ECE 2704
Signals and Systems
Spring 2006

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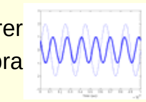
Lecture #5: Convolution Algebra



Overview

- What to read – Section 3.6 in the text
- This lecture (as well as the previous lecture) deals with the concept of convolution.
- Convolution allows us to determine the output of any linear time-invariant system through the system's impulse response.
  - We will discuss the impulse response next class
- Today we will examine convolution algebra as well as a few special convolutions.

Convolution Algebra

- Convolution properties
  - Commutative Property
  - Distributive Property
  - Associative Property
  - Derivative
  - Time-shifting
  - Convolution involving a periodic function
  - Duration
  - Location
  - Shape
- Special convolutions
  - Step
  - Impulse
- Area
- Centroid

Commutative Property

- Recall the definition of convolution

$$\begin{aligned} x_3(t) &= x_1(t) * x_2(t) \\ &= \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda \end{aligned}$$

- The commutative property says that the order of convolution is irrelevant. That is

$$\begin{aligned} x_3(t) &= x_1(t) * x_2(t) \\ &= x_2(t) * x_1(t) \end{aligned}$$

The proof was given last class.

Distributive Property

$$\begin{aligned} x_4(t) &= x_1(t) * \left[x_2(t) + x_3(t) \right] \\ &= x_1(t) * x_2(t) + x_1(t) * x_3(t) \end{aligned}$$

Proof:

$$\begin{aligned} x_4(t) &= x_1(t) * \left[x_2(t) + x_3(t) \right] \\ &= \int_{-\infty}^{\infty} x_1(\lambda) \left[x_2(t-\lambda) + x_3(t-\lambda) \right] d\lambda \\ &= \int_{-\infty}^{\infty} \left[x_1(\lambda) x_2(t-\lambda) + x_1(\lambda) x_3(t-\lambda) \right] d\lambda \\ &= \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda + \int_{-\infty}^{\infty} x_1(\lambda) x_3(t-\lambda)\, d\lambda \\ &= x_1(t) * x_2(t) + x_1(t) * x_3(t) \end{aligned}$$

Associative Property

$$\begin{aligned} x_4(t) &= x_1(t) * \left[x_2(t) * x_3(t) \right] \\ &= \left[x_1(t) * x_2(t) \right] * x_3(t) \end{aligned}$$

Proof

$$\begin{aligned} x_4(t) &= x_1(t) * \left[x_2(t) * x_3(t) \right] \\ &= x_1(t) * \left[\int_{-\infty}^{\infty} x_3(\lambda) x_2(t-\lambda)\, d\lambda \right] \\ &= \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_3(\lambda) x_2(t-\tau-\lambda)\, d\lambda\, d\tau \end{aligned}$$

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Associative (cont.)

- Let η=t-λ:

$$x_4(t) = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_3(t-\eta) x_2(\eta-\tau)\, d\eta\, d\tau$$
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(\eta-\tau)\, d\tau \right] x_3(t-\eta)\, d\eta$$
$$= \int_{-\infty}^{\infty} \left[x_1(\eta) * x_2(\eta) \right] x_3(t-\eta)\, d\eta$$
$$= \left[x_1(t) * x_2(t) \right] * x_3(t)$$

Derivative

$$\frac{d}{dt}\left[x_1(t) * x_2(t) \right] = \frac{dx_1(t)}{dt} * x_2(t)$$
$$= x_1(t) * \frac{dx_2(t)}{dt}$$

Proof:

$$\frac{d}{dt}\left[x_1(t) * x_2(t) \right] = \frac{d}{dt} \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda$$

Leibniz's Rule states that if the limits of integration are not a function of time, the order of integration and differentiation can be reversed.

$$\frac{d}{dt}\left[x_1(t) * x_2(t) \right] = \int_{-\infty}^{\infty} x_1(\lambda) \frac{d}{dt}\left\{ x_2(t-\lambda) \right\} d\lambda$$
$$= x_1(t) * \frac{d}{dt}\left\{ x_2(t) \right\}$$

