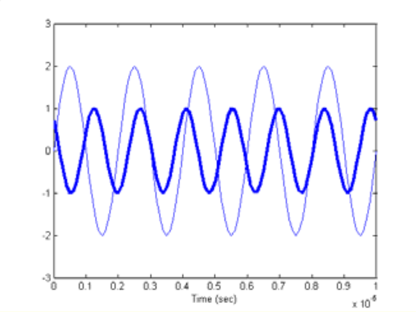
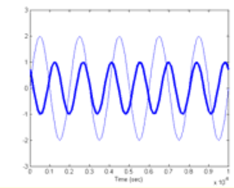
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer

Lecture #4: Convolution

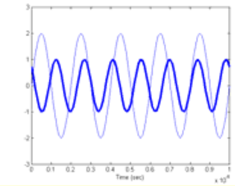


Overview



- What to read – Section 3.6 in the text
- This lecture (as well as the next few lectures) deals with the concept of convolution.
- Convolution allows us to determine the output of any linear time-invariant system through the system *impulse response*. Thus, it is an important concept.
- We will examine convolution both mathematically and graphically

Definition

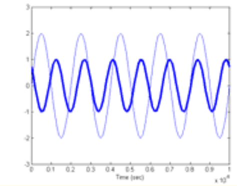


- The convolution of two functions is defined as

$$x_3(t) = x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

- Convolution is the integral of a function $x_1(t)$ and a second function $x_2(t)$ which is inverted in time and time shifted. Note that the result of convolution is a function where the argument of the function is equivalent to the time shift.

Symmetry of Convolution



$$x_3(t) = x_1(t) * x_2(t)$$
$$= x_2(t) * x_1(t)$$

- Proof:

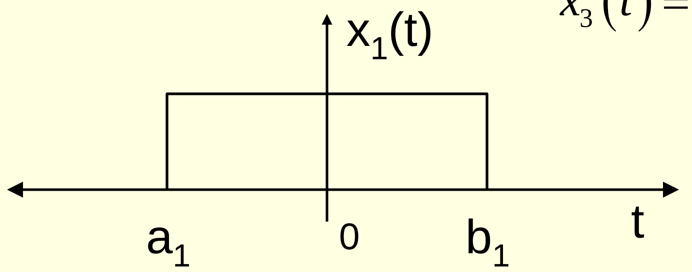
$$x_3(t) = x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$

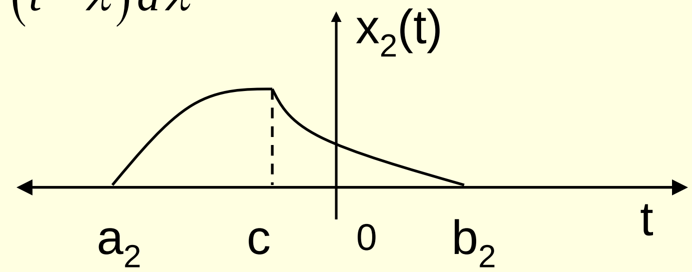
- Let $\beta = t - \lambda$ → $\lambda = t - \beta$, d λ = -dβ

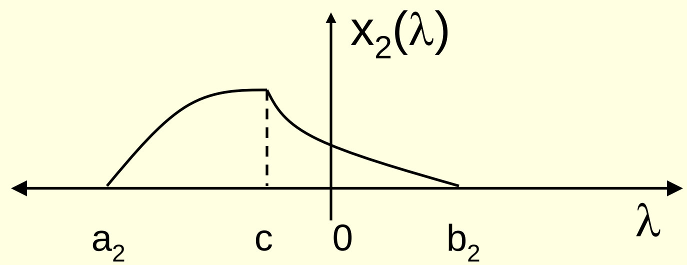
$$x_3(t) = -\int_{-\infty}^{\infty} x_1(t-\beta) x_2(\beta) d\beta$$
$$= \int_{-\infty}^{\infty} x_2(\beta) x_1(t-\beta) d\beta$$
$$= x_2(t) * x_1(t)$$

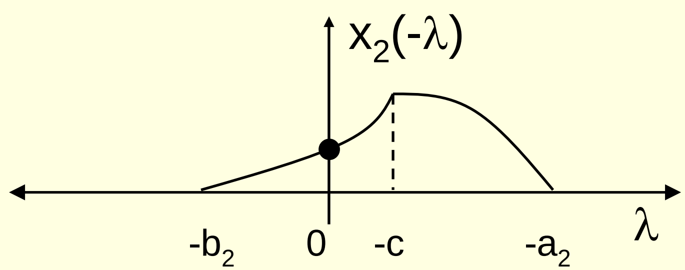
Graphical Illustration

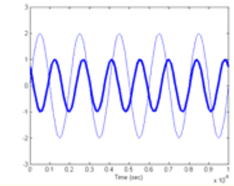
$$x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$



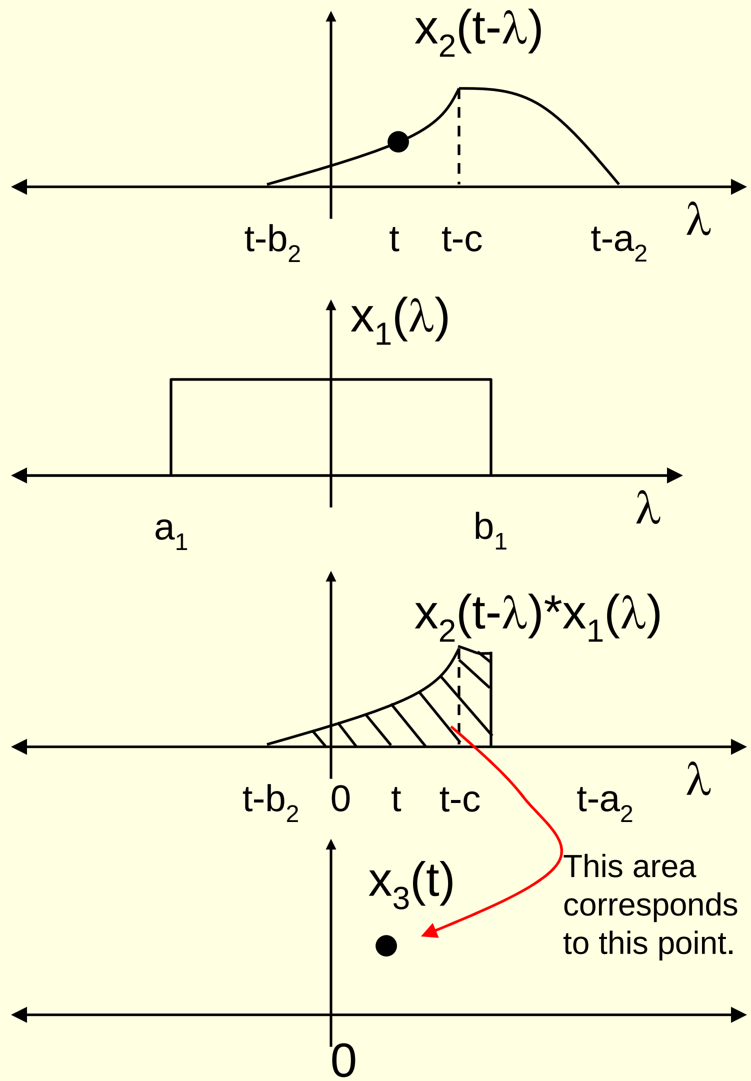








Graphical Illustration (cont.)

