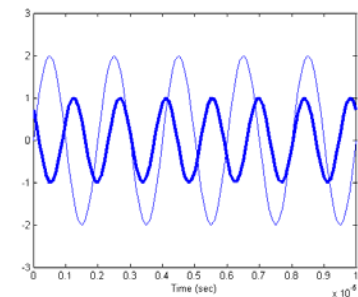


ECE 2704

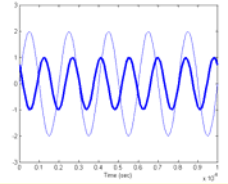
Signals and Systems

Spring 2006

Instructor: Dr. R. Michael Buehrer
Lecture #6: Convolution Applied to
LTI Systems

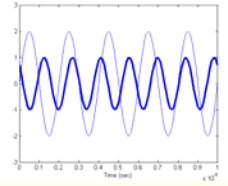


Overview

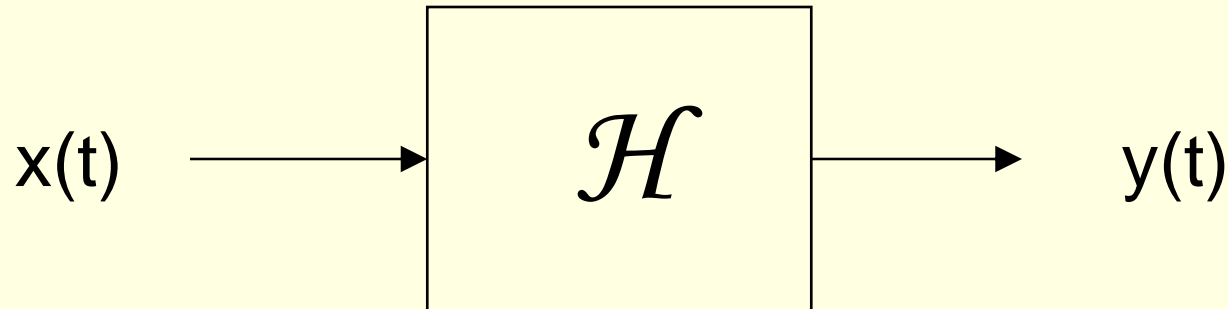


- What to read – Section 3.6 in the text
- This lecture connects the concept of convolution with linear system theory.
- Convolution allows us to determine the output of any linear time-invariant system through the system's impulse response.
- Today we will show this, discuss how we determine the impulse response and show an example of its use.

Linear Time-Invariant System



- Consider an LTI System

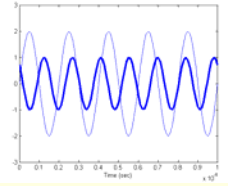


$$x(t) \xrightarrow{\mathcal{H}} y(t)$$

- If $x(t) = \delta(t)$, then the output $y(t) = h(t)$ is termed the *impulse response*

$$\delta(t) \xrightarrow{\mathcal{H}} h(t)$$

LTI System (cont.)



- Since the system is linear and time-invariant, we know that if

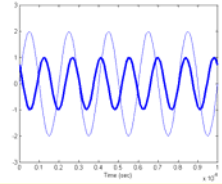
$$\delta(t) \xrightarrow{\mathcal{H}} h(t)$$

