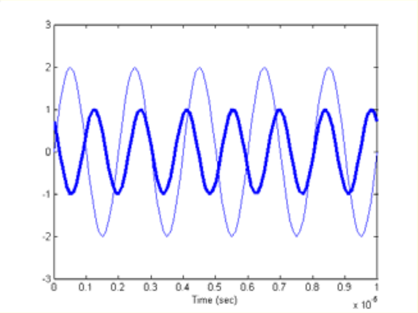
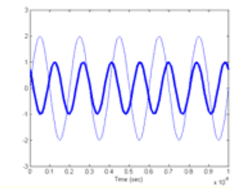
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer

Lecture #5: Convolution Algebra

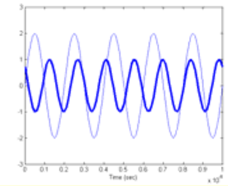




Overview

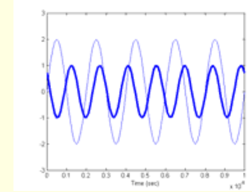
- What to read – Section 3.6 in the text
- This lecture (as well as the previous lecture) deals with the concept of convolution.
- Convolution allows us to determine the output of any linear time-invariant system through the system's impulse response.
  - We will discuss the impulse response next class
- Today we will examine convolution algebra as well as a few special convolutions.

Convolution Algebra



- Convolution properties
  - Commutative Property
  - Distributive Property
  - Associative Property
  - Derivative
  - Time-shifting
  - Convolution involving a periodic function
  - Duration
  - Location
  - Shape
- Special convolutions
  - Step
  - Impulse
- Area
- Centroid

Commutative Property



- Recall the definition of convolution

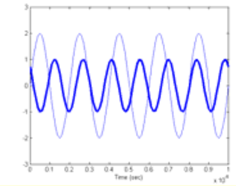
$$x_3(t) = x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$

- The commutative property says that the order of convolution is irrelevant. That is

$$x_3(t) = x_1(t) * x_2(t)$$
$$= x_2(t) * x_1(t)$$

The proof was given last class.

Distributive Property

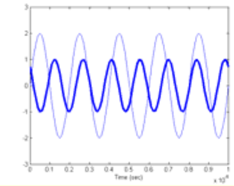


$$x_4(t) = x_1(t) * [x_2(t) + x_3(t)]$$
$$= x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Proof:

$$x_4(t) = x_1(t) * [x_2(t) + x_3(t)]$$
$$= \int_{-\infty}^{\infty} x_1(\lambda)[x_2(t-\lambda) + x_3(t-\lambda)]\, d\lambda$$
$$= \int_{-\infty}^{\infty} [x_1(\lambda) x_2(t-\lambda) + x_1(\lambda) x_3(t-\lambda)]\, d\lambda$$
$$= \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda + \int_{-\infty}^{\infty} x_1(\lambda) x_3(t-\lambda)\, d\lambda$$
$$= x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Associative Property



$$x_4(t) = x_1(t) * \left[x_2(t) * x_3(t) \right]$$
$$= \left[x_1(t) * x_2(t) \right] * x_3(t)$$

Proof

$$x_4(t) = x_1(t) * \left[x_2(t) * x_3(t) \right]$$
$$= x_1(t) * \left[\int_{-\infty}^{\infty} x_3(\lambda) x_2(t-\lambda) d\lambda \right]$$
$$= \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_3(\lambda) x_2(t-\tau-\lambda) d\lambda d\tau$$

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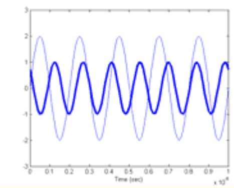
Associative (cont.)



