
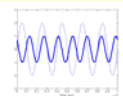


ECE 2704
Signals and Systems
Spring 2006


Instructor: Dr. R. Michael Buehrer
Lecture #7: System Analysis



Overview

- What to read – Section 3.6 in the text
- In this lecture we continue to discuss the idea of the impulse response.
- First we will examine a third technique for finding the impulse response of a system
- Second we will examine a few more examples of finding system impulse responses


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Finding the Impulse Response

- Method 3
- We reviewed two techniques previously, today we will review a third technique for finding the impulse response.
- The technique relies on an understanding of solving differential equations
 - Review your differential equations!

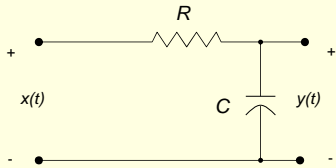
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Example



- Consider the following RC circuit:



- Using Kirchoff's Voltage Law we can relate the input to the output

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Example



- Using Kirchoff's Voltage law, the sum of the voltages around the circuit is zero:

$$x(t) = Ri(t) + y(t)$$

- The voltage across the capacitor is related to the current through the resistor

$$i(t) = C \frac{dy(t)}{dt}$$

- Eliminating $i(t)$ from the previous equation we obtain an equation relating the input to the output:

$$x(t) = RC \frac{dy(t)}{dt} + y(t)$$

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Impulse Response



- The system equation can be written as

$$RCy'(t) + y(t) = x(t)$$

- Now, let $x(t) = \delta(t)$

$$RCy'(t) + y(t) = \delta(t)$$

- 1. Since the input is zero for all $t < 0$, we know that

$$y(t) = 0 \quad t < 0$$

- 2. For $t > 0$, the input is also zero so that we can write

$$RCy'(t) + y(t) = 0$$

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Homogeneous Linear Differential Equation



Again, this is a first order linear homogeneous differential equation

$$RC \frac{dy}{dt} + y = 0$$

$$RC \frac{dy}{dt} = -y$$

$$\frac{dy}{y} = -\frac{1}{RC} dt$$

$$\int_{y_0}^{y(t)} \frac{dy}{y} = -\frac{1}{RC} \int_0^t dt$$

$$\ln \frac{y(t)}{y_0} = -\frac{t}{RC}$$

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Homogeneous Linear Differential Equation



$$\ln \frac{y(t)}{y_0} = -\frac{t}{RC}$$

$$\frac{y(t)}{y_0} = e^{-\frac{t}{RC}}$$

$$y(t) = y_0 e^{-\frac{t}{RC}}$$

This tells us the solution for $t > 0$, however, we need to determine y_0 . We also need to solve the equation for $t=0$. How do we do this?

From the system equation, we can see that $y(t)$ does not contain an impulse since the right hand of the equation does not have higher order impulse derivatives.

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Example (cont.)



- To determine the constant y_0 , we need to return to the original equation.
- By integrating through zero we can determine the constant.

$$\int_0^+ RCy'(t)dt + \int_0^+ y(t)dt = \int_0^+ \delta(t)dt$$

$$RC \{y(0^+) - y(0^-)\} + \int_0^+ y(t)dt = 1$$

- Now, since there is no impulse or derivative of an impulse in the output at $t=0$.

$$\int_0^+ y(t)dt = 0$$

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Example (cont.)



- Continuing

$$RC\{y(0^+) - y(0^-)\} = 1$$
$$y_0 RC = 1$$
$$y_0 = \frac{1}{RC}$$

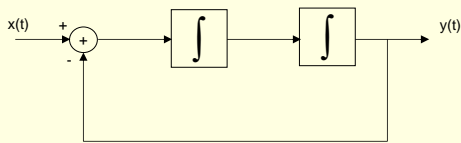
- Thus,

$$y(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

Example B



- Find the impulse response for the following system



- How do we write the system equation?

Example B (cont.)



- System equation

$$x(t) - y(t) = y''(t)$$
$$y''(t) + y(t) = x(t)$$

- If the input is an impulse

$$y''(t) + y(t) = \delta(t)$$

- For time less than zero

$$y(t) = 0 \quad t < 0$$

- For time greater than zero

$$y''(t) + y(t) = 0 \quad t > 0$$

2nd Order Linear Homogeneous Differential Equation



- Define a general second order linear differential equation as

$$ay''(t) + by'(t) + cy(t) = 0$$

- Consider a solution of the form

$$y(t) = e^{rt}$$

- Then

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$(ar^2 + br + c)e^{rt} = 0$$

Continued...