then

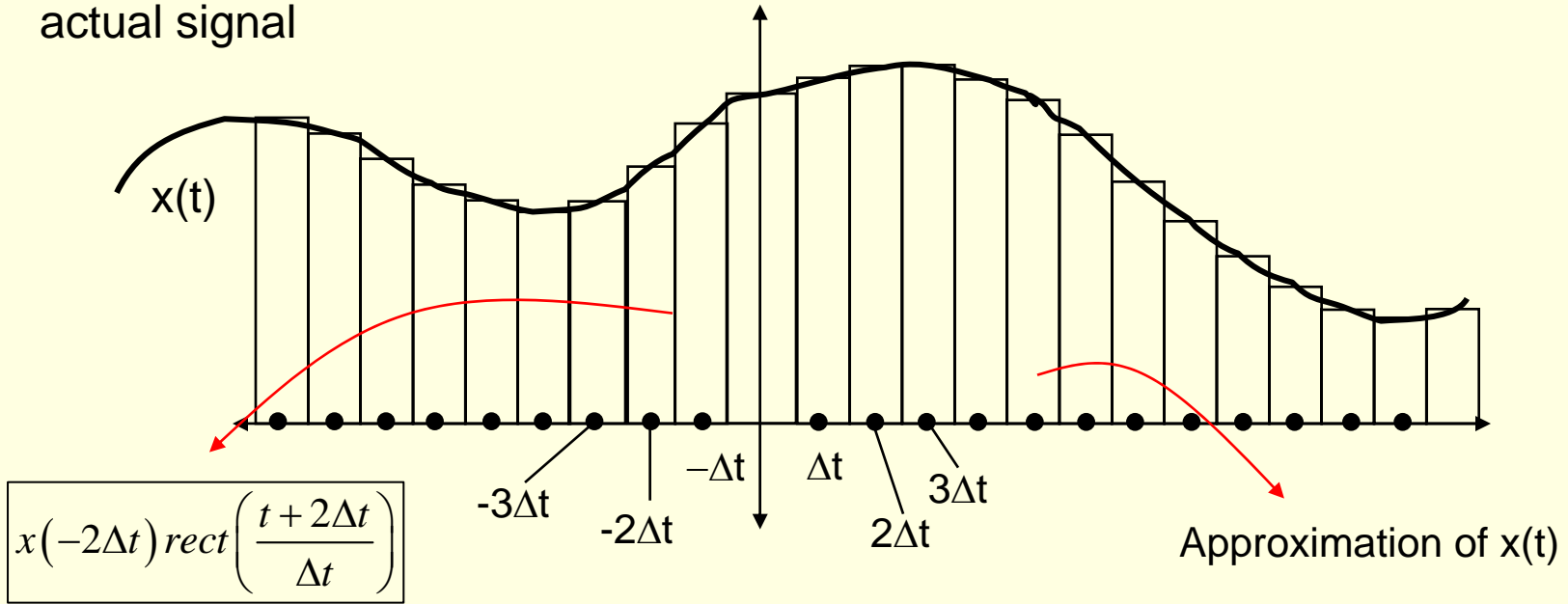
$$A\delta(t) \xrightarrow{\mathcal{H}} Ah(t)$$

$$\delta(t - t_o) \xrightarrow{\mathcal{H}} h(t - t_o)$$

Signal Representation



We may approximate a signal $x(t)$ as a weighted sum of pulses.
As the width of the pulses goes to zero, the approximation approaches the actual signal

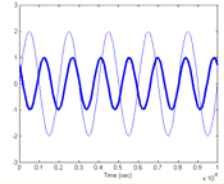


$$x(t) \approx \dots + x(-2\Delta t) \operatorname{rect}\left(\frac{t+2\Delta t}{\Delta t}\right) + x(-\Delta t) \operatorname{rect}\left(\frac{t+\Delta t}{\Delta t}\right) + \dots$$

$$x(0) \operatorname{rect}\left(\frac{t}{\Delta t}\right) + x(\Delta t) \operatorname{rect}\left(\frac{t-\Delta t}{\Delta t}\right) + x(2\Delta t) \operatorname{rect}\left(\frac{t-2\Delta t}{\Delta t}\right) + \dots$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta t) \operatorname{rect}\left(\frac{t-n\Delta t}{\Delta t}\right)$$

Signal Representation (cont.)