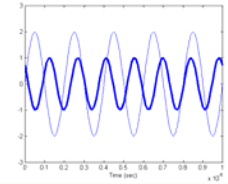


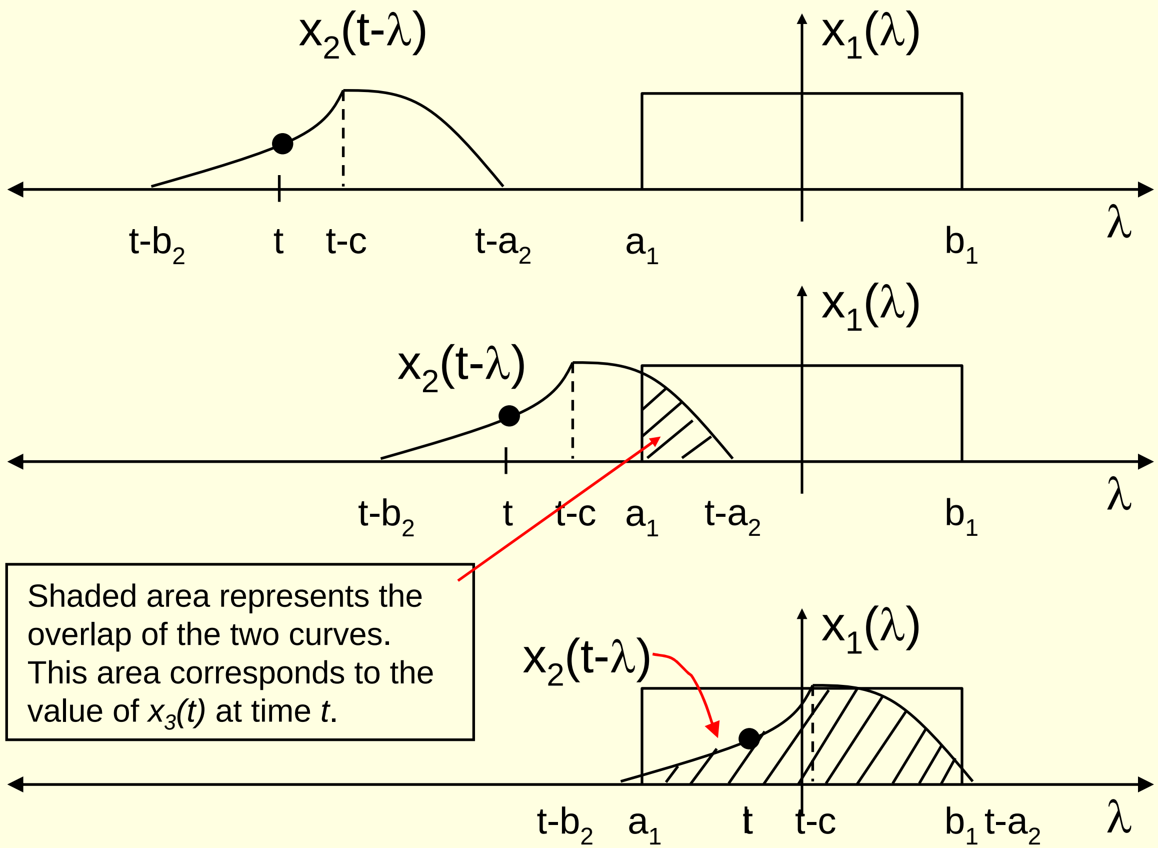
Graph is shifted in time by *t.* The point originally at 0 is at *t*

Convolution integral evaluated at *t* is simply the area of under the multiplication of the two curves

$x_3(t)$ evaluated at a single point in time. Note, that we must evaluate this for all values of *t*

Calculating Intervals



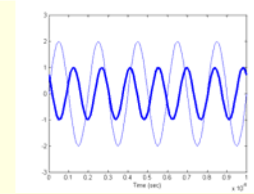


$$x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

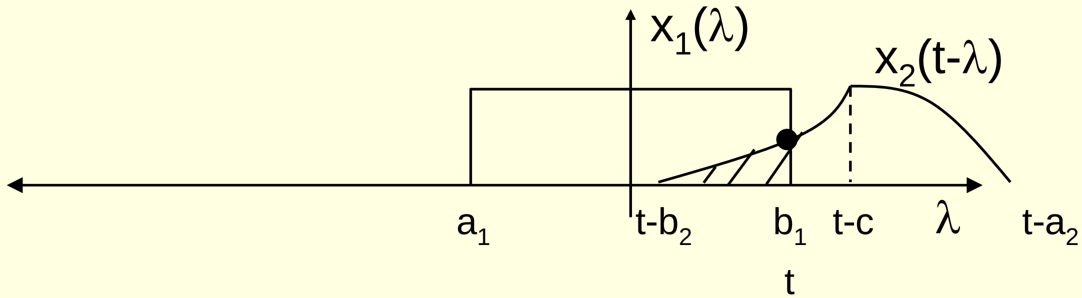
$$-\infty < t - a_2 \le a_1$$
$$-\infty < t \le a_1 + a_2 \qquad x_3(t) = 0$$

$$a_1 \le t - a_2 \le b_1$$
$$a_1 + a_2 \le t \le b_1 + a_2$$
$$x_3(t) = \int_{a_1}^{t-a_2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

$$t - a_2 \ge b_1, \quad t - b_2 \le a_1,$$
$$x_3(t) = \int_{a_1}^{b_1} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$



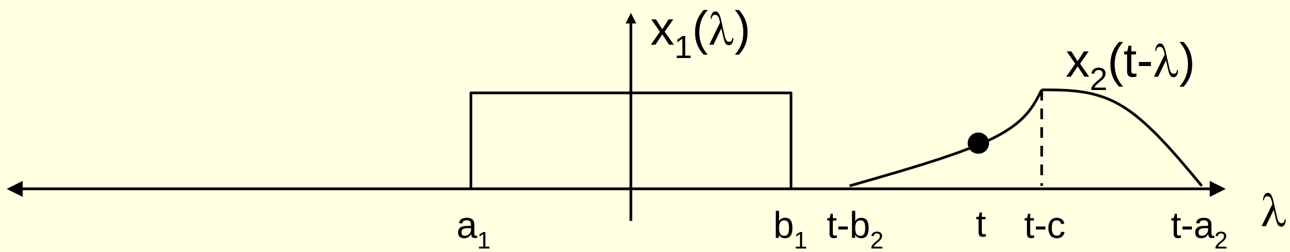
Calculating Intervals (cont.)