- The equation

$$(ar^2 + br + c)e^{rt} = 0$$

- will hold when

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Thus, if we define

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- Then all solutions will be of the form

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$$

Proof



- The general solution is of the form

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$$

- Substituting into the original equation

$$ay''(t) + by'(t) + cy(t) = 0$$

$$a \frac{d^2}{dt^2}(c_1e^{r_1t} + c_2e^{r_2t}) + b \frac{d}{dt}(c_1e^{r_1t} + c_2e^{r_2t}) + c(c_1e^{r_1t} + c_2e^{r_2t}) = 0$$

$$a(c_1r_1^2e^{r_1t} + c_2r_2^2e^{r_2t}) + b(c_1r_1e^{r_1t} + c_2r_2e^{r_2t}) + c(c_1e^{r_1t} + c_2e^{r_2t}) = 0$$

$$c_1(ar_1^2 + br_1 + c)e^{r_1t} + c_2(ar_2^2 + br_2 + c)e^{r_2t} = 0$$

- Which can only be true if

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Example (cont.)



- Returning to our example for time > 0

$$a=1 \quad b=0 \quad c=1$$

$$r_1 = \frac{\sqrt{-4}}{2} = j, \quad r_2 = \frac{-\sqrt{-4}}{2} = -j$$

$$y(t) = \{c_1 e^{jt} + c_2 e^{-jt}\} u(t)$$

- To determine the constants we can integrate the system over time = 0

$$\int_0^+ y''(t) dt + \int_0^+ y(t) dt = \int_0^+ \delta(t) dt$$

$$\{y'(0^+) - y'(0^-)\} + \int_0^+ y(t) dt = 1$$

$$y'(0^+) = 1$$

Example (cont.)



- Taking the derivative of the solution

$$y'(t) = \{jc_1 e^{jt} - jc_2 e^{-jt}\} u(t) + \{c_1 e^{jt} + c_2 e^{-jt}\} \delta(t)$$

$$y'(0^+) = \{jc_1 e^{j0} - jc_2 e^{-j0}\}$$

$$= jc_1 - jc_2$$

$$c_1 - c_2 = \frac{1}{j}$$

- We can also plug the solution into the original equation

$$y''(t) + y(t) = \delta(t)$$

Example B (cont.)



- Substituting:

$$\frac{d^2}{dt^2} \{ (c_1 e^{jt} + c_2 e^{-jt}) u(t) \} + (c_1 e^{jt} + c_2 e^{-jt}) u(t) = \delta(t)$$

$$\frac{d}{dt} \{ (c_1 e^{jt} + c_2 e^{-jt}) \delta(t) + (c_1 j e^{jt} - c_2 j e^{-jt}) u(t) \} + (c_1 e^{jt} + c_2 e^{-jt}) u(t) = \delta(t)$$

$$(c_1 e^{jt} + c_2 e^{-jt}) \delta'(t) + (c_1 j e^{jt} - c_2 j e^{-jt}) \delta(t) + (c_1 j e^{jt} - c_2 j e^{-jt}) \delta(t) + \dots$$

$$+ (-c_1 e^{jt} - c_2 e^{-jt}) u(t) + (c_1 e^{jt} + c_2 e^{-jt}) u(t) = \delta(t)$$

$$(c_1 + c_2) \delta'(t) - (c_1 j - c_2 j) \delta(t) + (2c_1 j e^{jt} - 2c_2 j e^{-jt}) \delta(t) = \delta(t)$$

$$(c_1 + c_2) \delta'(t) + (c_1 j - c_2 j) \delta(t) = \delta(t)$$

Note: $x(0)\delta'(t) = x(0)\delta'(t) - x'(0)\delta(t)$
 $x(0)\delta(t) = x(0)\delta(t)$

- Thus, we know that $c_1 = -c_2$ and $jc_1 - jc_2 = 1$

$$c_1 - c_2 = \frac{1}{j}$$

Solution



- Substituting $c_2 = -c_1$

$$c_1 + c_1 = \frac{1}{j}$$

$$c_1 = \frac{1}{2j}$$

- Thus,

$$y(t) = \left\{ \frac{1}{2j} e^{jt} - \frac{1}{2j} e^{-jt} \right\} u(t) \\ = \sin(t) u(t)$$

Checking



- Returning to the system equation

$$y''(t) + y(t) = \delta(t)$$

$$\frac{d^2}{dt^2} \{ \sin(t) u(t) \} + \sin(t) u(t) = \delta(t)$$

$$\frac{d}{dt} \{ \cos(t) u(t) + \sin(t) \delta(t) \} + \sin(t) u(t) = \delta(t)$$

$$-\sin(t) u(t) + \cos(t) \delta(t) + \dots$$

$$\cos(t) \delta(t) + \sin(t) \delta'(t) + \sin(t) u(t) = \delta(t)$$

$$\cos(0) \delta(t) + \cos(0) \delta(t) + \sin(0) \delta'(t) - \cos(0) \delta(t) = \delta(t)$$

$$\delta(t) = \delta(t)$$

Note: $x(t)\delta'(t) = x(0)\delta'(t) - x'(0)\delta(t)$
 $x(t)\delta(t) = x(0)\delta(t)$

Example C



- Consider a pendulum of length L and mass m . For small changes in the angle $\theta(t)$, the angle can be related to the force applied to the mass tangential to the direction of motion as

$$mL \theta''(t) + mg \theta(t) \approx x(t)$$

- (a) Determine the impulse response of this system
- (b) If the mass of the pendulum is 2kg and the length is 0.5m, at what frequency will the pendulum oscillate?

