ECE 2704
Signals and Systems
Spring 2006

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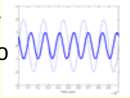
Lecture #6: Convolution Applied to LTI Systems

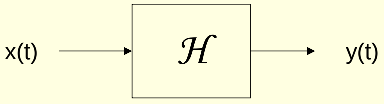


Overview

- What to read – Section 3.6 in the text
- This lecture connects the concept of convolution with linear system theory.
- Convolution allows us to determine the output of any linear time-invariant system through the system's impulse response.
- Today we will show this, discuss how we determine the impulse response and show an example of its use.

Linear Time-Invariant System

- Consider an LTI System



$$x(t)\xrightarrow{\mathcal{H}}y(t)$$

- If *x(t)* = *δ(t)*, then the output *y(t)* = *h(t)* is termed the *impulse response*

$$\delta(t)\xrightarrow{\mathcal{H}}h(t)$$

LTI System (cont.)

- Since the system is linear and time-invariant, we know that if

$$\delta(t)\xrightarrow{\mathcal{H}}h(t)$$

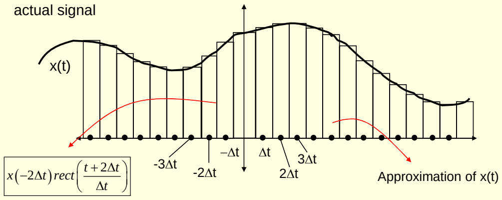
then

$$A\delta(t)\xrightarrow{\mathcal{H}}Ah(t)$$

$$\delta(t-t_o)\xrightarrow{\mathcal{H}}h(t-t_o)$$

Signal Representation

We may approximate a signal x(t) as a weighted sum of pulses.
As the width of the pulses goes to zero, the approximation approaches the actual signal



$$x(t)\approx \ldots + x(-2\Delta t)\,rect\left(\frac{t+2\Delta t}{\Delta t}\right)+x(-\Delta t)\,rect\left(\frac{t+\Delta t}{\Delta t}\right)+\ldots$$

$$x(0)\,rect\left(\frac{t}{\Delta t}\right)+x(\Delta t)\,rect\left(\frac{t-\Delta t}{\Delta t}\right)+x(2\Delta t)\,rect\left(\frac{t-2\Delta t}{\Delta t}\right)+\ldots$$

$$=\sum_{n=-\infty}^{\infty}x(n\Delta t)\,rect\left(\frac{t-n\Delta t}{\Delta t}\right)$$

Signal Representation (cont.)

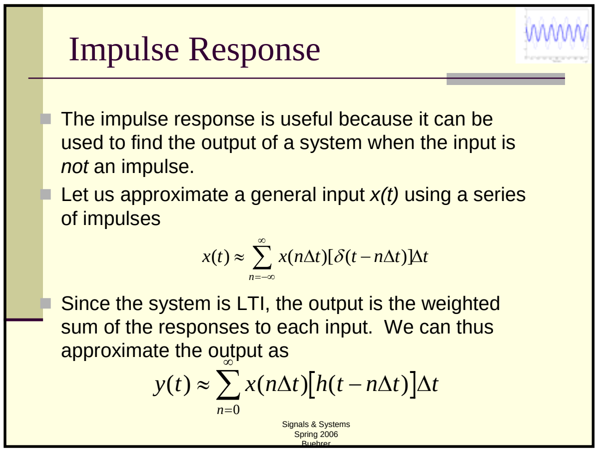
$$x(t)\approx\sum_{n=-\infty}^{\infty}x(n\Delta t)\,rect\left(\frac{t-n\Delta t}{\Delta t}\right)$$

$$=\sum_{n=-\infty}^{\infty}x(n\Delta t)\Delta t\underbrace{\left[\frac{1}{\Delta t}rect\left(\frac{t-n\Delta t}{\Delta t}\right)\right]}_{\text{unit area pulse}}$$

$$\lim_{\Delta t\to 0}\left\{\sum_{n=-\infty}^{\infty}x(n\Delta t)\Delta t\underbrace{\left[\frac{1}{\Delta t}rect\left(\frac{t-n\Delta t}{\Delta t}\right)\right]}_{\text{unit area pulse}}\right\}=\int_{-\infty}^{\infty}x(\lambda)\delta(t-\lambda)\,d\lambda$$

$$=x(t)$$

Thus, we can approximate a function *x(t)* as a series of weighted, time-shifted impulses.