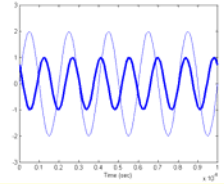


$$\begin{aligned}x(t) &\approx \sum_{n=-\infty}^{\infty} x(n\Delta t) \operatorname{rect}\left(\frac{t-n\Delta t}{\Delta t}\right) \\ &= \sum_{n=-\infty}^{\infty} x(n\Delta t) \Delta t \underbrace{\left[\frac{1}{\Delta t} \operatorname{rect}\left(\frac{t-n\Delta t}{\Delta t}\right)\right]}_{\text{unit area pulse}}\end{aligned}$$

$$\begin{aligned}\lim_{\Delta t \rightarrow 0} \left\{ \sum_{n=-\infty}^{\infty} x(n\Delta t) \Delta t \underbrace{\left[\frac{1}{\Delta t} \operatorname{rect}\left(\frac{t-n\Delta t}{\Delta t}\right)\right]}_{\text{unit area pulse}} \right\} &= \int_{-\infty}^{\infty} x(\lambda) \delta(t-\lambda) d\lambda \\ &= x(t)\end{aligned}$$

Thus, we can approximate a function $x(t)$ as a series of weighted, time-shifted impulses.

Impulse Response



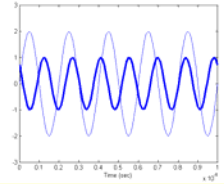
- The impulse response is useful because it can be used to find the output of a system when the input is *not* an impulse.
- Let us approximate a general input $x(t)$ using a series of impulses

$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta t)[\delta(t - n\Delta t)]\Delta t$$

- Since the system is LTI, the output is the weighted sum of the responses to each input. We can thus approximate the output as

$$y(t) \approx \sum_{n=0}^{\infty} x(n\Delta t)[h(t - n\Delta t)]\Delta t$$

Impulse Response (cont.)

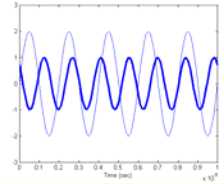


- If we let Δt go to zero we get:

$$y(t) = \lim_{\Delta t \rightarrow 0} \left\{ \sum_{n=-\infty}^{\infty} x(n\Delta t) [h(t - n\Delta t)] \Delta t \right\}$$
$$= \int_{-\infty}^{\infty} x(\lambda) h(t - \lambda) d\lambda$$

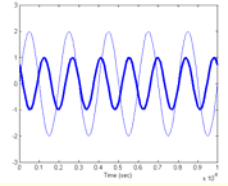
- This says that the output of an LTI system is the *convolution* of the input and the impulse response. Thus we can find the output based on any input if we know the impulse response.

Finding the Impulse Response

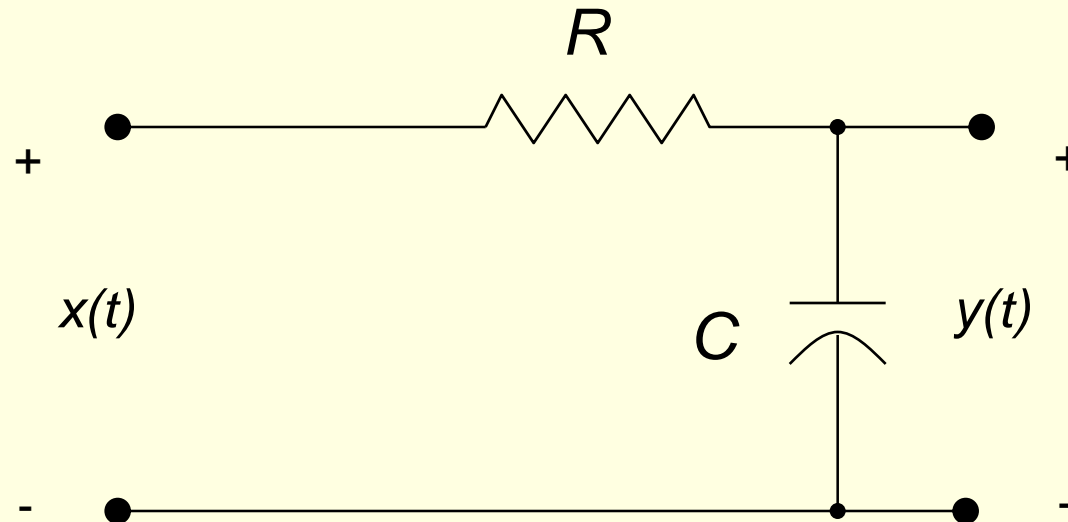


- There are two general techniques to find the impulse response of an LTI system:
 1. Find the response of the system to a unit area pulse of width a and determine the limit as $a \rightarrow 0$.
 - A unit area pulse can be written as the combination of weighted and time-delayed step functions.
 2. Find the response to a unit step function and take the derivative.
- Let us examine both of these approaches with a couple of examples.
- Note that there is a third technique provided in the textbook that we will not review in class.