- Let $\eta=t-\lambda$:

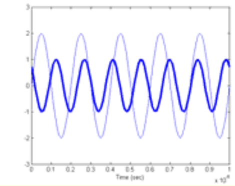
$$x_4(t)=\int_{-\infty}^{\infty} x_1(\tau)\int_{-\infty}^{\infty} x_3(t-\eta)\,x_2(\eta-\tau)\,d\eta\,d\tau$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)\,x_2(\eta-\tau)\,d\tau\right]x_3(t-\eta)\,d\eta$$

$$=\int_{-\infty}^{\infty}\left[x_1(\eta)*x_2(\eta)\right]x_3(t-\eta)\,d\eta$$

$$=\left[x_1(t)*x_2(t)\right]*x_3(t)$$

Derivative



$$\frac{d}{dt}\left[x_1(t) * x_2(t)\right] = \frac{dx_1(t)}{dt} * x_2(t)$$

$$= x_1(t) * \frac{dx_2(t)}{dt}$$

Proof:

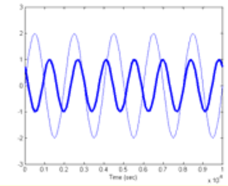
$$\frac{d}{dt}\left[x_1(t) * x_2(t)\right] = \frac{d}{dt}\int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Leibniz's Rule states that if the limits of integration are not a function of time, the order of integration and differentiation can be reversed.

$$\frac{d}{dt}\left[x_1(t) * x_2(t)\right] = \int_{-\infty}^{\infty} x_1(\lambda) \frac{d}{dt}\left\{x_2(t-\lambda)\right\} d\lambda$$

$$= x_1(t) * \frac{d}{dt}\left\{x_2(t)\right\}$$

Derivative – Property II



$$x_1(t)*x_2(t)=\frac{dx_1(t)}{dt}*\int_{-\infty}^{t}x_2(\lambda)d\lambda$$

$$=\frac{dx_2(t)}{dt}*\int^{t}x_1(\lambda)d\lambda$$

Proof:

$$x_1(t)*x_2(t)=\int_{-\infty}^{t}\frac{d}{d\eta}\left\{x_1(\eta)*x_2(\eta)\right\}d\eta$$

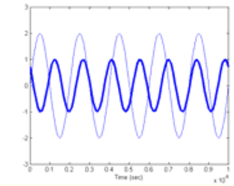
$$=\int_{-\infty}^{t}\left\{\frac{dx_1(\eta)}{d\eta}*x_2(\eta)\right\}d\eta$$

$$=\int_{-\infty}^{t}\left\{\int_{-\infty}^{\infty}\frac{dx_1(\lambda)}{d\lambda}x_2(\eta-\lambda)d\lambda\right\}d\eta$$

$$=\int_{-\infty}^{\infty}\frac{dx_1(\lambda)}{d\lambda}\int_{-\infty}^{t}x_2(\eta-\lambda)d\eta d\lambda$$

(cont.)

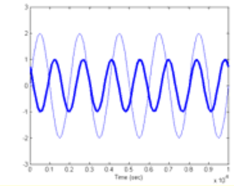
Derivative – Property II (cont.)



Let $\tau = \eta - \lambda$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \frac{dx_1(\lambda)}{d\lambda} \int_{-\infty}^{t-\lambda} x_2(\tau)\, d\tau\, d\lambda$$

$$= \frac{dx_1(\lambda)}{d\lambda} * \int_{-\infty}^{t} x_2(\tau)\, d\tau$$



Time Shift

- If $$x_3(t) = x_1(t) * x_2(t)$$

then $$x_1(t-a) * x_2(t-b) = x_3(t-a-b)$$

Proof: $$x_1(t-a) * x_2(t-b) = \int_{-\infty}^{\infty} x_1(\lambda - a)\, x_2(t-\lambda-b)\, d\lambda$$

$\eta = \lambda$-a

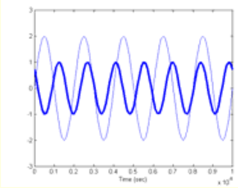
$$= \int_{-\infty}^{\infty} x_1(\eta)\, x_2(t-a-b-\eta)\, d\eta$$

$$= \int_{-\infty}^{\infty} x_1(\eta)\, x_2(t-\eta)\, d\eta \Big|_{t \to t-a-b}$$

$$= x_3(t)\Big|_{t \to t-a-b}$$

$$= x_3(t-a-b)$$

Periodic Functions



- If $x_1(t)$ is periodic with period T, *i.e.*, $x_1(t) = x_1(t+T)$ then $x_3(t) = x_1(t) * x_2(t)$ is periodic with period T.

