ECE 2704
Signals and Systems
Spring 2006

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Lecture #4: Convolution

Overview

- What to read – Section 3.6 in the text
- This lecture (as well as the next few lectures) deals with the concept of convolution.
- Convolution allows us to determine the output of any linear time-invariant system through the system *impulse response*. Thus, it is an important concept.
- We will examine convolution both mathematically and graphically

Definition

- The convolution of two functions is defined as

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

- Convolution is the integral of a function $x_1(t)$ and a second function $x_2(t)$ which is inverted in time and time shifted. Note that the result of convolution is a function where the argument of the function is equivalent to the time shift.

Symmetry of Convolution

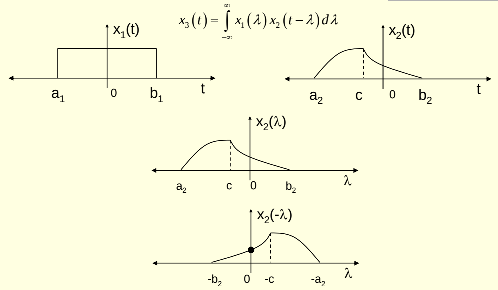
$$x_3(t) = x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

- Proof: $x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$
- Let $\beta = t - \lambda \rightarrow \lambda = t - \beta$, $d\lambda = -d\beta$

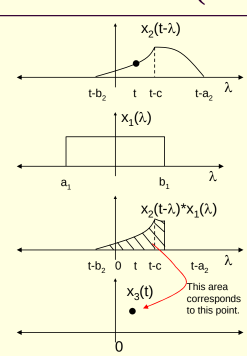
$$x_3(t) = -\int_{\infty}^{-\infty} x_1(t-\beta)\, x_2(\beta)\, d\beta = \int_{-\infty}^{\infty} x_2(\beta)\, x_1(t-\beta)\, d\beta = x_2(t) * x_1(t)$$

Graphical Illustration

$$x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$



Graphical Illustration (cont.)



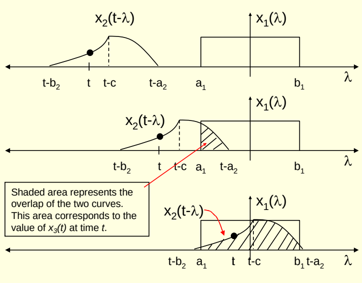
Graph is shifted in time by *t*. The point originally at 0 is at *t*

Convolution integral evaluated at *t* is simply the area of under the multiplication of the two curves

$x_3(t)$ evaluated at a single point in time. Note, that we must evaluate this for all values of *t*

Calculating Intervals

$$x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$



$-\infty < t - a_2 \le a_1$
$-\infty < t \le a_1 + a_2$ $\quad x_3(t) = 0$

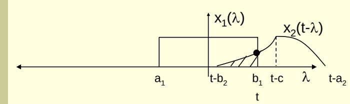
$a_1 \le t - a_2 \le b_1$
$a_1 + a_2 \le t \le b_1 + a_2$

$$x_3(t) = \int_{a_1}^{t-a_2} x_1(\lambda) x_2(t-\lambda) d\lambda$$

$t - a_2 \ge b_1, \quad t - b_2 \le a_1,$

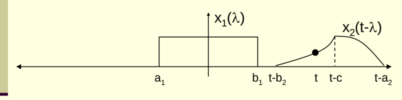
$$x_3(t) = \int_{a_1}^{b_1} x_1(\lambda) x_2(t-\lambda) d\lambda$$

Calculating Intervals (cont.)



$a_1 \le t - b_2 \le b_1$
$a_1 + b_2 \le t \le b_1 + b_2$

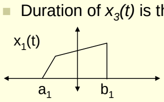
$$x_3(t) = \int_{t-b_2}^{b_1} x_1(\lambda) x_2(t-\lambda) d\lambda$$

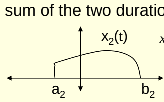


$b_1 \le t - b_2 < \infty$
$b_1 + b_2 \le t < \infty$
$x_3(t) = 0$

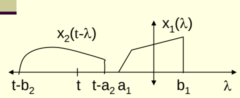
Duration of Convolution

- For two pulse-type (finite length) signals of $x_1(t)$ and $x_2(t)$:
  - Starting time of $x_3(t)$ is the sum of the starting time of the two signals $x_1(t)$ and $x_2(t)$
  - Ending time of $x_3(t)$ is the sum of the end of the two signals $x_1(t)$ and $x_2(t)$
  - Duration of $x_3(t)$ is the sum of the two durations





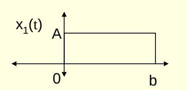
$$x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$