Example

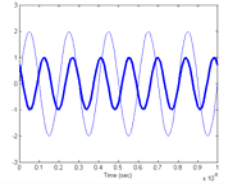


- Consider the following RC circuit:



- Using Kirchoff's Voltage Law we can relate the input to the output

Example



- Using Kirchhoff's Voltage law, the sum of the voltages around the circuit is zero:

$$x(t) = Ri(t) + y(t)$$

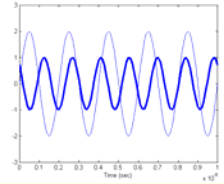
- The voltage across the capacitor is related to the current through the resistor

$$i(t) = C \frac{dy(t)}{dt}$$

- Eliminating $i(t)$ from the previous equation we obtain an equation relating the input to the output:

$$x(t) = RC \frac{dy(t)}{dt} + y(t)$$

Unit Step Response



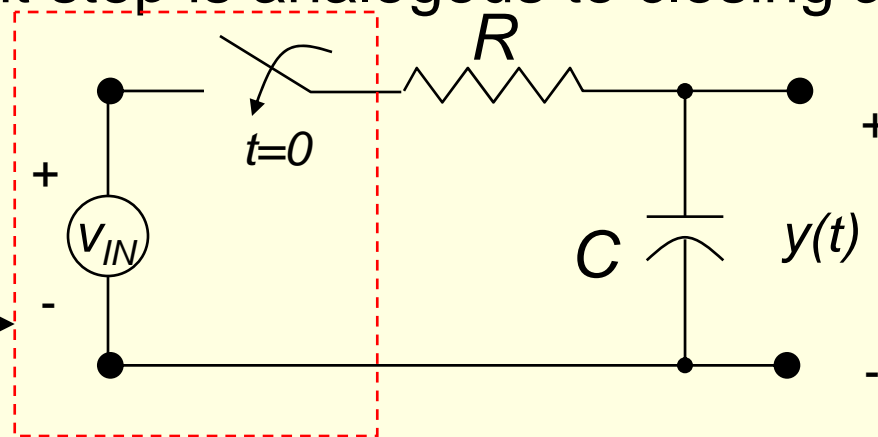
- Both ways to obtain the impulse response require us to first obtain the step response. So, let us do this first.
- We assume that the circuit has had no input for $t < 0$ and all initial conditions are zero. In other words we assume

$$x(t) = 0 \quad t \leq 0$$

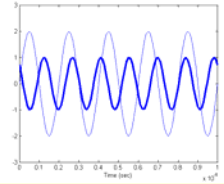
$$y(t) = 0 \quad t \leq 0$$

- The unit step is analogous to closing a switch at time $= 0$.

This box represents the application of a step function. V_{in} is the height of the step.



Linear First Order Diff. Eq.



- Rewriting the equation for the circuit after time = 0:

$$V_{in} = RC \frac{dy(t)}{dt} + y(t)$$

- This is a first-order linear differential equation. To solve it, we first separate the variables y and t .

$$RC \frac{dy}{dt} = V_{in} - y$$
$$\frac{dy}{y - V_{in}} = -\frac{1}{RC} dt$$

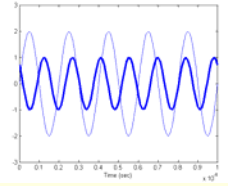
We drop the dependence of y on t for notational simplicity. It is however, still dependent on t .

- Next, we integrate both sides using λ and τ as our variables of integration:

$$\int_{y_0}^{y(t)} \frac{d\lambda}{\lambda - V_{in}} = -\int_0^t \frac{1}{RC} d\tau$$

y_0 represents the initial value of y at time = 0. We will eventually set this to zero since the initial conditions are 0.

