{s+10}\frac{s}{s^2+(4\pi)^2}(s+10)\right]\Bigg|_{s=-10}$$

$$=\left[\frac{10s}{s^2+(4\pi)^2}\right]\Bigg|_{s=-10}$$

$$=-0.388$$

$$Y(s)=-\frac{0.388}{s+10}+0.388\frac{s}{s^2+(4\pi)^2}+0.487\frac{4\pi}{s^2+(4\pi)^2}$$

- Taking the inverse Laplace Transform

$$y(t)=\left(-0.388e^{-10t}+0.388\cos(4\pi t)+0.487\sin(4\pi t)\right)u(t)$$

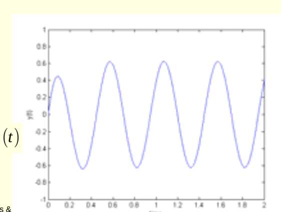
Example – cont.

- Repeating the solution

$$y(t)=\left(-0.388e^{-10t}+0.388\cos(4\pi t)+0.487\sin(4\pi t)\right)u(t)$$

steady-state response

transient response



$$y(t)=\left(-0.388e^{-10t}+0.62\cos(4\pi t-0.9)\right)u(t)$$

Comparison with Fourier Domain

- Consider the same system in the Fourier Domain:

$$H(f)=\frac{10}{10+j2\pi f}$$

- A cosine (that has been on for all time) has Fourier Transform

$$X(f)=\frac{1}{2}\{\delta(f-2)+\delta(f+2)\}$$

- The system output is

$$H(f)=\left(\frac{10}{10+j2\pi f}\right)\frac{1}{2}\{\delta(f-2)+\delta(f+2)\}$$

$$=\frac{1}{2}\left\{\left(\frac{10}{10+j4\pi}\right)\delta(f-2)+\left(\frac{10}{10-j4\pi}\right)\delta(f+2)\right\}$$

Comparison with the FT – cont.

- Continuing

$$H(f)=\frac{1}{2}\left\{\left(\frac{10}{10+j4\pi}\right)\delta(f-2)+\left(\frac{10}{10-j4\pi}\right)\delta(f+2)\right\}$$

$$=\frac{1}{2}\frac{10\left([10-j4\pi]\delta(f-2)+[10+j4\pi]\delta(f+2)\right)}{\left(100+16\pi^2\right)}$$

$$=\frac{1}{2}\frac{100\left(\delta(f-2)+\delta(f+2)\right)}{\left(100+16\pi^2\right)}+\frac{1}{2}\frac{j40\pi\left(\delta(f+2)-\delta(f-2)\right)}{\left(100+16\pi^2\right)}$$

$$y(t)=0.388\cos(4\pi t)+0.48\sin(4\pi t)$$

- Which is the same as the *steady state output.*
- This makes sense since we ignored the “turning on” of the sinusoid.

Frequency Response

- For a *stable system* the frequency response $H_F(f)$ can be found from the Laplace transform *H(s)* by substituting s = j2πf:

$$H_F(f)=H(j2\pi f)$$

- As stated previously, we can factor the transfer function of a system as

$$H(s)=\frac{(s-z_1)(s-z_2)\dots(s-z_N)}{(s-p_1)(s-p_2)\dots(s-p_D)}$$

where $N < D$

- Thus, the frequency response can be written as

$$H_F(f)=\frac{(j2\pi f-z_1)(j2\pi f-z_2)\dots(j2\pi f-z_N)}{(j2\pi f-p_1)(j2\pi f-p_2)\dots(j2\pi f-p_D)}$$

Frequency Response

- It is also common to express the frequency response in terms of the radian frequency $\omega=2\pi f$

$$H(j\omega)=\frac{(j\omega-z_1)(j\omega-z_2)\ldots(j\omega-z_N)}{(j\omega-p_1)(j\omega-p_2)\ldots(j\omega-p_D)}$$

- We can graph the frequency response using this form by examining vectors in the *s*-plane.
- Specifically,

$$|H(j\omega)|=\frac{|j\omega-z_1||j\omega-z_2|\ldots|j\omega-z_N|}{|j\omega-p_1||j\omega-p_2|\ldots|j\omega-p_D|}$$

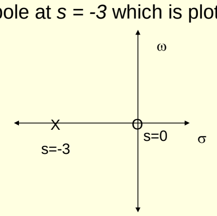
$$\angle H(j\omega)=\angle(j\omega-z_1)+\angle(j\omega-z_2)+\ldots\angle(j\omega-z_N)$$
$$-\angle(j\omega-p_1)-\angle(j\omega-p_2)-\ldots\angle(j\omega-p_D)$$

Example

- Find the frequency response of the transfer function

$$H(s)=\frac{3s}{s+3}$$

- Solution: This transfer function has one zero at *s = 0*, and one pole at *s = -3* which is plotted below.