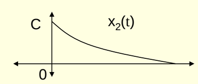
We can see that the integral will be zero until t-a_2 >= a_1 or t >= a_1+a_2
Further, the integral will be zero when t-b_2>= b_1 or t >= b_1+b_2
Duration = (b_1+b_2) – (a_1 +a_2) or (b_1-a_1) + (b_2 - a_2)

Example A

$$x_3(t) = x_1(t) * x_2(t)$$

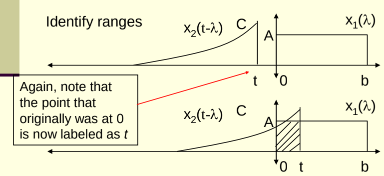


$x_1(t) = A[u(t) - u(t-b)]$



$x_2(t) = Ce^{-at}u(t)$

Identify ranges

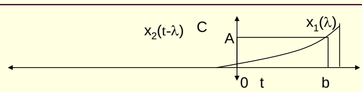


$-\infty < t \le 0 \quad x_3(t) = 0$

$0 \le t \le b$

$$x_3(t) = \int_0^t x_1(\lambda) x_2(t-\lambda) d\lambda$$

Example A (cont.)



$t \ge b$

$$x_3(t) = \int_0^b x_1(\lambda) x_2(t-\lambda) d\lambda$$

- Summarizing

$$0 \qquad -\infty < t \le 0$$

$$x_3(t) = \int_0^t x_1(\lambda) x_2(t-\lambda) d\lambda \qquad 0 \le t \le b$$

$$x_3(t) = \int_0^b x_1(\lambda) x_2(t-\lambda) d\lambda \qquad t \ge b$$

Example A (cont.)

- Evaluating the second interval ($0 \le t \le b$)

$$x_3(t) = \int_0^t x_1(\lambda) x_2(t-\lambda) d\lambda$$

$$= \int_0^t ACe^{-a(t-\lambda)} d\lambda$$

$$= ACe^{-at} \int_0^t e^{a\lambda} d\lambda$$

$$= ACe^{-at} \frac{1}{a} e^{a\lambda} \Big|_0^t$$

$$= \frac{AC}{a} e^{-at} \left(e^{at} - 1\right)$$

$$= \frac{AC}{a} \left(1 - e^{-at}\right)$$

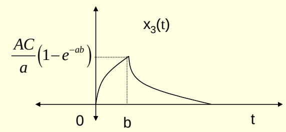
Example A (cont.)

- Evaluating the third interval (t>=b)

$$\begin{aligned} x_3(t) &= \int_0^b x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \\ &= \int_0^b ACe^{-a(t-\lambda)}\,d\lambda \\ &= ACe^{-at}\int_0^b e^{a\lambda}\,d\lambda \\ &= ACe^{-at}\,\frac{1}{a}e^{a\lambda}\Big|_0^b \\ &= \frac{AC}{a}e^{-at}\left(e^{ab}-1\right) \\ &= \frac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)} \end{aligned}$$

Example A (cont.)

$$x_3(t)=\begin{cases} 0 & -\infty < t \le 0 \\ \frac{AC}{a}\left(1-e^{-at}\right) & 0 \le t \le b \\ \frac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)} & t \ge b \end{cases}$$



Example A (cont.)

A good double check: The function $x_3(t)$ should be continuous unless there are impulses in $x_2(t)$ or $x_1(t)$. As a result, the function give the same value at the endpoints regardless of which interval I use to evaluate the point.

Ex:

$$x_3(t)=\begin{cases} 0 & -\infty < t \le 0 \\ \frac{AC}{a}\left(1-e^{-at}\right) & 0 \le t \le b \\ \frac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)} & t \ge b \end{cases}$$

Both should provide the same value at *t*=0

Both should provide the same value at *t=b*

$$\frac{AC}{a}\left(1-e^{-at}\right)\Big|_{t=0}=0$$

$$\frac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)}\Big|_{t=b}=\frac{AC}{a}\left(1-e^{-at}\right)\Big|_{t=b}$$

Example A (final)

Point 1

$$\frac{AC}{a}\left(1-e^{-at}\right)\Big|_{t=0}=0$$

$$\frac{AC}{a}\left(1-e^{0}\right)=0$$

$$0=0$$

checks!

Point 2

$$\frac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)}\Big|_{t=b}=\frac{AC}{a}\left(1-e^{-at}\right)\Big|_{t=b}$$

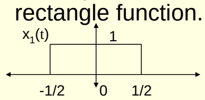
$$\frac{AC}{a}\left(1-e^{-ab}\right)\underbrace{e^{-a(b-b)}}_{=1}=\frac{AC}{a}\left(1-e^{-ab}\right)$$

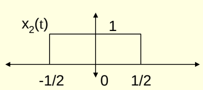
$$\frac{AC}{a}\left(1-e^{-ab}\right)=\frac{AC}{a}\left(1-e^{-ab}\right)$$

checks!