Derivative – Property II

$$x_1(t) * x_2(t) = \frac{dx_1(t)}{dt} * \int_{-\infty}^{t} x_2(\lambda) d\lambda$$
$$= \frac{dx_2(t)}{dt} * \int_{-\infty}^{t} x_1(\lambda) d\lambda$$

Proof:

$$x_1(t) * x_2(t) = \int_{-\infty}^{t} \frac{d}{d\eta}\left\{ x_1(\eta) * x_2(\eta) \right\} d\eta$$
$$= \int_{-\infty}^{t} \left\{ \frac{dx_1(\eta)}{d\eta} * x_2(\eta) \right\} d\eta$$
$$= \int_{-\infty}^{t} \left\{ \int_{-\infty}^{\infty} \frac{dx_1(\lambda)}{d\lambda} x_2(\eta-\lambda)\, d\lambda \right\} d\eta$$
$$= \int_{-\infty}^{\infty} \frac{dx_1(\lambda)}{d\lambda} \int_{-\infty}^{t} x_2(\eta-\lambda)\, d\eta\, d\lambda$$

(cont.)

Derivative – Property II (cont.)

Let τ = η – λ

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \frac{dx_1(\lambda)}{d\lambda} \int_{-\infty}^{t-\lambda} x_2(\tau)\, d\tau\, d\lambda$$
$$= \frac{dx_1(\lambda)}{d\lambda} * \int_{-\infty}^{t} x_2(\tau)\, d\tau$$

Time Shift

- If $x_3(t) = x_1(t) * x_2(t)$

then $x_1(t-a) * x_2(t-b) = x_3(t-a-b)$

Proof:

$$x_1(t-a) * x_2(t-b) = \int_{-\infty}^{\infty} x_1(\lambda-a) x_2(t-\lambda-b)\, d\lambda$$

η = λ-a

$$= \int_{-\infty}^{\infty} x_1(\eta) x_2(t-a-b-\eta)\, d\eta$$
$$= \left. \int_{-\infty}^{\infty} x_1(\eta) x_2(t-\eta)\, d\eta \right|_{t \to t-a-b}$$
$$= \left. x_3(t) \right|_{t \to t-a-b}$$
$$= x_3(t-a-b)$$

Periodic Functions

- If $x_1(t)$ is periodic with period T, *i.e.*, $x_1(t) = x_1(t+T)$ then $x_3(t) = x_1(t) * x_2(t)$ is periodic with period T.

Proof:

$$x_3(t) = x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_2(\lambda) x_1(t-\lambda)\, d\lambda$$
$$x_3(t+T) = \int_{-\infty}^{\infty} x_2(\lambda) x_1(t+T-\lambda)\, d\lambda$$
$$= \int_{-\infty}^{\infty} x_2(\lambda) x_1(t-\lambda)\, d\lambda$$
$$= x_3(t)$$

Duration and Location

- Duration
  - If $x_1(t)$ is of duration T_1 and $x_2(t)$ is of duration T_2, then $x_3(t) = x_1(t)*x_2(t)$ is of duration T_1+T_2.
- Location
  - If $x_1(t) = 0$ for all $t < a_1$ and all $t>b_1$, and $x_2(t)=0$ for all $t < a_2$ and all $t > b_2$, then $x_3(t)=x_1(t)*x_2(t)=0$ for all $t < a_1+a_2$ and all $t>b_1+b_2$

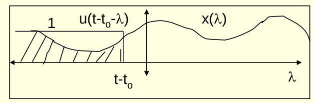
Convolution with Unit Step

$$x_1(t)*u(t-t_o) = \int_{-\infty}^{t-t_o} x(\lambda)\,d\lambda$$

Proof:

$$x_1(t)*u(t-t_o) = \int_{-\infty}^{\infty} x(\lambda)\underbrace{u(t-t_o-\lambda)}_{\text{zero for } \lambda>t-t_o}d\lambda$$

$$= \int_{-\infty}^{t-t_o} x(\lambda)\,d\lambda$$



Convolution with an Impulse

$$x_1(t)*\delta(t-t_o) = x(t-t_o)$$

Proof

$$x_1(t)*\delta(t-t_o) = \int_{-\infty}^{\infty} x(\lambda)\underbrace{\delta(t-t_o-\lambda)}_{\text{zero for } \lambda\neq t-t_o}d\lambda$$