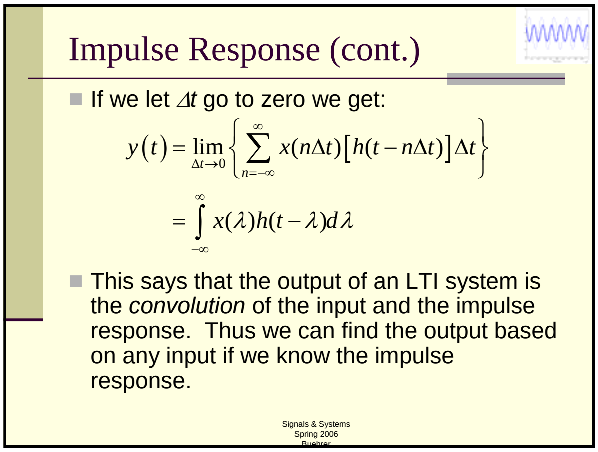
Impulse Response

- The impulse response is useful because it can be used to find the output of a system when the input is *not* an impulse.
- Let us approximate a general input *x(t)* using a series of impulses

$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta t)[\delta(t-n\Delta t)]\Delta t$$

- Since the system is LTI, the output is the weighted sum of the responses to each input. We can thus approximate the output as

$$y(t) \approx \sum_{n=0}^{\infty} x(n\Delta t)\big[h(t-n\Delta t)\big]\Delta t$$



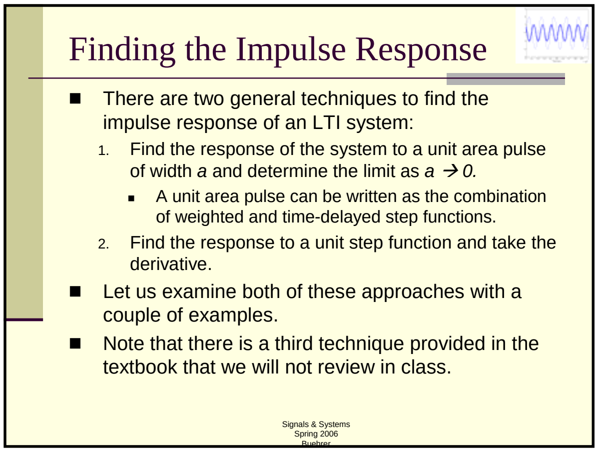
Impulse Response (cont.)

- If we let *Δt* go to zero we get:

$$y(t) = \lim_{\Delta t \to 0}\left\{\sum_{n=-\infty}^{\infty} x(n\Delta t)\big[h(t-n\Delta t)\big]\Delta t\right\}$$

$$= \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

- This says that the output of an LTI system is the *convolution* of the input and the impulse response. Thus we can find the output based on any input if we know the impulse response.

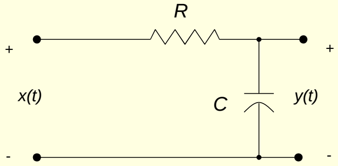


Finding the Impulse Response

- There are two general techniques to find the impulse response of an LTI system:
    1. Find the response of the system to a unit area pulse of width *a* and determine the limit as *a → 0.*
        - A unit area pulse can be written as the combination of weighted and time-delayed step functions.
    2. Find the response to a unit step function and take the derivative.
- Let us examine both of these approaches with a couple of examples.
- Note that there is a third technique provided in the textbook that we will not review in class.

Example

- Consider the following RC circuit:



- Using Kirchoff's Voltage Law we can relate the input to the output

Example

- Using Kirchhoff's Voltage law, the sum of the voltages around the circuit is zero:

$$x(t) = Ri(t) + y(t)$$

- The voltage across the capacitor is related to the current through the resistor

$$i(t) = C\frac{dy(t)}{dt}$$

- Eliminating *i(t)* from the previous equation we obtain an equation relating the input to the output:

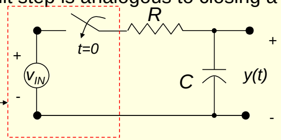
$$x(t) = RC\frac{dy(t)}{dt} + y(t)$$

Unit Step Response

- Both ways to obtain the impulse response require us to first obtain the step response. So, let us do this first.
- We assume that the circuit has had no input for t<0 and all initial conditions are zero. In other words we assume

$$x(t) = 0 \quad t \le 0$$
$$y(t) = 0 \quad t \le 0$$

- The unit step is analogous to closing a switch at time = 0.



This box represents the application of a step function. V_{in} is the height of the step.

Linear First Order Diff. Eq.

- Rewriting the equation for the circuit after time = 0:

$$V_{in} = RC\frac{dy(t)}{dt} + y(t)$$

- This is a first-order linear differential equation. To solve it, we first separate the variables *y* and *t*:

$$RC\frac{dy}{dt} = V_{in} - y$$

$$\frac{dy}{y - V_{in}} = -\frac{1}{RC}dt$$

We drop the dependence of y on t for notational simplicity. It is however, still dependent on t.