$$a_1 \le t - b_2 \le b_1$$

$$a_1 + b_2 \le t \le b_1 + b_2$$

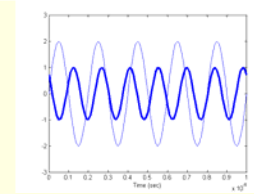
$$x_3(t) = \int_{t-b_2}^{b_1} x_1(\lambda) x_2(t-\lambda) \, d\lambda$$



$$b_1 \le t - b_2 < \infty$$

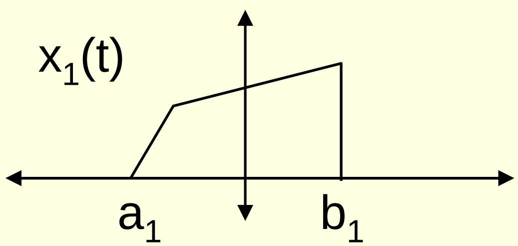
$$b_1 + b_2 \le t < \infty$$

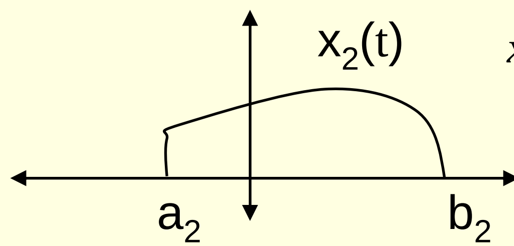
$$x_3(t) = 0$$



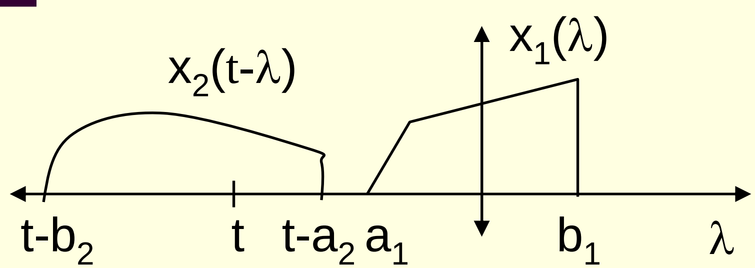
Duration of Convolution

- For two pulse-type (finite length) signals of $x_1(t)$ and $x_2(t)$:
  - Starting time of $x_3(t)$ is the sum of the starting time of the two signals $x_1(t)$ and $x_2(t)$
  - Ending time of $x_3(t)$ is the sum of the end of the two signals $x_1(t)$ and $x_2(t)$
  - Duration of $x_3(t)$ is the sum of the two durations

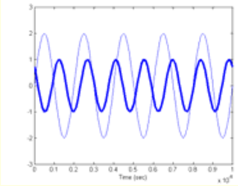




$$x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

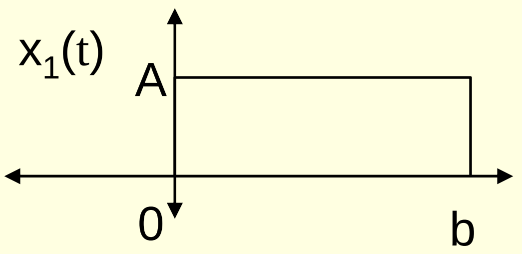


We can see that the integral will be zero until $t-a_2 >= a_1$ or $t >= a_1+a_2$
Further, the integral will be zero when $t-b_2 >= b_1$ or $t >= b_1+b_2$
Duration = $(b_1+b_2) – (a_1 +a_2)$ or $(b_1-a_1) + (b_2 - a_2)$

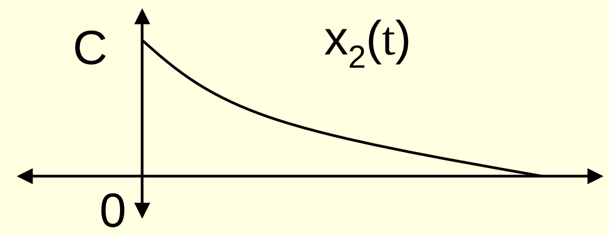


Example A

$$x_3(t) = x_1(t) * x_2(t)$$

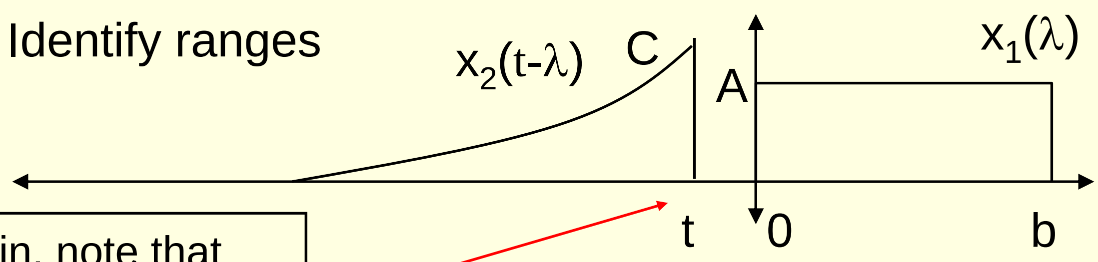


$$x_1(t) = A\left[u(t) - u(t-b)\right]$$



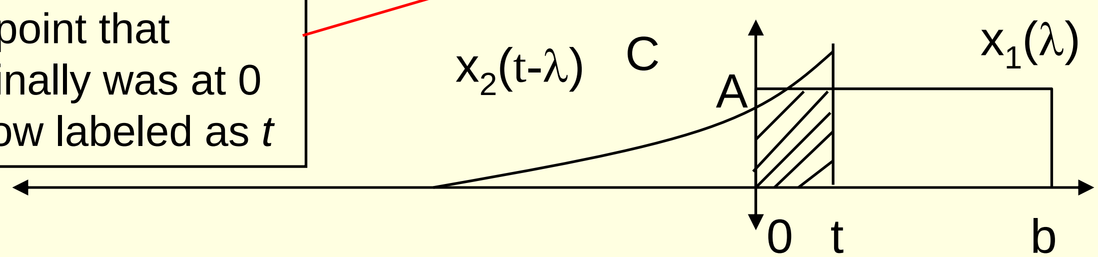
$$x_2(t) = Ce^{-at}u(t)$$

Identify ranges



$$-\infty < t \le 0 \qquad x_3(t) = 0$$

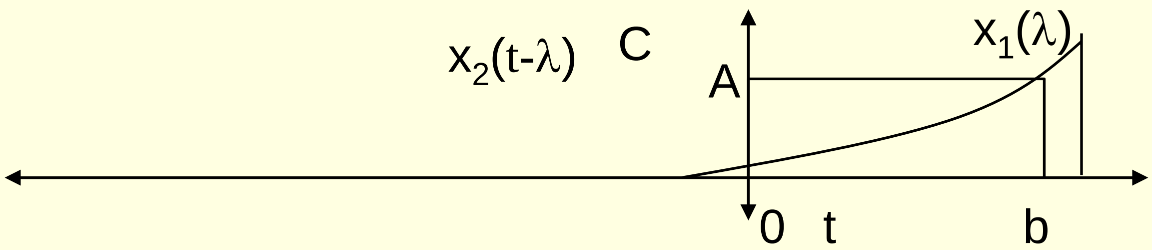
Again, note that the point that originally was at 0 is now labeled as *t*



$$0 \le t \le b$$

$$x_3(t) = \int_0^t x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Example A (cont.)



$$t \geq b$$

$$x_3(t)=\int_0^b x_1(\lambda)\,x_2(t-\lambda)\,d\lambda$$

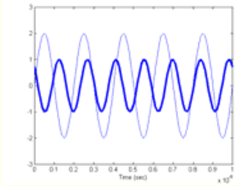
- Summarizing

$$0 \qquad -\infty < t \leq 0$$

$$x_3(t)=\int_0^t x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \qquad 0 \leq t \leq b$$

$$x_3(t)=\int_0^b x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \qquad t \geq b$$

Example A (cont.)