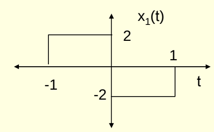
$$= x_1(t-t_o)$$

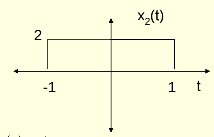
Recall that when we integrate any function times an impulse, we simply evaluate the function at the location of the impulse.

This is also called the *sifting property* of the unit impulse.

Example

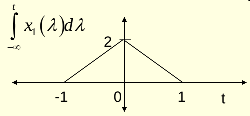
- Determine $x_3(t) = x_1(t)*x_2(t)$



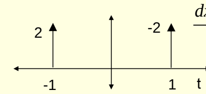


- Using $x_1(t)*x_2(t) = \frac{dx_1(t)}{dt} * \int_{-\infty}^{t} x_2(\lambda)d\lambda$

$\int_{-\infty}^{t} x_1(\lambda)d\lambda$



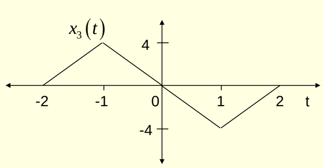
$\frac{dx_2(t)}{dt}$



Example (cont.)

- Now using

$$x_1(t)*\delta(t-t_o) = x(t-t_o)$$



Example (cont.)

Doing it the hard way:

$$x_3(t) = x_1(t)*x_2(t)$$

$$x_1(t) = 2u(t+1) - 4u(t) + 2u(t-1)$$

$$x_2(t) = 2u(t+1) - 2u(t-1)$$

$$x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)\,d\lambda$$

$$= \int_{-\infty}^{\infty} \left\{2u(\lambda+1) - 4u(\lambda) + 2u(\lambda-1)\right\}\ldots$$

$$\left[2u\left(t-(\lambda+1)\right) - 2u\left(t-(\lambda-1)\right)\right]d\lambda$$

Example (cont.)

$$x_1(t)*x_2(t)=\int_{-\infty}^{\infty}\{2u(\lambda+1)-4u(\lambda)+2u(\lambda-1)\}\ldots$$

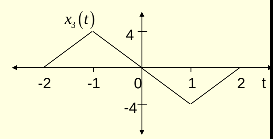
$$\left[2u\left(t-(\lambda+1)\right)-2u\left(t-(\lambda-1)\right)\right]d\lambda$$

$$=\begin{cases} 0 & t\le -2 \\ \int_{-1}^{t+1}4d\lambda & -2\le t\le -1 \\ \int_{-1}^{0}4d\lambda-\int_{0}^{t+1}4d\lambda & -1\le t\le 0 \\ \int_{t-1}^{0}4d\lambda-\int_{0}^{1}4d\lambda & 0\le t\le 1 \\ -\int_{t-1}^{1}4d\lambda & 1\le t\le 2 \\ 0 & t\ge 2 \end{cases}$$

Example (cont.)

$$x_1(t)*x_2(t)=\begin{cases} 0 & t\le -2 \\ 4\lambda\big|_{-1}^{t+1} & -2\le t\le -1 \\ 4-4\lambda\big|_{0}^{t+1} & -1\le t\le 0 \\ 4\lambda\big|_{t-1}^{0}-4 & 0\le t\le 1 \\ -4\lambda\big|_{t-1}^{1} & 1\le t\le 2 \\ 0 & t\ge 2 \end{cases}$$

$$=\begin{cases} 0 & t\le -2 \\ 4t+8 & -2\le t\le -1 \\ -4t & -1\le t\le 0 \\ -4t & 0\le t\le 1 \\ 4t-8 & 1\le t\le 2 \\ 0 & t\ge 2 \end{cases}$$



Area

- Let $\quad x_3(t)=x_1(t)*x_2(t)$

and $\quad A_i=\int_{-\infty}^{\infty}x_i(\lambda)d\lambda$

Then, $A_3 = A_1A_2$

Area (cont.)