- Next, we integrate both sides using λ and τ as our variables of integration:

$$\int_{y_o}^{y(t)} \frac{d\lambda}{\lambda - V_{in}} = -\int_0^t \frac{1}{RC}d\tau$$

y_o represents the intial value of *y* at time = 0. We will eventually set this to zero since the initial conditions are 0.

Example (cont.)

$$\int_{y_o}^{y(t)} \frac{d\lambda}{\lambda - V_{in}} = -\int_0^t \frac{1}{RC}d\tau$$

$$\ln(\lambda - V_{in})\Big|_{y_o}^{y(t)} = -\frac{1}{RC}\tau\Big|_0^t$$

$$\ln\left\{\frac{y(t) - V_{in}}{y_o - V_{in}}\right\} = -\frac{1}{RC}t$$

- Taking *exp(x)* of both sides:

$$\frac{y(t) - V_{in}}{y_o - V_{in}} = e^{-\frac{1}{RC}t}$$

$$y(t) = (y_o - V_{in})e^{-\frac{1}{RC}t} + V_{in}$$

Example (cont.)

- Now, since we have assumed that the initial conditions are zero, $y_o = 0$. Further, since we are examining the *unit* step, $V_{in} = 1$. Thus:

$$y(t) = (y_o - V_{in})e^{-\frac{1}{RC}t} + V_{in}$$

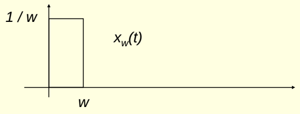
$$= 1 - e^{-\frac{1}{RC}t}$$

- Finally, since the output is zero for t<=0, and this solution only applies for t>=0, we can finally write the step response $y_s(t)$ as

$$y_s(t) = \left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

Impulse Response – Method 1

- The first method of finding the impulse response involves finding the response to a unit area pulse of width *w:*



- This pulse can be written in terms of the unit pulse as

$$x_w(t)=\frac{1}{w}u(t)-\frac{1}{w}u(t-w)$$

- Due to the properties of linearity and time-invariance:

$$y_w(t)=\frac{1}{w}y_s(t)-\frac{1}{w}y_s(t-w)$$

Method 1 (cont.)

$$y_s(t)=\left(1-e^{-\frac{1}{RC}t}\right)u(t)$$

$$y_w(t)=\frac{1}{w}y_s(t)-\frac{1}{w}y_s(t-w)$$
$$=\frac{1}{w}\left(1-e^{-\frac{1}{RC}t}\right)u(t)-\frac{1}{w}\left(1-e^{-\frac{1}{RC}(t-w)}\right)u(t-w)$$

- The impulse response can be found by taking the limit as w→0:

$$h(t)=\lim_{w\to 0}y_w(t)$$
$$=\lim_{w\to 0}\left[\frac{1}{w}\left(1-e^{-\frac{1}{RC}t}\right)u(t)-\frac{1}{w}\left(1-e^{-\frac{1}{RC}(t-w)}\right)u(t-w)\right]$$

Method 1 (cont.)

$$h(t)=\lim_{w\to 0}\left[\frac{1}{w}\left(1-e^{-\frac{1}{RC}t}\right)u(t)-\frac{1}{w}\left(1-e^{-\frac{1}{RC}(t-w)}\right)u(t-w)\right]$$
$$=\lim_{w\to 0}\left[\frac{1}{w}\left(1-e^{-\frac{1}{RC}t}\right)-\frac{1}{w}\left(1-e^{-\frac{1}{RC}(t-w)}\right)\right]u(t)$$

u(t-w) → *u(t)* as *w*→0

$$=\lim_{w\to 0}\left[\frac{1}{w}\left(e^{-\frac{1}{RC}(t-w)}-e^{-\frac{1}{RC}t}\right)\right]u(t)$$
$$=\frac{0}{0}$$

Thus, we must use L'Hopital's rule to find the solution.

Continuing:

Using L'Hopital's Rule:

$$h(t)=\lim_{w\to 0}\left[\frac{\frac{d}{dw}\left(e^{-\frac{1}{RC}(t-w)}-e^{-\frac{1}{RC}t}\right)}{\frac{d}{dw}w}\right]u(t)$$

$$=\lim_{w\to 0}\left[\frac{\frac{1}{RC}\left(e^{-\frac{1}{RC}(t-w)}\right)}{1}\right]u(t)$$

$$=\frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

Thus, we have the impulse response for this circuit.