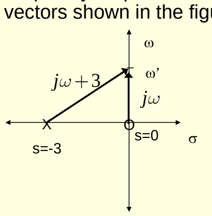


Example – cont.

- Now the magnitude of the frequency response is

$$|H(j\omega)|=\frac{|3j\omega|}{|j\omega+3|}$$

- For an arbitrary value of ω', the magnitude of the frequency response is the ratio of the length of two vectors shown in the figure below



As w goes from –infinity to infinity, the length of the vector jω starts with a very large magnitude and decreases until it reaches its smallest value (0) at ω = 0 and starts increasing again.
The vector jω+3 has similar behavior except that its minimum is 3.

Example – cont.

- The phase response can be written as

$$\angle H(j\omega) = \angle 3j\omega - \angle(j\omega + 3)$$

- For negative values of ω, we have

$$\angle H(j\omega) = -\frac{\pi}{2} - \angle(j\omega + 3)$$

which will be zero for sufficiently large negative values of ω and will approach -π/2 as w approaches zero

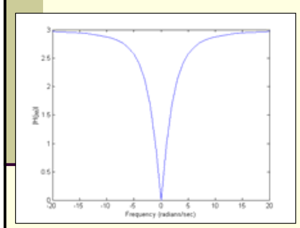
- For positive values of ω, we have

$$\angle H(j\omega) = \frac{\pi}{2} - \angle(j\omega + 3)$$

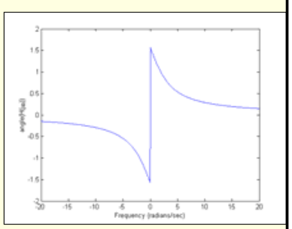
which will be zero for sufficiently large negative values of ω and will approach π/2 as w approaches zero

Example – cont.

$$|H(j\omega)|$$



$$\angle H(j\omega)$$



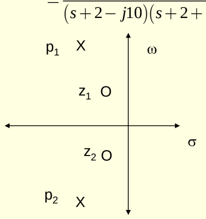
Example 2

- Consider a transfer function

$$H(s) = \frac{s^2 + 2s + 17}{s^2 + 4s + 104}$$

$$= \frac{(s + 1 - j4)(s + 1 + j4)}{(s + 2 - j10)(s + 2 + j10)}$$

z_1 = -1+j4
z_2 = -1-j4
p_1 = -2+j10
p_2 = -2-j10



Example – cont.

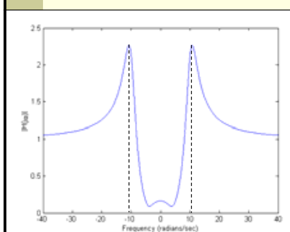
- The magnitude response can be written as

$$|H(j\omega)| = \frac{|j\omega + 1 - j4||j\omega + 1 + j4|}{|j\omega + 2 - j10||j\omega + 2 - j10|}$$

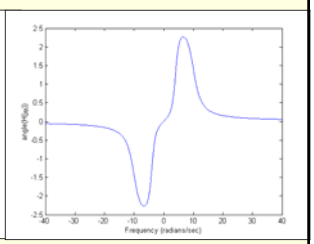
- When the frequency is large and negative the gain is unity.
- When the frequency is near zero, the gain is small (approximately 0.15)
- The minimum values should occur near ω=+/-4 while the maximum values should occur near the poles ω = +/-10.

Example – cont.

$|H(j\omega)|$



$\angle H(j\omega)$



Summary

- In this lecture we have examined the application of the Laplace Transform to positive and negative feedback system analysis
- We also examined the output of systems to standard inputs (unit step, sinusoid)
- Finally we examined the analysis of the frequency response using a graphical interpretation of the Laplace Transform