Proof:

$$A_3=\int_{-\infty}^{\infty}x_3(\tau)d\tau$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x_1(\lambda)x_2(\tau-\lambda)d\lambda\right]d\tau$$

$$=\int_{-\infty}^{\infty}x_1(\lambda)\underbrace{\left[\int_{-\infty}^{\infty}x_2(\tau-\lambda)d\tau\right]}_{A_2}d\lambda$$

$$=A_2\int_{-\infty}^{\infty}x_1(\lambda)d\lambda$$

$$=A_2A_1$$

Centroid

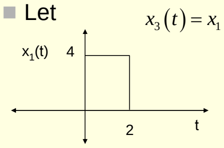
- Let $\quad x_3(t)=x_1(t)*x_2(t)$

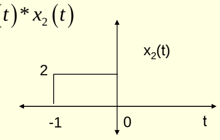
Define the centroid as $\quad \Upsilon_i=\dfrac{\int_{-\infty}^{\infty}\lambda x_i(\lambda)d\lambda}{\int_{-\infty}^{\infty}x_i(\lambda)d\lambda}$

Then, $\quad \Upsilon_3=\Upsilon_1+\Upsilon_2$

Example

- Let $\quad x_3(t)=x_1(t)*x_2(t)$





- Determine the duration, location, area and centroid of $x_3(t)$
- Duration: $\quad T_3=T_1+T_2$ $\;=2+1$ $\;=3$

Example (cont.)

- Location:
  - starting point = 0+(-1) = -1
  - ending point = 2+0 = 2
- Area:

$$A_3 = A_1 A_2 = 8 * 2 = 16$$

- Centroid

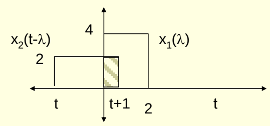
$$\Upsilon_1 = \frac{\int_0^2 4\lambda d\lambda}{8} = \frac{1}{8}\frac{4\lambda^2}{2}\Big|_0^2 = 1$$

$$\Upsilon_2 = \frac{\int_{-1}^0 2\lambda d\lambda}{2} = \frac{1}{2}\frac{2\lambda^2}{2}\Big|_{-1}^0 = -\frac{1}{2}$$

$$\Upsilon_3 = 1 - \frac{1}{2} = \frac{1}{2}$$

Example (cont.)

$$x_3(t) = x_1(t) * x_2(t)$$



$-\infty < t \le -1$: $x_3(t) = 0$

$-1 \le t \le 0$: $x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda = \int_0^{t+1} 8 d\lambda = 8\lambda\big|_0^{t+1} = 8t + 8$

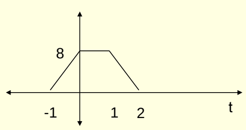
$0 \le t \le 1$: $x_3(t) = 8$

$1 \le t \le 2$: $x_3(t) = \int_t^2 8 d\lambda = 8\lambda\big|_t^2 = 16 - 8t$

$t > 2$: $x_3(t) = 0$

Example (cont.)

$$x_3(t) = x_1(t) * x_2(t)$$



$$T_3 = 3 = T_1 + T_2$$

$$A_3 = 16 = A_1 A_2$$

$-\infty < t \le -1$: $x_3(t) = 0$

$-1 \le t \le 0$: $x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda = \int_0^{t+1} 8 d\lambda = 8\lambda\big|_0^{t+1} = 8t + 8$

$0 \le t \le 1$: $x_3(t) = 8$

$1 \le t \le 2$: $x_3(t) = \int_t^2 8 d\lambda = 8\lambda\big|_t^2 = 16 - 8t$

$t > 2$: $x_3(t) = 0$

Conclusions

- In this lecture we have examined several properties of convolution.
- These properties can be used to simplify more complicated convolution integrals.
- In the next class we will examine the application of convolution to determine the response of a system.