Example (cont.)



$$\int_{y_o}^{y(t)} \frac{d\lambda}{\lambda - V_{in}} = -\int_0^t \frac{1}{RC} d\tau$$

$$\ln(\lambda - V_{in}) \Big|_{y_o}^{y(t)} = -\frac{1}{RC} \tau \Big|_0^t$$

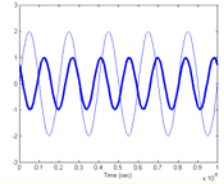
$$\ln \left\{ \frac{y(t) - V_{in}}{y_o - V_{in}} \right\} = -\frac{1}{RC} t$$

- Taking $\exp(x)$ of both sides:

$$\frac{y(t) - V_{in}}{y_o - V_{in}} = e^{-\frac{1}{RC}t}$$

$$y(t) = (y_o - V_{in}) e^{-\frac{1}{RC}t} + V_{in}$$

Example (cont.)



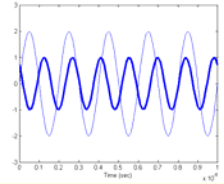
- Now, since we have assumed that the initial conditions are zero, $y_o = 0$. Further, since we are examining the *unit* step, $V_{in} = 1$. Thus:

$$\begin{aligned}y(t) &= (y_o - V_{in})e^{-\frac{1}{RC}t} + V_{in} \\ &= 1 - e^{-\frac{1}{RC}t}\end{aligned}$$

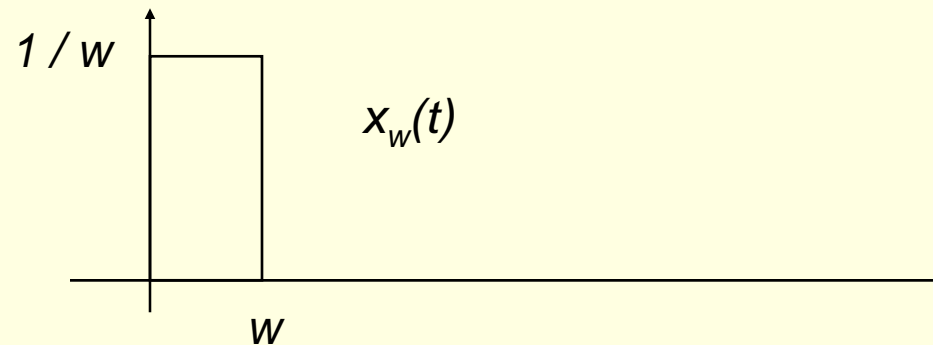
- Finally, since the output is zero for $t \leq 0$, and this solution only applies for $t \geq 0$, we can finally write the step response $y_s(t)$ as

$$y_s(t) = \left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

Impulse Response – Method 1



- The first method of finding the impulse response involves finding the response to a unit area pulse of width w :



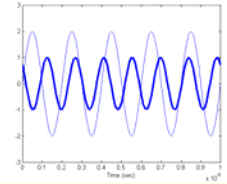
- This pulse can be written in terms of the unit pulse as

$$x_w(t) = \frac{1}{w} u(t) - \frac{1}{w} u(t - w)$$

- Due to the properties of linearity and time-invariance:

$$y_w(t) = \frac{1}{w} y_s(t) - \frac{1}{w} y_s(t - w)$$

Method 1 (cont.)



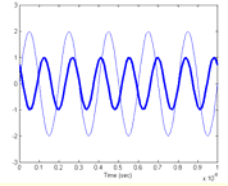
$$y_s(t) = \left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

$$\begin{aligned}y_w(t) &= \frac{1}{w}y_s(t) - \frac{1}{w}y_s(t-w) \\ &= \frac{1}{w}\left(1 - e^{-\frac{1}{RC}t}\right)u(t) - \frac{1}{w}\left(1 - e^{-\frac{1}{RC}(t-w)}\right)u(t-w)\end{aligned}$$