- Evaluating the second interval ($0 \le t \le b$)

$$x_3(t) = \int_0^t x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

$$= \int_0^t ACe^{-a(t-\lambda)} d\lambda$$

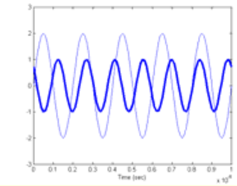
$$= ACe^{-at} \int_0^t e^{a\lambda} d\lambda$$

$$= ACe^{-at} \frac{1}{a} e^{a\lambda} \Big|_0^t$$

$$= \frac{AC}{a} e^{-at} \left(e^{at} - 1 \right)$$

$$= \frac{AC}{a} \left(1 - e^{-at} \right)$$

Example A (cont.)



- Evaluating the third interval (t>=b)

$$x_3(t) = \int_0^b x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

$$= \int_0^b ACe^{-a(t-\lambda)} d\lambda$$

$$= ACe^{-at} \int_0^b e^{a\lambda} d\lambda$$

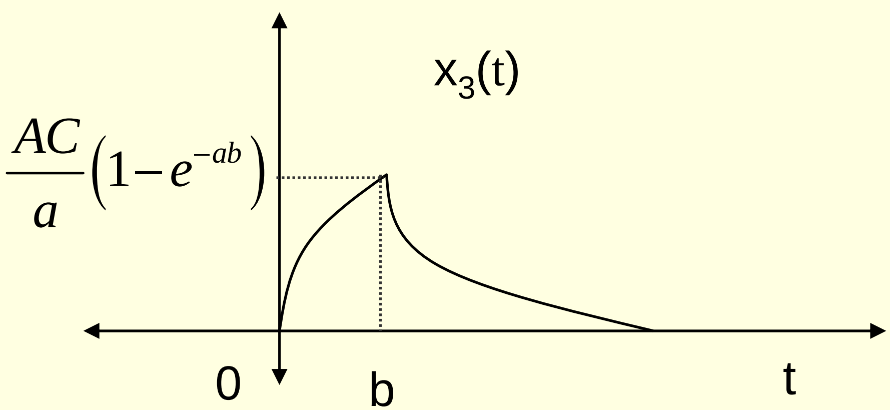
$$= ACe^{-at} \frac{1}{a} e^{a\lambda} \Big|_0^b$$

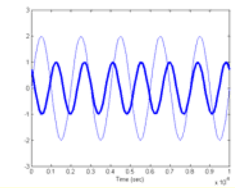
$$= \frac{AC}{a} e^{-at} \left(e^{ab} - 1\right)$$

$$= \frac{AC}{a} \left(1 - e^{-ab}\right) e^{-a(t-b)}$$

Example A (cont.)

$$x_3(t) = \begin{cases} 0 & -\infty < t \le 0 \\ \dfrac{AC}{a}\left(1 - e^{-at}\right) & 0 \le t \le b \\ \dfrac{AC}{a}\left(1 - e^{-ab}\right)e^{-a(t-b)} & t \ge b \end{cases}$$





Example A (cont.)

A good double check: The function $x_3(t)$ should be continuous unless there are impulses in $x_2(t)$ or $x_1(t)$. As a result, the function give the same value at the endpoints regardless of which interval I use to evaluate the point.

Ex:

$$x_3(t) = \begin{cases} 0 & -\infty < t \le 0 \\ \dfrac{AC}{a}\left(1-e^{-at}\right) & 0 \le t \le b \\ \dfrac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)} & t \ge b \end{cases}$$

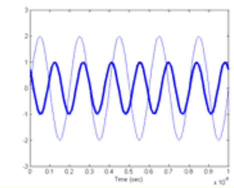
Both should provide the same value at t=0

Both should provide the same value at $t=b$

$$\left.\frac{AC}{a}\left(1-e^{-at}\right)\right|_{t=0} = 0$$

$$\left.\frac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)}\right|_{t=b} = \left.\frac{AC}{a}\left(1-e^{-at}\right)\right|_{t=b}$$

Example A (final)



Point 1

$$\left.\frac{AC}{a}\left(1-e^{-at}\right)\right|_{t=0}=0$$

$$\frac{AC}{a}\left(1-e^{0}\right)=0$$

$$0=0$$

checks!

Point 2

$$\left.\frac{AC}{a}\left(1-e^{-ab}\right)e^{-a(t-b)}\right|_{t=b}=\left.\frac{AC}{a}\left(1-e^{-at}\right)\right|_{t=b}$$

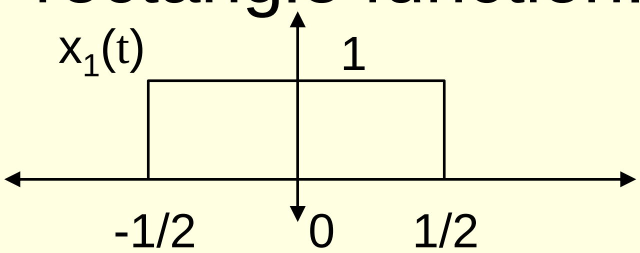
$$\frac{AC}{a}\left(1-e^{-ab}\right)\underbrace{e^{-a(b-b)}}_{=1}=\frac{AC}{a}\left(1-e^{-ab}\right)$$

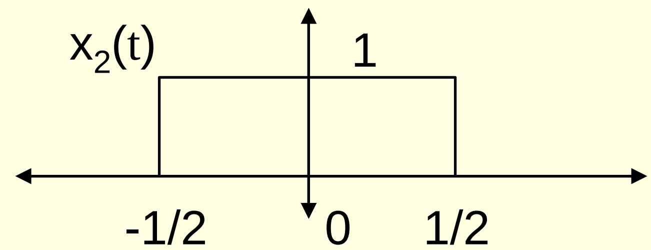
$$\frac{AC}{a}\left(1-e^{-ab}\right)=\frac{AC}{a}\left(1-e^{-ab}\right)$$

checks!

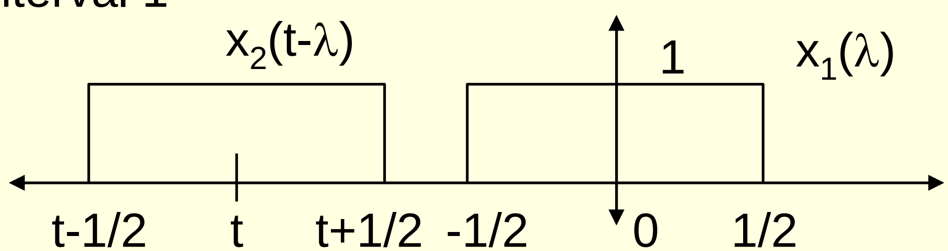
Example B

- Determine the convolution of the unit rectangle function with a second unit rectangle function.