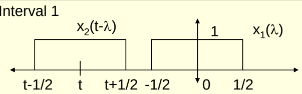
Example B

- Determine the convolution of the unit rectangle function with a second unit rectangle function.





Interval 1

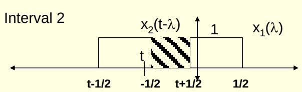


$-\infty < t \le -1 \quad x_3(t)=0$

No overlap and thus no response for *t+1/2 < -1/2*

Example B (cont.)

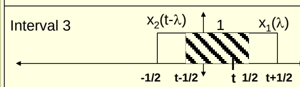
Interval 2



$-1 \le t \le 0$

$$\begin{aligned} x_3(t) &= \int_{-1/2}^{t+1/2} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \\ &= \int_{-1/2}^{t+1/2} d\lambda \\ &= t+1/2-(-1/2) \\ &= t+1 \end{aligned}$$

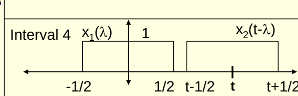
Interval 3



$0 \le t \le 1$

$$\begin{aligned} x_3(t) &= \int_{t-1/2}^{1/2} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \\ &= \int_{t-1/2}^{1/2} d\lambda \\ &= 1/2-(t-1/2) \\ &= 1-t \end{aligned}$$

Interval 4

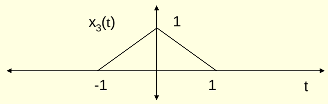


$t \ge 1 \quad x_3(t)=0$

Example B (cont.)

- Thus, finally we have

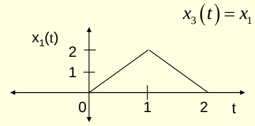
$$x_3(t) = \begin{cases} 0 & -\infty < t < -1 \\ t+1 & -1 \le t \le 0 \\ 1-t & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

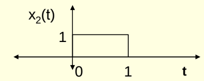


The convolution of two unit rectangle functions is the unit triangle function.

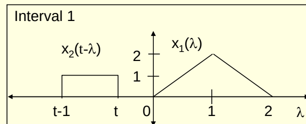
Example C

$$x_3(t) = x_1(t) * x_2(t)$$





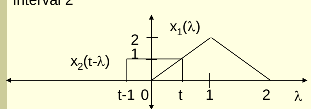
Interval 1



$-\infty < t \le 0 \quad x_3(t) = 0$

Example C (cont.)

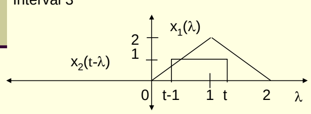
Interval 2



$0 \le t \le 1$

$$x_3(t) = \int_0^t x_1(\lambda) x_2(t-\lambda)\, d\lambda = \int_0^t 2\lambda\, d\lambda = \lambda^2 \Big|_0^t = t^2$$

Interval 3

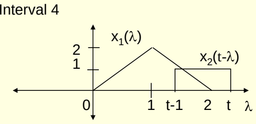


$1 \le t \le 2$

$$x_3(t) = \int_{t-1}^{1} 2\lambda\, d\lambda + \int_1^t (4-2\lambda)\, d\lambda = \lambda^2 \Big|_{t-1}^{1} + 4\lambda - \lambda^2 \Big|_1^t = 1-(t-1)^2 + 4t - t^2 - 4 + 1 = -2t^2 + 6t - 3 = \frac{3}{2} - 2\left(t - \frac{3}{2}\right)^2$$

Example C (cont.)

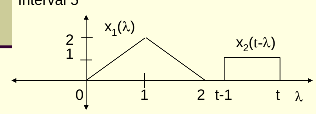
Interval 4



$2 \le t \le 3$

$$x_3(t) = \int_{t-1}^{2} (4-2\lambda)\, d\lambda = 4\lambda - \lambda^2 \Big|_{t-1}^{2} = 8 - 4 - 4(t-1) + (t-1)^2 = t^2 - 6t + 9 = (t-3)^2$$

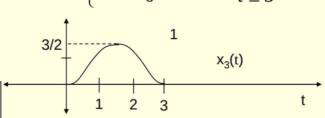
Interval 5



$t \ge 3 \quad x_3(t) = 0$

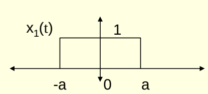
Example C (cont.)