- The impulse response can be found by taking the limit as $w \rightarrow 0$:

$$\begin{aligned}h(t) &= \lim_{w \rightarrow 0} y_w(t) \\ &= \lim_{w \rightarrow 0} \left[\frac{1}{w}\left(1 - e^{-\frac{1}{RC}t}\right)u(t) - \frac{1}{w}\left(1 - e^{-\frac{1}{RC}(t-w)}\right)u(t-w) \right]\end{aligned}$$

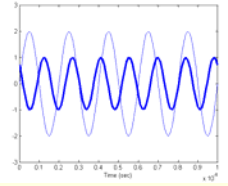
Method 1 (cont.)



$$\begin{aligned}h(t) &= \lim_{w \rightarrow 0} \left[\frac{1}{w} \left(1 - e^{-\frac{1}{RC}t} \right) u(t) - \frac{1}{w} \left(1 - e^{-\frac{1}{RC}(t-w)} \right) u(t-w) \right] \\&= \lim_{w \rightarrow 0} \left[\frac{1}{w} \left(1 - e^{-\frac{1}{RC}t} \right) - \frac{1}{w} \left(1 - e^{-\frac{1}{RC}(t-w)} \right) \right] u(t) \quad \boxed{u(t-w) \rightarrow u(t) \text{ as } w \rightarrow 0} \\&= \lim_{w \rightarrow 0} \left[\frac{1}{w} \left(e^{-\frac{1}{RC}(t-w)} - e^{-\frac{1}{RC}t} \right) \right] u(t) \\&= \frac{0}{0}\end{aligned}$$

Thus, we must use L'Hopital's rule to find the solution.

Continuing:

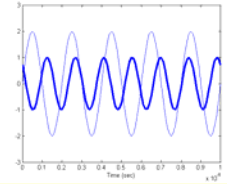


Using L'Hopital's Rule:

$$\begin{aligned} h(t) &= \lim_{w \rightarrow 0} \left[\frac{\frac{d}{dw} \left(e^{-\frac{1}{RC}(t-w)} - e^{-\frac{1}{RC}t} \right)}{\frac{d}{dw} w} \right] u(t) \\ &= \lim_{w \rightarrow 0} \left[\frac{\frac{1}{RC} \left(e^{-\frac{1}{RC}(t-w)} \right)}{1} \right] u(t) \\ &= \frac{1}{RC} e^{-\frac{1}{RC}t} u(t) \end{aligned}$$

Thus, we have the impulse response for this circuit.

Method 2



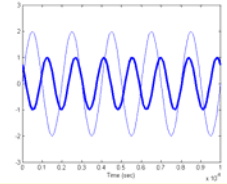
- A second method to calculate the impulse response comes directly from the step response.

$$y_s(t) = u(t) * h(t) = \int_{-\infty}^t h(\lambda) d\lambda$$
$$\frac{d}{dt} \{y_s(t)\} = h(t)$$

- Returning to the step response of this circuit:

$$y_s(t) = \left(1 - e^{-\frac{1}{RC}t}\right) u(t)$$
$$h(t) = \frac{d}{dt} \left\{ \left(1 - e^{-\frac{1}{RC}t}\right) u(t) \right\}$$

Method 2 (cont.)



$$y_s(t) = \left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

$$h(t) = \frac{d}{dt} \left\{ \left(1 - e^{-\frac{1}{RC}t}\right)u(t) \right\}$$

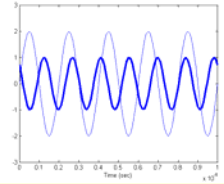
$$= \frac{d}{dt} \left(1 - e^{-\frac{1}{RC}t}\right)u(t) + \left(1 - e^{-\frac{1}{RC}t}\right) \frac{d}{dt}u(t)$$

$$= \frac{1}{RC} e^{-\frac{1}{RC}t} u(t) + \left(1 - e^{-\frac{1}{RC}t}\right) \delta(t)$$

$$= \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

This term is zero at the only point where impulse is non-zero (t=0)

Example



- Assume that the input to this circuit is $\cos(2\pi ft)$
- Find the output
- From our relationship between input and output:

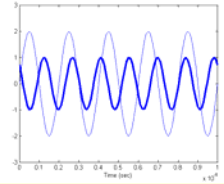
$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \\ &= \int_0^{\infty} e^{-\lambda/RC} \cos[2\pi f(t-\lambda)]d\lambda\end{aligned}$$

- We will want to use the following integrals from an integral table

$$\int e^{au} \cos(bu)du = \frac{e^{au}}{a^2 + b^2} (a \cos(bu) + b \sin(bu))$$

$$\int e^{au} \sin(bu)du = \frac{e^{au}}{a^2 + b^2} (a \sin(bu) - b \cos(bu))$$

Example (cont.)