Interval 1

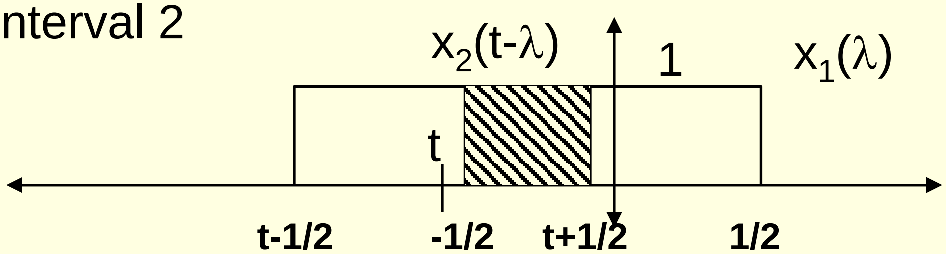


$$-\infty < t \le -1 \quad x_3(t) = 0$$

No overlap and thus no response for *t+1/2 < -1/2*

Example B (cont.)

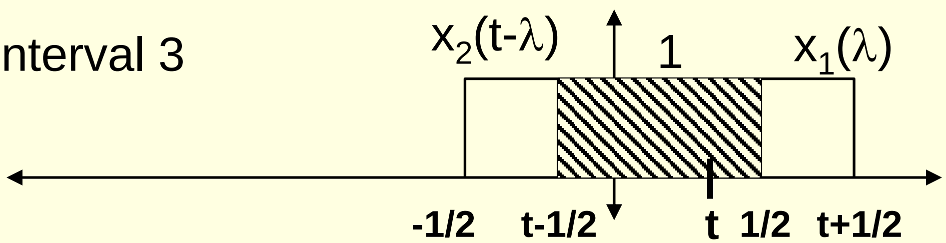
Interval 2



$-1 \le t \le 0$

$$x_3(t) = \int_{-1/2}^{t+1/2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$
$$= \int_{-1/2}^{t+1/2} d\lambda$$
$$= t + 1/2 - (-1/2)$$
$$= t + 1$$

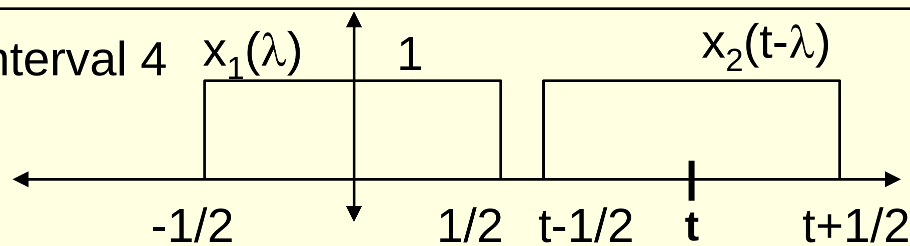
Interval 3



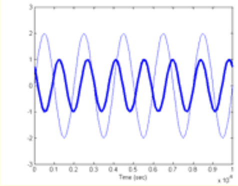
$0 \le t \le 1$

$$x_3(t) = \int_{t-1/2}^{1/2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$
$$= \int_{t-1/2}^{1/2} d\lambda$$
$$= 1/2 - (t - 1/2)$$
$$= 1 - t$$

Interval 4



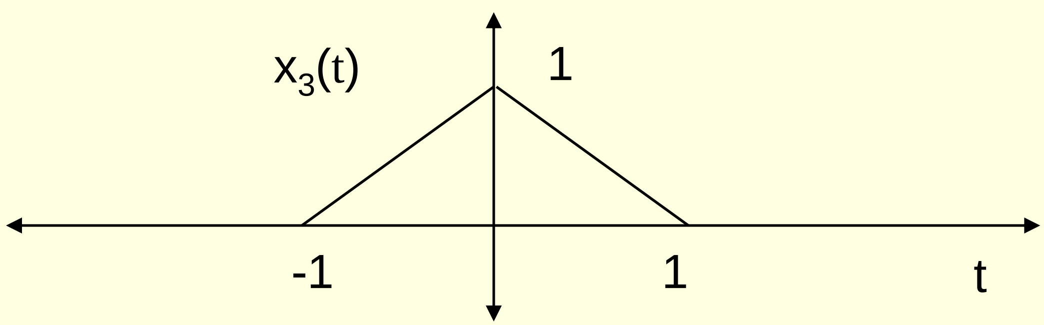
$t \ge 1 \quad x_3(t) = 0$



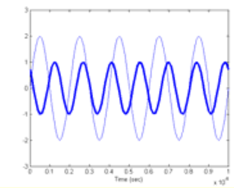
Example B (cont.)

- Thus, finally we have

$$x_3(t) = \begin{cases} 0 & -\infty < t < -1 \\ t+1 & -1 \le t \le 0 \\ 1-t & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

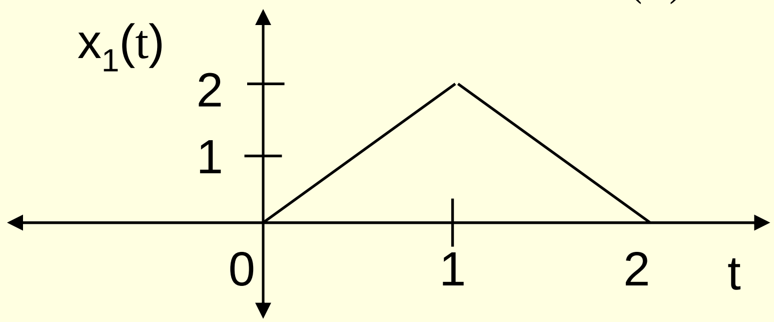


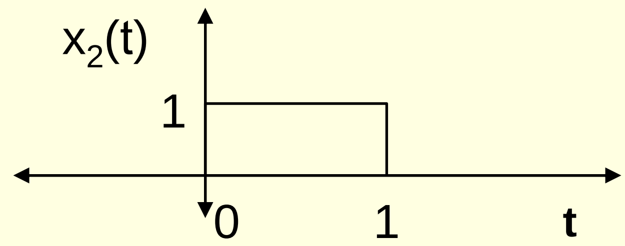
The convolution of two unit rectangle functions is the unit triangle function.