Proof:

$$x_3(t) = x_1(t) * x_2(t)$$

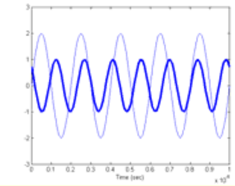
$$= \int_{-\infty}^{\infty} x_2(\lambda)\, x_1(t-\lambda)\, d\lambda$$

$$x_3(t+T) = \int_{-\infty}^{\infty} x_2(\lambda)\, x_1(t+T-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x_2(\lambda)\, x_1(t-\lambda)\, d\lambda$$

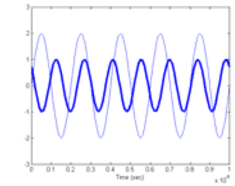
$$= x_3(t)$$

Duration and Location



- Duration
  - If $x_1(t)$ is of duration T_1 and $x_2(t)$ is of duration T_2, then $x_3(t) = x_1(t)*x_2(t)$ is of duration T_1+T_2.
- Location
  - If $x_1(t) = 0$ for all $t < a_1$ and all $t>b_1$, and $x_2(t)=0$ for all $t < a_2$ and all $t > b_2$, then $x_3(t)=x_1(t)*x_2(t)=0$ for all $t < a_1+a_2$ and all $t>b_1+b_2$

Convolution with Unit Step

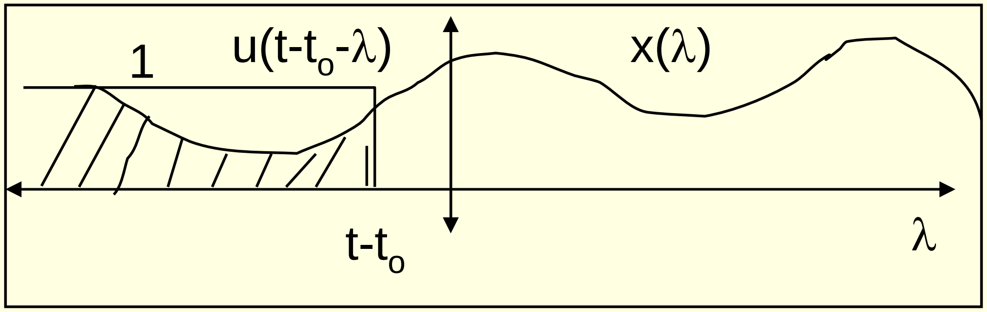


$$x_1(t)*u(t-t_o)=\int_{-\infty}^{t-t_o} x(\lambda)\,d\lambda$$

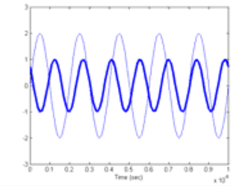
Proof:

$$x_1(t)*u(t-t_o)=\int_{-\infty}^{\infty} x(\lambda)\underbrace{u(t-t_o-\lambda)}_{\text{zero for } \lambda>\text{t-t}_\text{o}}\,d\lambda$$

$$=\int_{-\infty}^{t-t_o} x(\lambda)\,d\lambda$$



Convolution with an Impulse



$$x_1(t) * \delta(t - t_o) = x(t - t_o)$$

Proof

$$x_1(t) * \delta(t - t_o) = \int_{-\infty}^{\infty} x(\lambda) \underbrace{\delta(t - t_o - \lambda)}_{\text{zero for } \lambda \neq t - t_o} d\lambda$$