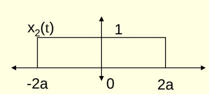
$$x_3(t) = \begin{cases} 0 & t \le 0 \\ t^2 & 0 \le t \le 1 \\ \frac{3}{2} - 2\left(t - \frac{3}{2}\right)^2 & 1 \le t \le 2 \\ (t-3)^2 & 2 \le t \le 3 \\ 0 & t \ge 3 \end{cases}$$



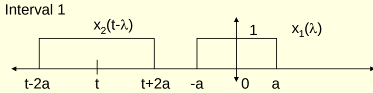
The result of convolution is typically a smoother function

Example D





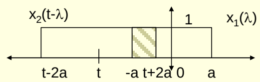
Interval 1



$-\infty < t \le -3a \quad x_3(t) = 0$

Example D (cont.)

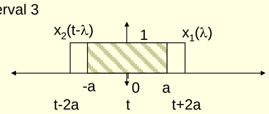
Interval 2



$-3a \le t \le -a$

$$x_3(t) = \int_{-a}^{t+2a} x_1(\lambda) x_2(t-\lambda)\, d\lambda = \int_{-a}^{t+2a} d\lambda = \lambda\Big|_{-a}^{t+2a} = t + 3a$$

Interval 3

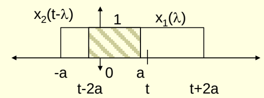


$-a \le t \le a$

$$x_3(t) = \int_{-a}^{a} x_1(\lambda) x_2(t-\lambda)\, d\lambda = \int_{-a}^{a} d\lambda = \lambda\Big|_{-a}^{a} = a + a = 2a$$

Example D (cont.)

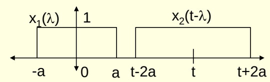
Interval 4



$a \le t \le 3a$

$$x_3(t) = \int_{t-2a}^{a} x_1(\lambda) x_2(t-\lambda)\, d\lambda = \int_{t-2a}^{a} d\lambda = \lambda\Big|_{t-2a}^{a} = a - (t - 2a) = 3a - t$$

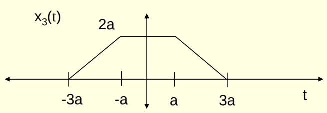
Interval 5



$t \ge 3a \quad x_3(t) = 0$

Example D (final solution)

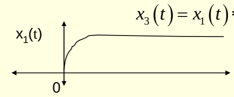
$$x_3(t) = \begin{cases} 0 & t \le -3a \\ t + 3a & -3a \le t \le -a \\ 2a & -a \le t \le a \\ 3a - t & a \le t \le 3a \\ 0 & t \ge 3a \end{cases}$$



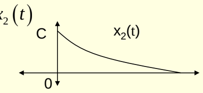
Note on intervals

- If both $x_1(t)$ and $x_2(t)$ are non-zero only on the interval {0,infinity} than the result of the convolution $x_3(t) = x_1(t)*x_2(t)$ also exist on {0,infinity}.

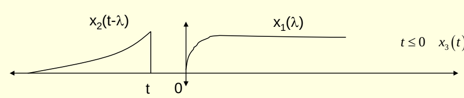
$$x_3(t) = x_1(t) * x_2(t)$$



$x_1(t) = g(t)u(t)$

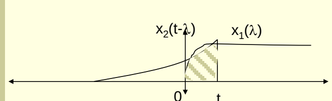


$x_2(t) = h(t)u(t)$



$t \le 0 \quad x_3(t) = 0$

Note on intervals (cont.)



$t \ge 0$

$$x_3(t) = \int_0^t x_1(\lambda) x_2(t-\lambda)\, d\lambda = \int_0^t g(\lambda) h(t-\lambda)\, d\lambda$$

- Thus, there are two intervals: t<=0 and t >= 0. The function $x_3(t)$ is zero for t<0 and equal to some non-zero value for t < infinity.

Example E

- Determine the following convolution

$$x_3(t) = x_1(t) * x_2(t)$$
$$x_1(t) = e^{-At} u(t)$$
$$x_2(t) = u(t)$$

Example E (solution)

$t \le 0 \quad x_3(t) = 0$

$t \ge 0$

$$\begin{aligned} x_3(t) &= \int_0^t x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \\ &= \int_0^t e^{-A(t-\lambda)}\,d\lambda \\ &= e^{-At}\int_0^t e^{\lambda A}\,d\lambda \\ &= \frac{e^{-At}}{A} e^{\lambda A}\Big|_0^t \\ &= \frac{e^{-At}}{A}\left(e^{At}-1\right) \\ &= \frac{1}{A}\left(1-e^{-At}\right) \end{aligned}$$

$$x_3(t) = \frac{1}{A}\left(1-e^{-At}\right)u(t)$$

Conclusions

- Convolution is a somewhat complicated but very important operation in system analysis.
- Steps
  - Determine the intervals for *t*
  - Determine the limits of integration
  - Calculate each integral
  - Write a summary of the total function $x_3(t)$
- Next class we will discuss more properties of convolution.