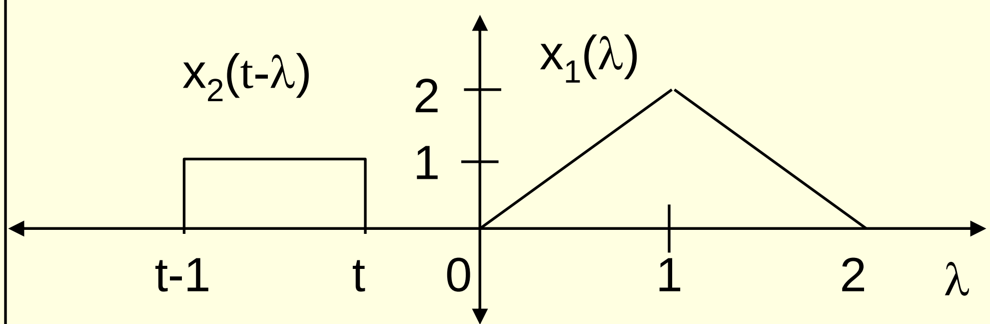
Example C

$$x_3(t) = x_1(t) * x_2(t)$$

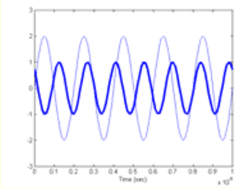




Interval 1

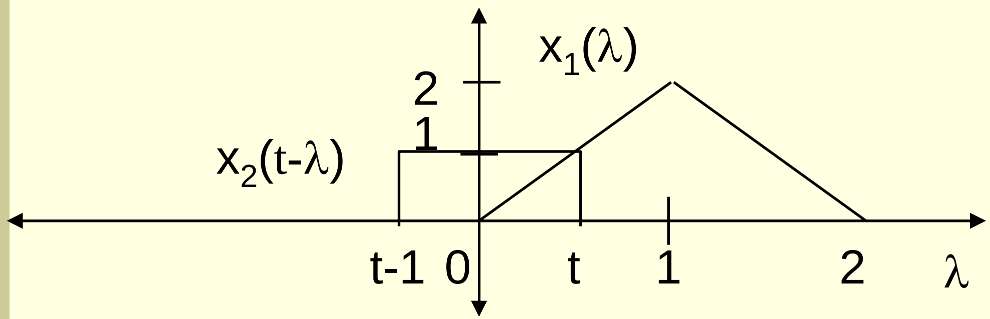


$-\infty < t \le 0 \quad x_3(t) = 0$



Example C (cont.)

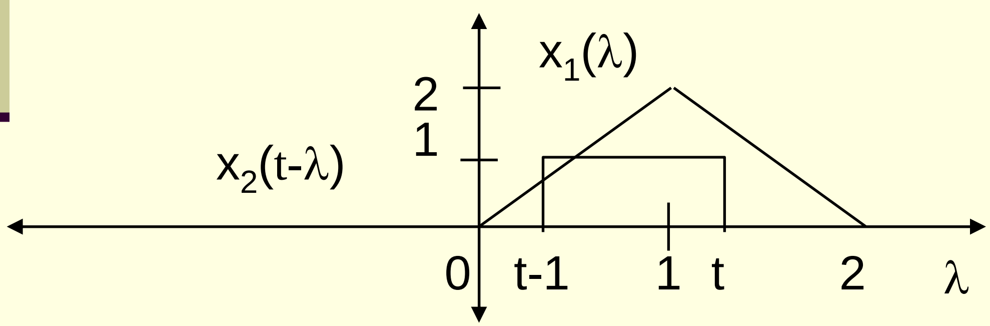
Interval 2



$0 \le t \le 1$

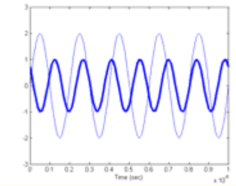
$$x_3(t) = \int_0^t x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$
$$= \int_0^t 2\lambda\, d\lambda$$
$$= \lambda^2 \Big|_0^t$$
$$= t^2$$

Interval 3



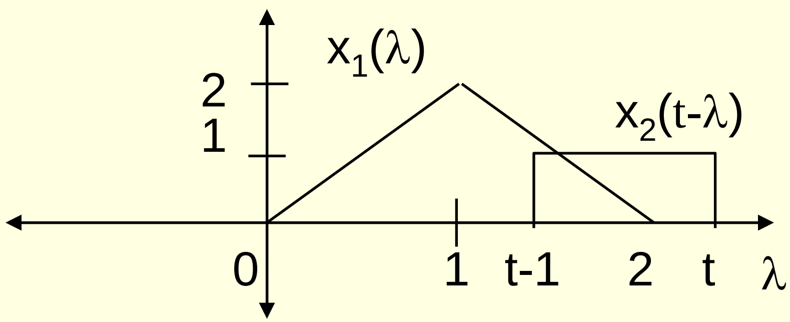
$1 \le t \le 2$

$$x_3(t) = \int_{t-1}^{1} 2\lambda\, d\lambda + \int_1^t (4-2\lambda)\, d\lambda$$
$$= \lambda^2 \Big|_{t-1}^{1} + 4\lambda - \lambda^2 \Big|_1^t$$
$$= 1-(t-1)^2 + 4t - t^2 - 4 + 1$$
$$= -2t^2 + 6t - 3$$
$$= \frac{3}{2} - 2\left(t - \frac{3}{2}\right)^2$$



Example C (cont.)

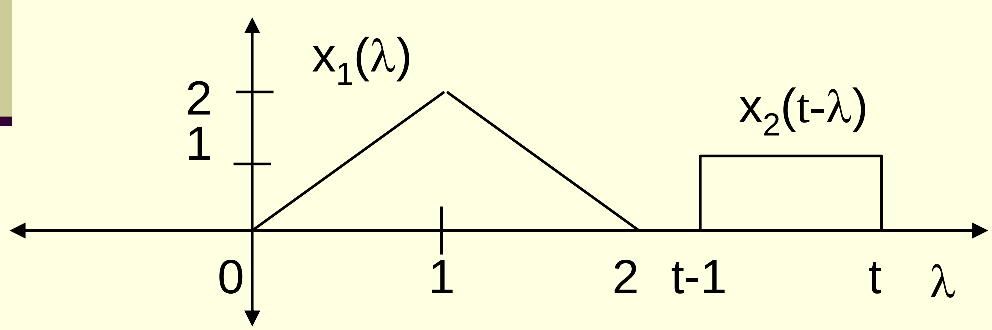
Interval 4



$2 \le t \le 3$

$$
\begin{aligned}
x_3(t) &= \int_{t-1}^{2} (4-2\lambda)\,d\lambda \\
&= 4\lambda - \lambda^2 \Big|_{t-1}^{2} \\
&= 8-4-4(t-1)+(t-1)^2 \\
&= t^2-6t+9 \\
&= (t-3)^2
\end{aligned}
$$

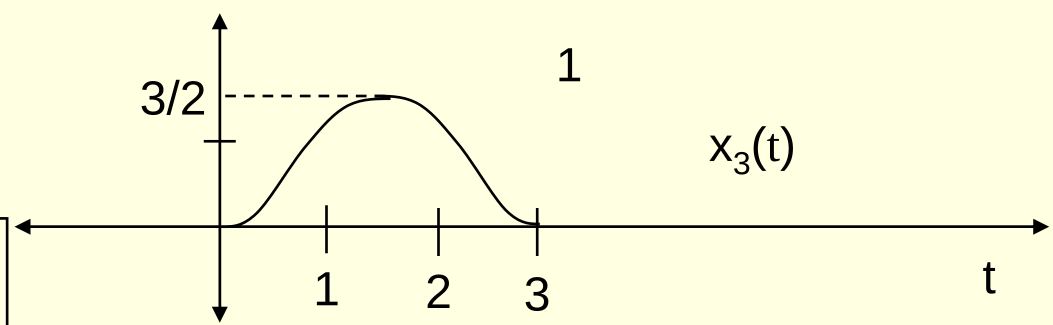
Interval 5



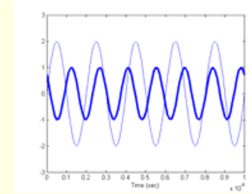
$t \ge 3 \quad x_3(t) = 0$

Example C (cont.)