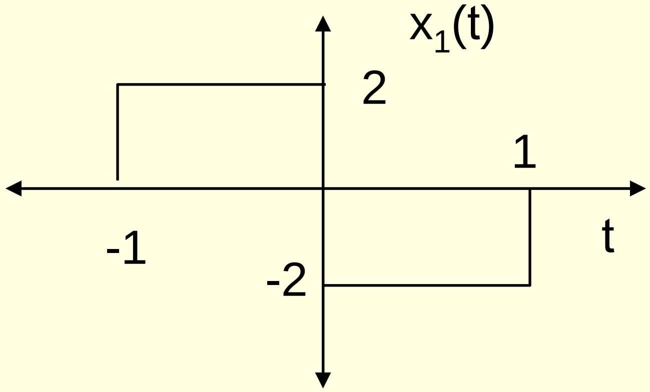
$$= x_1(t - t_o)$$

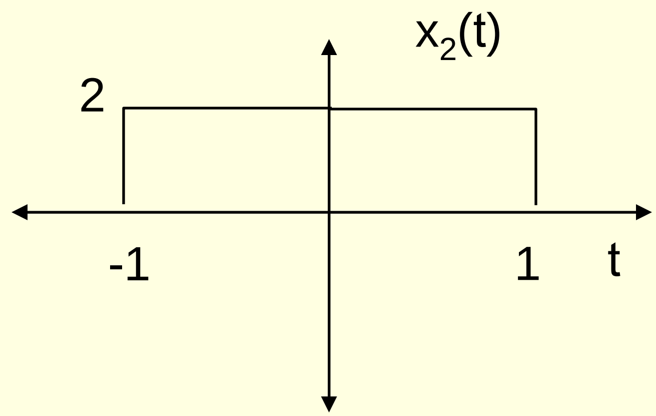
Recall that when we integrate any function times an impulse, we simply evaluate the function at the location of the impulse.

This is also called the *sifting property* of the unit impulse.

Example

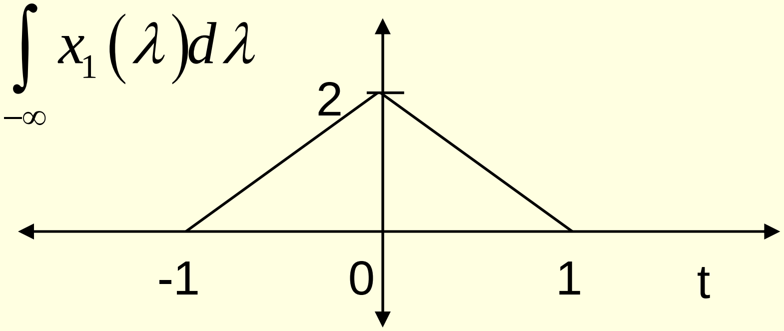
- Determine $x_3(t) = x_1(t) * x_2(t)$



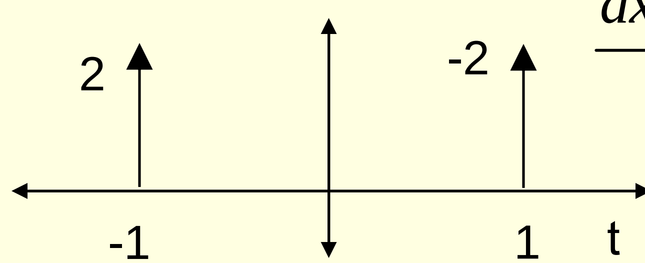


- Using $$x_1(t) * x_2(t) = \frac{dx_1(t)}{dt} * \int_{-\infty}^{t} x_2(\lambda)\,d\lambda$$

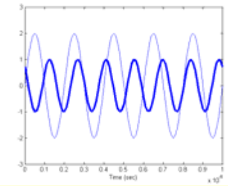
$$\int_{-\infty}^{t} x_1(\lambda)\,d\lambda$$



$$\frac{dx_2(t)}{dt}$$

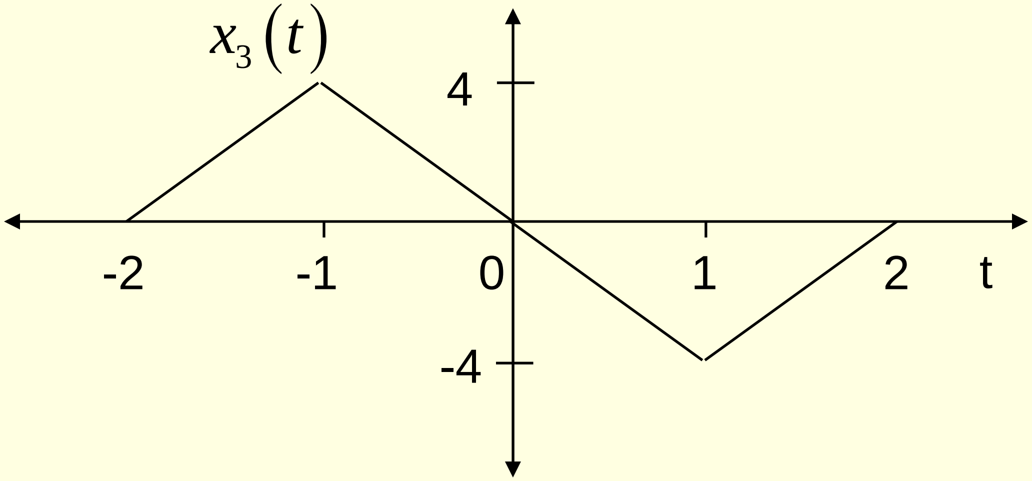


Example (cont.)