Method 2

- A second method to calculate the impulse response comes directly from the step response.

$$y_s(t)=u(t)*h(t)=\int_{-\infty}^{t}h(\lambda)d\lambda$$

$$\frac{d}{dt}\{y_s(t)\}=h(t)$$

- Returning to the step response of this circuit:

$$y_s(t)=\left(1-e^{-\frac{1}{RC}t}\right)u(t)$$

$$h(t)=\frac{d}{dt}\left\{\left(1-e^{-\frac{1}{RC}t}\right)u(t)\right\}$$

Method 2 (cont.)

$$y_s(t)=\left(1-e^{-\frac{1}{RC}t}\right)u(t)$$

$$h(t)=\frac{d}{dt}\left\{\left(1-e^{-\frac{1}{RC}t}\right)u(t)\right\}$$

$$=\frac{d}{dt}\left(1-e^{-\frac{1}{RC}t}\right)u(t)+\left(1-e^{-\frac{1}{RC}t}\right)\frac{d}{dt}u(t)$$

$$=\frac{1}{RC}e^{-\frac{1}{RC}t}u(t)+\boxed{\left(1-e^{-\frac{1}{RC}t}\right)}\delta(t)$$

$$=\frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

This term is zero at the only point where impulse is non-zero (t=0)

Example

- Assume that the input to this circuit is $\cos(2\pi ft)$
- Find the output
- From our relationship between input and output:

$$y(t)=\int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$
$$=\int_{0}^{\infty} e^{-\lambda/RC}\cos\left[2\pi f\left(t-\lambda\right)\right]d\lambda$$

- We will want to use the following integrals from an integral table

$$\int e^{au}\cos(bu)du=\frac{e^{au}}{a^2+b^2}\left(a\cos(bu)+b\sin(bu)\right)$$
$$\int e^{au}\sin(bu)du=\frac{e^{au}}{a^2+b^2}\left(a\sin(bu)-b\cos(bu)\right)$$

Example (cont.)

- Continuing:

$$y(t)=\int_{0}^{\infty} e^{-\lambda/RC}\cos\left[2\pi f\left(t-\lambda\right)\right]d\lambda$$
$$=\int_{0}^{\infty} e^{-\lambda/RC}\left\{\cos[2\pi ft]\cos[-2\pi f\lambda]-\sin[2\pi ft]\sin[-2\pi f\lambda]\right\}d\lambda$$
$$=\int_{0}^{\infty} e^{-\lambda/RC}\left\{\cos[2\pi ft]\cos[2\pi f\lambda]+\sin[2\pi ft]\sin[2\pi f\lambda]\right\}d\lambda$$
$$=\cos[2\pi ft]\int_{0}^{\infty} e^{-\lambda/RC}\cos[2\pi f\lambda]d\lambda+\sin[2\pi ft]\int_{0}^{\infty} e^{-\lambda/RC}\sin[2\pi f\lambda]d\lambda$$

- Using our integral tables:

$$y(t)=\cos[2\pi ft]\frac{e^{-\lambda/RC}}{(1/RC)^2+(2\pi f)^2}\left(\frac{-1}{RC}\cos(2\pi f\lambda)+2\pi f\sin(2\pi f\lambda)\right)\Bigg|_0^\infty$$
$$+\sin[2\pi ft]\frac{e^{-\lambda/RC}}{(1/RC)^2+(2\pi f)^2}\left(\frac{-1}{RC}\sin(2\pi f\lambda)-2\pi f\cos(2\pi f\lambda)\right)\Bigg|_0^\infty$$

Example (cont.)

$$y(t)=\cos[2\pi ft]\frac{e^{-\lambda/RC}}{(1/RC)^2+(2\pi f)^2}\left(\frac{-1}{RC}\cos(2\pi f\lambda)+2\pi f\sin(2\pi f\lambda)\right)\Bigg|_0^\infty$$
$$+\sin[2\pi ft]\frac{e^{-\lambda/RC}}{(1/RC)^2+(2\pi f)^2}\left(\frac{-1}{RC}\sin(2\pi f\lambda)-2\pi f\cos(2\pi f\lambda)\right)\Bigg|_0^\infty$$
$$=\cos[2\pi ft]\left\{0-\frac{1}{(1/RC)^2+(2\pi f)^2}\left(\frac{-1}{RC}\cos(0)+2\pi f\sin(0)\right)\right\}$$
$$+\sin[2\pi ft]\left\{0-\frac{1}{(1/RC)^2+(2\pi f)^2}\left(\frac{-1}{RC}\sin(0)-2\pi f\cos(0)\right)\right\}$$
$$=\frac{1/RC}{(1/RC)^2+(2\pi f)^2}\cos[2\pi ft]+\frac{2\pi f}{(1/RC)^2+(2\pi f)^2}\sin[2\pi ft]$$

Conclusions

- For linear, time-invariant systems the input can be represented as the sum of scaled and time-shifted impulses
- This permits the output to be determined as the convolution of the input with the system's impulse response.
- The impulse response can be determined by either taking the response of a pulse and letting the width approach zero or by taking the derivative of the unit step response.