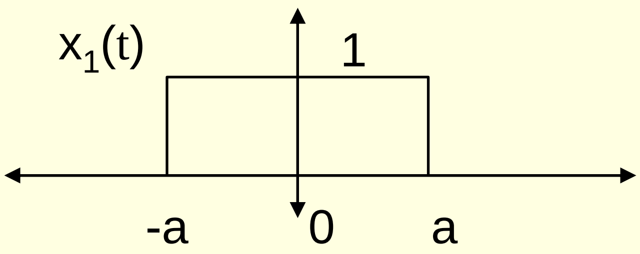
$$x_3(t)=\begin{cases}0 & t\le 0\\ t^2 & 0\le t\le 1\\ \frac{3}{2}-2\left(t-\frac{3}{2}\right)^2 & 1\le t\le 2\\ (t-3)^2 & 2\le t\le 3\\ 0 & t\ge 3\end{cases}$$

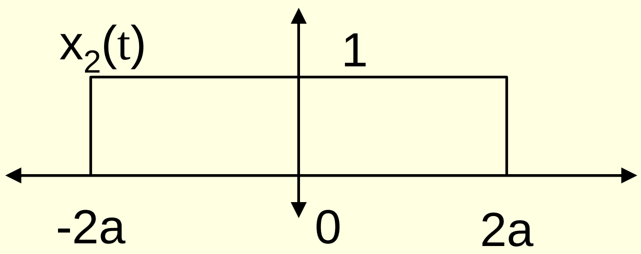


The result of convolution is typically a smoother function

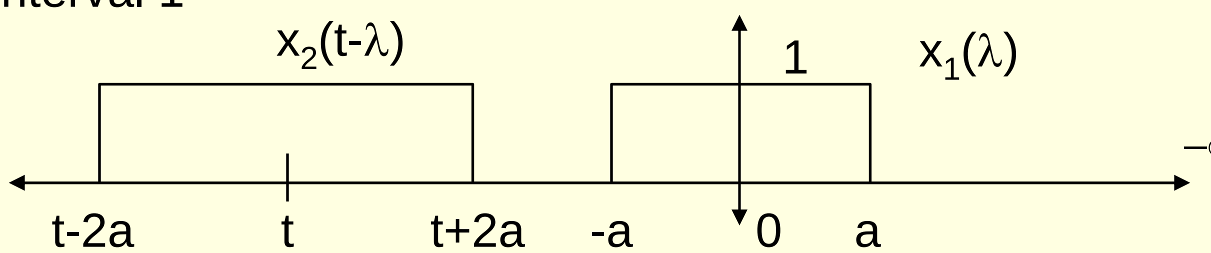


Example D

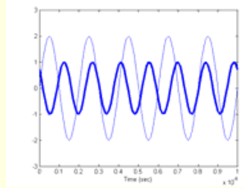




Interval 1

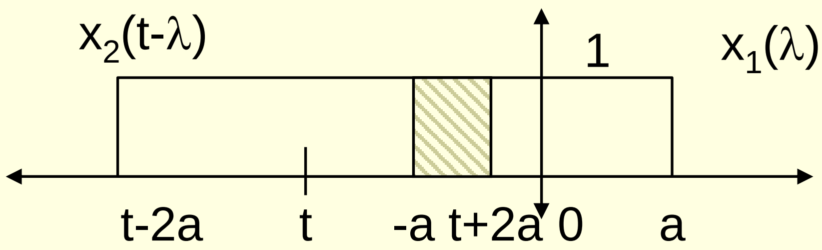


$$-\infty < t \leq -3a \quad x_3(t) = 0$$



Example D (cont.)

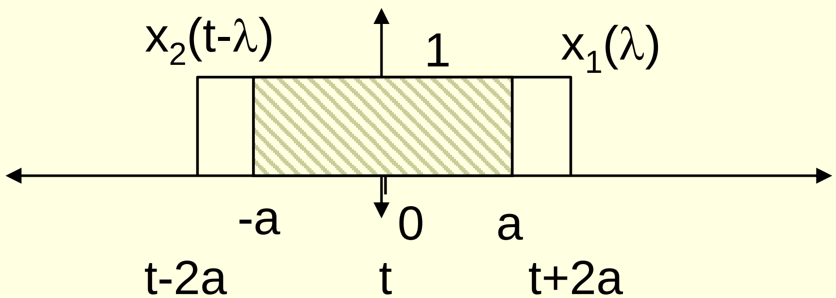
Interval 2



$-3a \le t \le -a$

$$x_3(t) = \int_{-a}^{t+2a} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$
$$= \int_{-a}^{t+2a} d\lambda$$
$$= \lambda \Big|_{-a}^{t+2a}$$
$$= t + 3a$$

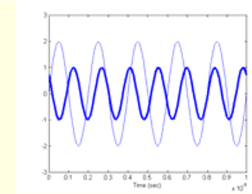
Interval 3



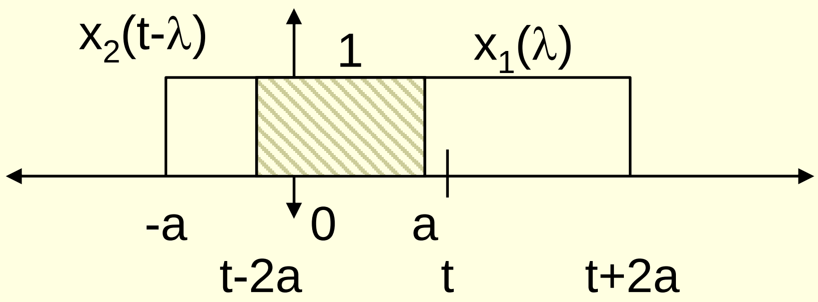
$-a \le t \le a$

$$x_3(t) = \int_{-a}^{a} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$
$$= \int_{-a}^{a} d\lambda$$
$$= \lambda \Big|_{-a}^{a}$$
$$= a + a$$
$$= 2a$$

Example D (cont.)



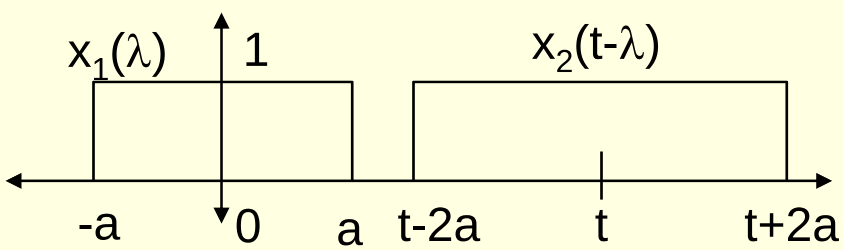
Interval 4



$$a \le t \le 3a$$

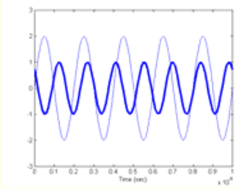
$$\begin{aligned} x_3(t) &= \int_{t-2a}^{a} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda \\ &= \int_{t-2a}^{a} d\lambda \\ &= \lambda\Big|_{t-2a}^{a} \\ &= a-(t-2a) \\ &= 3a-t \end{aligned}$$