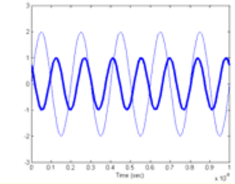


- Now using

$$x_1(t) * \delta(t - t_o) = x(t - t_o)$$



Example (cont.)



Doing it the hard way:

$$x_3(t) = x_1(t) * x_2(t)$$

$$x_1(t) = 2u(t+1) - 4u(t) + 2u(t-1)$$

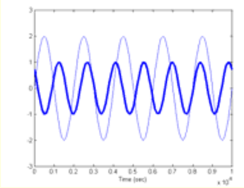
$$x_2(t) = 2u(t+1) - 2u(t-1)$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} \left\{2u(\lambda+1) - 4u(\lambda) + 2u(\lambda-1)\right\} \ldots$$

$$\left[2u\big(t-(\lambda+1)\big) - 2u\big(t-(\lambda-1)\big)\right] d\lambda$$

Example (cont.)



$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \left\{ 2u(\lambda+1) - 4u(\lambda) + 2u(\lambda-1) \right\} \ldots$$

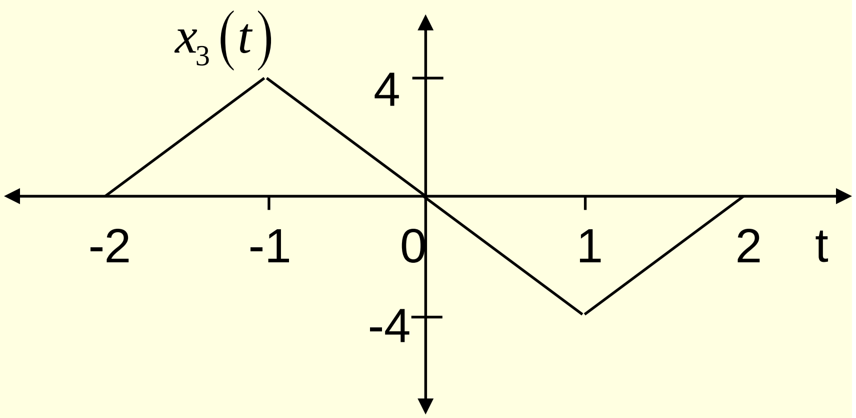
$$\left[2u\left(t-(\lambda+1)\right) - 2u\left(t-(\lambda-1)\right) \right] d\lambda$$

$$= \begin{cases} 0 & t \le -2 \\ \int_{-1}^{t+1} 4 d\lambda & -2 \le t \le -1 \\ \int_{-1}^{0} 4d\lambda - \int_{0}^{t+1} 4d\lambda & -1 \le t \le 0 \\ \int_{t-1}^{0} 4d\lambda - \int_{0}^{1} 4d\lambda & 0 \le t \le 1 \\ -\int_{t-1}^{1} 4d\lambda & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

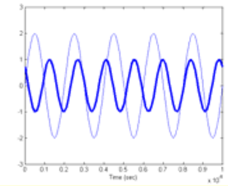
Example (cont.)

$$x_1(t) * x_2(t) = \begin{cases} 0 & t \le -2 \\ 4\lambda\Big|_{-1}^{t+1} & -2 \le t \le -1 \\ 4 - 4\lambda\Big|_{0}^{t+1} & -1 \le t \le 0 \\ 4\lambda\Big|_{t-1}^{0} - 4 & 0 \le t \le 1 \\ -4\lambda\Big|_{t-1}^{1} & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

$$= \begin{cases} 0 & t \le -2 \\ 4t + 8 & -2 \le t \le -1 \\ -4t & -1 \le t \le 0 \\ -4t & 0 \le t \le 1 \\ 4t - 8 & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$



Area

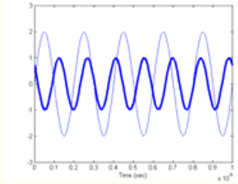


- Let $$x_3(t) = x_1(t) * x_2(t)$$

and $$A_i = \int_{-\infty}^{\infty} x_i(\lambda)\, d\lambda$$

Then, $A_3 = A_1 A_2$

Area (cont.)