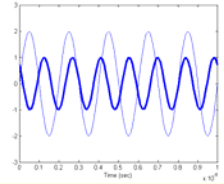
■ Continuing:

$$\begin{aligned}y(t) &= \int_0^{\infty} e^{-\lambda/RC} \cos[2\pi f(t-\lambda)] d\lambda \\&= \int_0^{\infty} e^{-\lambda/RC} \{ \cos[2\pi ft] \cos[-2\pi f\lambda] - \sin[2\pi ft] \sin[-2\pi f\lambda] \} d\lambda \\&= \int_0^{\infty} e^{-\lambda/RC} \{ \cos[2\pi ft] \cos[2\pi f\lambda] + \sin[2\pi ft] \sin[2\pi f\lambda] \} d\lambda \\&= \cos[2\pi ft] \int_0^{\infty} e^{-\lambda/RC} \cos[2\pi f\lambda] d\lambda + \sin[2\pi ft] \int_0^{\infty} e^{-\lambda/RC} \sin[2\pi f\lambda] d\lambda\end{aligned}$$

■ Using our integral tables:

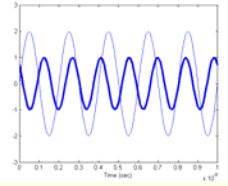
$$\begin{aligned}y(t) &= \cos[2\pi ft] \frac{e^{-\lambda/RC}}{(1/RC)^2 + (2\pi f)^2} \left(\frac{-1}{RC} \cos(2\pi f\lambda) + 2\pi f \sin(2\pi f\lambda) \right) \Bigg|_0^{\infty} \\&\quad + \sin[2\pi ft] \frac{e^{-\lambda/RC}}{(1/RC)^2 + (2\pi f)^2} \left(\frac{-1}{RC} \sin(2\pi f\lambda) - 2\pi f \cos(2\pi f\lambda) \right) \Bigg|_0^{\infty}\end{aligned}$$

Example (cont.)



$$\begin{aligned}y(t) &= \cos[2\pi ft] \frac{e^{-\lambda/RC}}{(1/RC)^2 + (2\pi f)^2} \left(\frac{-1}{RC} \cos(2\pi f \lambda) + 2\pi f \sin(2\pi f \lambda) \right) \Bigg|_0^\infty \\ &+ \sin[2\pi ft] \frac{e^{-\lambda/RC}}{(1/RC)^2 + (2\pi f)^2} \left(\frac{-1}{RC} \sin(2\pi f \lambda) - 2\pi f \cos(2\pi f \lambda) \right) \Bigg|_0^\infty \\ &= \cos[2\pi ft] \left\{ 0 - \frac{1}{(1/RC)^2 + (2\pi f)^2} \left(\frac{-1}{RC} \cos(0) + 2\pi f \sin(0) \right) \right\} \\ &+ \sin[2\pi ft] \left\{ 0 - \frac{1}{(1/RC)^2 + (2\pi f)^2} \left(\frac{-1}{RC} \sin(0) - 2\pi f \cos(0) \right) \right\} \\ &= \frac{1/RC}{(1/RC)^2 + (2\pi f)^2} \cos[2\pi ft] + \frac{2\pi f}{(1/RC)^2 + (2\pi f)^2} \sin[2\pi ft]\end{aligned}$$

Conclusions



- For linear, time-invariant systems the input can be represented as the sum of scaled and time-shifted impulses
- This permits the output to be determined as the convolution of the input with the system's impulse response.
- The impulse response can be determined by either taking the response of a pulse and letting the width approach zero or by taking the derivative of the unit step response.