Interval 5

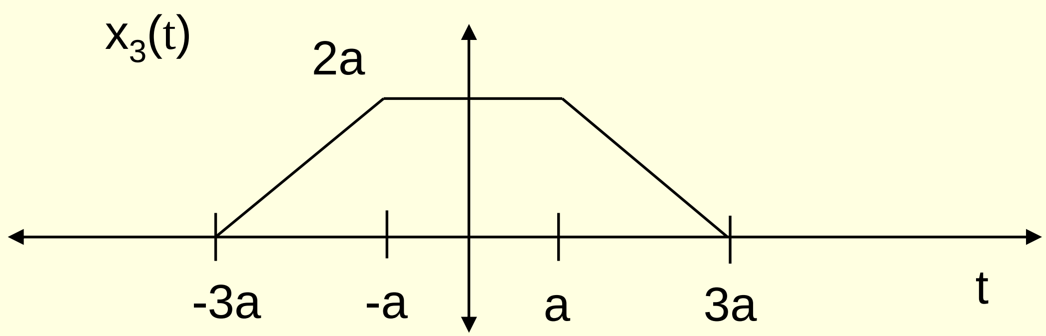


$$t \ge 3a \quad x_3(t)=0$$

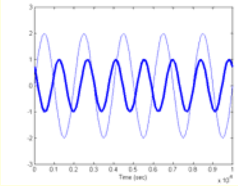
Example D (final solution)



$$x_3(t)=\begin{cases} 0 & t\le -3a \\ t+3a & -3a\le t\le -a \\ 2a & -a\le t\le a \\ 3a-t & a\le t\le 3a \\ 0 & t\ge 3a \end{cases}$$

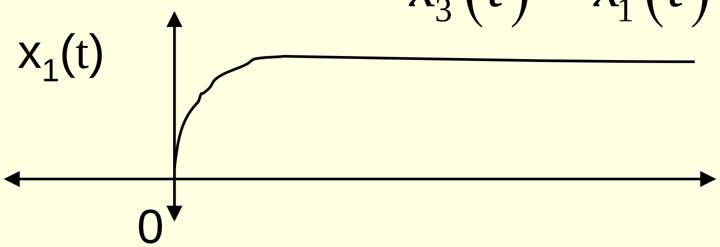


Note on intervals

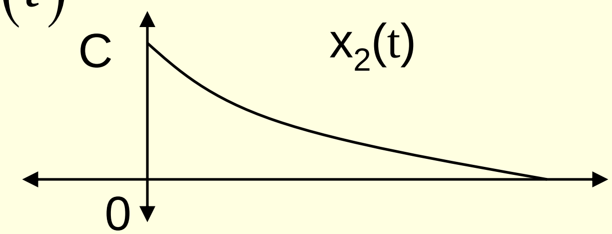


- If both $x_1(t)$ and $x_2(t)$ are non-zero only on the interval {0,infinity} than the result of the convolution $x_3(t) = x_1(t)*x_2(t)$ also exist on {0,infinity}.

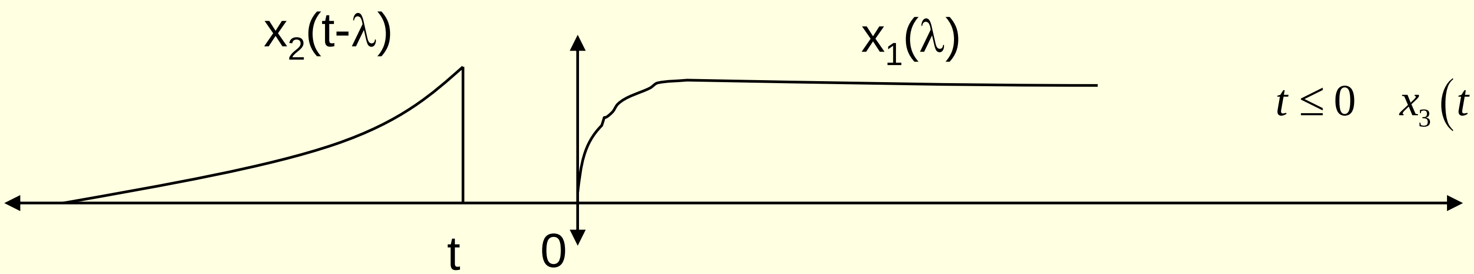
$$x_3(t) = x_1(t) * x_2(t)$$



$$x_1(t) = g(t)u(t)$$

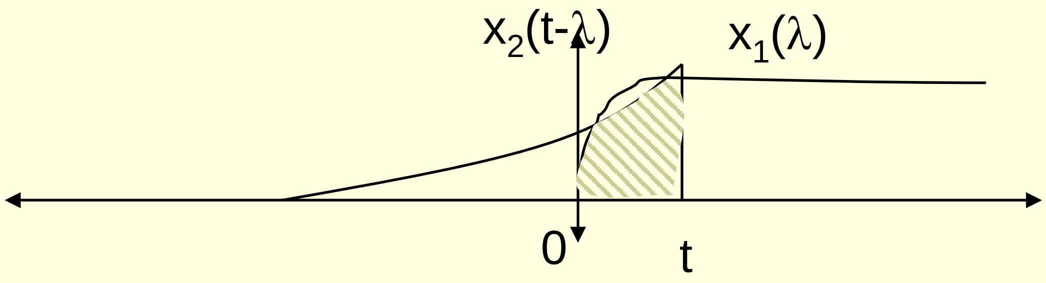


$$x_2(t) = h(t)u(t)$$



$t \le 0 \quad x_3(t) = 0$

Note on intervals (cont.)

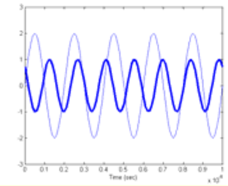


$t \geq 0$

$$x_3(t) = \int_0^t x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

$$= \int_0^t g(\lambda)\, h(t-\lambda)\, d\lambda$$

- Thus, there are two intervals: t<=0 and t >= 0. The function $x_3(t)$ is zero for t<0 and equal to some non-zero value for t < infinity.



Example E

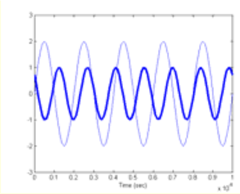
- Determine the following convolution

$$x_3(t) = x_1(t) * x_2(t)$$

$$x_1(t) = e^{-At}u(t)$$

$$x_2(t) = u(t)$$

Example E (solution)



$t \le 0 \quad x_3(t) = 0$

$t \ge 0$

$$x_3(t) = \int_0^t x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

$$= \int_0^t e^{-A(t-\lambda)} d\lambda$$

$$= e^{-At} \int_0^t e^{\lambda A} d\lambda$$

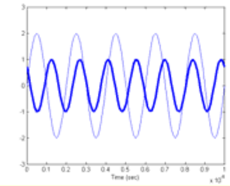
$$= \frac{e^{-At}}{A} e^{\lambda A} \Big|_0^t$$

$$= \frac{e^{-At}}{A}\left(e^{At} - 1\right)$$

$$= \frac{1}{A}\left(1 - e^{-At}\right)$$

$$x_3(t) = \frac{1}{A}\left(1 - e^{-At}\right) u(t)$$

Conclusions



- Convolution is a somewhat complicated but very important operation in system analysis.
- Steps
  - Determine the intervals for *t*
  - Determine the limits of integration
  - Calculate each integral
  - Write a summary of the total function $x_3(t)$
- Next class we will discuss more properties of convolution.