Proof:

$$A_3 = \int_{-\infty}^{\infty} x_3(\tau)\, d\tau$$

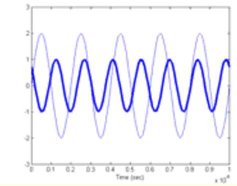
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\lambda)\, x_2(\tau-\lambda)\, d\lambda \right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\lambda) \underbrace{\left[\int_{-\infty}^{\infty} x_2(\tau-\lambda)\, d\tau \right]}_{A_2} d\lambda$$

$$= A_2 \int_{-\infty}^{\infty} x_1(\lambda)\, d\lambda$$

$$= A_2 A_1$$

Centroid



- Let $x_3(t) = x_1(t) * x_2(t)$

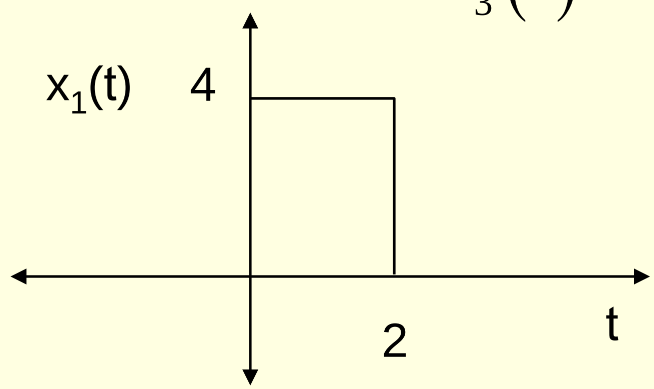
Define the centroid as
$$\Upsilon_i = \frac{\int_{-\infty}^{\infty} \lambda x_i(\lambda)\, d\lambda}{\int_{-\infty}^{\infty} x_i(\lambda)\, d\lambda}$$

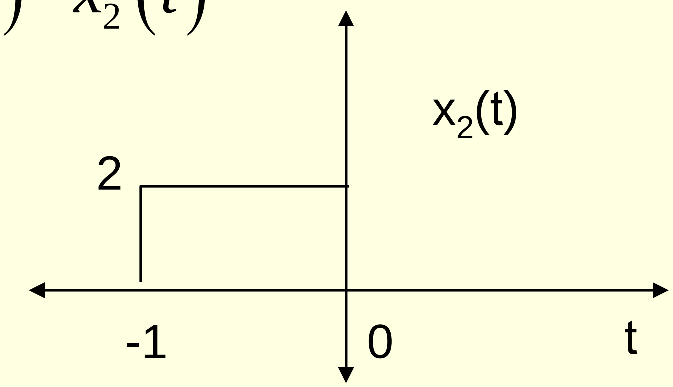
Then, $\Upsilon_3 = \Upsilon_1 + \Upsilon_2$

Example

- Let

$$x_3(t) = x_1(t) * x_2(t)$$

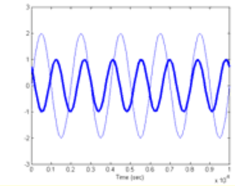




- Determine the duration, location, area and centroid of $x_3(t)$
- Duration:

$$\begin{aligned} T_3 &= T_1 + T_2 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

Example (cont.)



- Location:
  - starting point = 0+(-1) = -1
  - ending point = 2+0 = 2
- Area:

$$A_3 = A_1 A_2$$
$$= 8 * 2$$
$$= 16$$

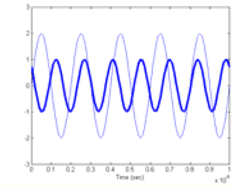
- Centroid

$$\Upsilon_1 = \frac{\int_0^2 4\lambda d\lambda}{8} = \frac{1}{8}\frac{4\lambda^2}{2}\Big|_0^2 = 1$$

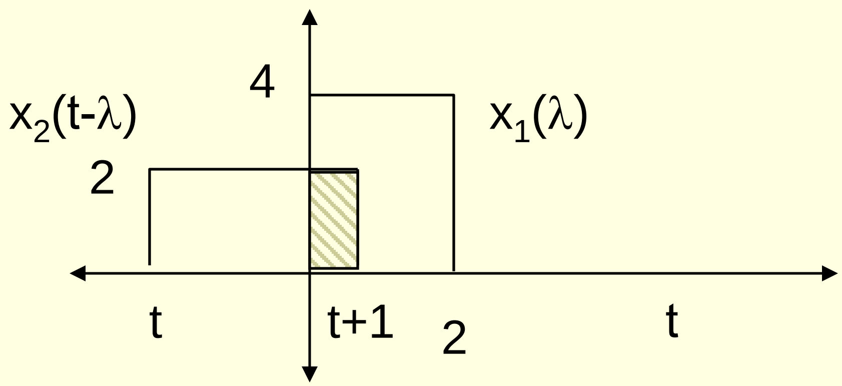
$$\Upsilon_2 = \frac{\int_{-1}^0 2\lambda d\lambda}{2} = \frac{1}{2}\frac{2\lambda^2}{2}\Big|_{-1}^0 = -\frac{1}{2}$$

$$\Upsilon_3 = 1 - \frac{1}{2} = \frac{1}{2}$$

Example (cont.)



$$x_3(t) = x_1(t) * x_2(t)$$



$-\infty < t \le -1 \qquad x_3(t) = 0$

$$-1 \le t \le 0 \qquad x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$

$$= \int_0^{t+1} 8 d\lambda$$

$$= 8\lambda\Big|_0^{t+1}$$

$$= 8t + 8$$

$0 \le t \le 1 \qquad x_3(t) = 8$

$$1 \le t \le 2 \qquad x_3(t) = \int_t^2 8 d\lambda$$

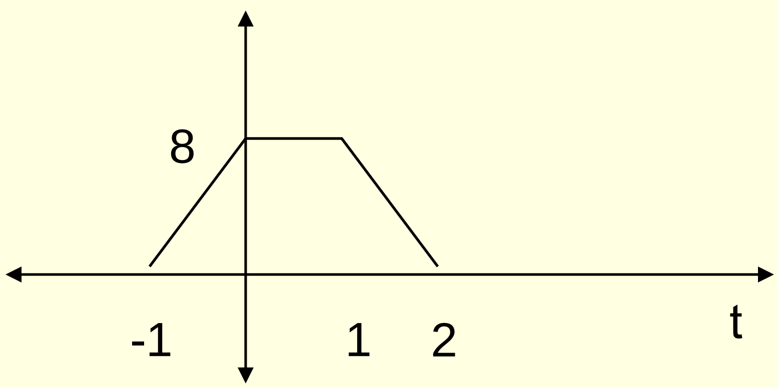
$$= 8\lambda\Big|_t^2$$

$$= 16 - 8t$$

$t > 2 \qquad x_3(t) = 0$

Example (cont.)

$$x_3(t) = x_1(t) * x_2(t)$$



$$T_3 = 3$$
$$= T_1 + T_2$$
$$A_3 = 16$$
$$= A_1 A_2$$

$-\infty < t \le -1$: $\quad x_3(t) = 0$

$-1 \le t \le 0$:

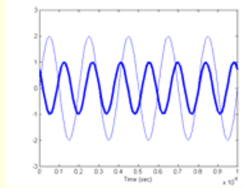
$$x_3(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda$$
$$= \int_{0}^{t+1} 8\, d\lambda$$
$$= 8\lambda \Big|_{0}^{t+1}$$
$$= 8t + 8$$

$0 \le t \le 1$: $\quad x_3(t) = 8$

$1 \le t \le 2$:

$$x_3(t) = \int_{t}^{2} 8\, d\lambda$$
$$= 8\lambda \Big|_{t}^{2}$$
$$= 16 - 8t$$

$t > 2$: $\quad x_3(t) = 0$



Conclusions

- In this lecture we have examined several properties of convolution.
- These properties can be used to simplify more complicated convolution integrals.
- In the next class we will examine the application of convolution to determine the response of a system.