Solution



- We again have a second order linear differential equation. As before, we can solve this by examining the solution for $t < 0$, $t > 0$ and then $t=0$.

- For $t < 0$

$$y(t) = 0 \quad t < 0$$

- For $t > 0$

$$mL \theta''(t) + mg \theta(t) = 0$$

- We can determine the solution from our previous problem using $a = mL$ $b = 0$ $c = mg$

Solution (cont.)



- This results in

$$a = mL \quad b = 0 \quad c = mg$$

$$r_1 = \frac{\sqrt{-4mLmg}}{2mL} = j\sqrt{\frac{g}{L}}, \quad r_2 = \frac{-\sqrt{-4mLmg}}{2mL} = -j\sqrt{\frac{g}{L}}$$

$$y(t) = \left\{ c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right\} u(t)$$

- Thus, we must solve for c_1 and c_2 . Recalling our results from the last problem, we can integrate over zero.

Finding the constants



- Integrating over $t=0$

$$mL \int_{0^-}^{0^+} \theta''(t) dt + mg \int_{0^-}^{0^+} \theta(t) dt = \int_{0^-}^{0^+} \delta(t) dt$$

$$mL [\theta'(0^+) - \theta'(0^-)] + 0 = 1$$

$$\theta'(0^+) = \frac{1}{mL}$$

$$\theta'(t) = \frac{d}{dt} \left[c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right] u(t)$$

$$= \left[c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right] \delta(t) + \left[j\sqrt{\frac{g}{L}} c_1 e^{j\sqrt{\frac{g}{L}}t} - j\sqrt{\frac{g}{L}} c_2 e^{-j\sqrt{\frac{g}{L}}t} \right] u(t)$$

Finding the constants (cont.)



- Substituting for $t = 0^+$

$$\theta'(0^+) = \left[j\sqrt{\frac{g}{L}}c_1 - j\sqrt{\frac{g}{L}}c_2 \right] = \frac{1}{mL}$$

$$c_1 - c_2 = \frac{1}{jmL} \sqrt{\frac{L}{g}}$$

- Next we can substitute the solution into the system equation

Returning to the System Equation



$$mL \frac{d^2}{dt^2} \left\{ \left(c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) u(t) \right\} + mg \left(c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) u(t) = \delta(t)$$

$$mL \frac{d}{dt} \left\{ \left(c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) \delta(t) + j\sqrt{\frac{g}{L}} \left(c_1 e^{j\sqrt{\frac{g}{L}}t} - c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) u(t) \right\} + \dots$$

$$mg \left(c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) u(t) = \delta(t)$$

$$mL \left(c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) \delta'(t) + jmL \sqrt{\frac{g}{L}} \left(c_1 e^{j\sqrt{\frac{g}{L}}t} - c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) \delta(t) + \dots$$

$$jmL \sqrt{\frac{g}{L}} \left(c_1 e^{j\sqrt{\frac{g}{L}}t} - c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) \delta(t) - \dots$$

$$mL \frac{g}{L} \left(c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) u(t) + mg \left(c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right) u(t) = \delta(t)$$

At time $t=0^+$



$$mL \frac{g}{L} (c_1 + c_2) + mg (c_1 + c_2) = 0$$

$$\left\{ mL \frac{g}{L} + mg \right\} (c_1 + c_2) = 0$$

$$c_1 = -c_2$$

- From the previous calculation

$$c_1 - c_2 = \frac{1}{jmL} \sqrt{\frac{L}{g}}$$

$$2c_1 = \frac{1}{jmL} \sqrt{\frac{L}{g}}$$

$$c_1 = \frac{1}{2jmL} \sqrt{\frac{L}{g}}$$

Solution



- Substituting the constants

$$\begin{aligned}y(t) &= \left\{ \frac{1}{2jmL} \sqrt{\frac{L}{g}} e^{j\sqrt{\frac{g}{L}}t} - \frac{1}{2jmL} \sqrt{\frac{L}{g}} e^{-j\sqrt{\frac{g}{L}}t} \right\} u(t) \\ &= \frac{1}{mL} \sqrt{\frac{L}{g}} \left\{ \frac{1}{2j} e^{j\sqrt{\frac{g}{L}}t} - \frac{1}{2j} e^{-j\sqrt{\frac{g}{L}}t} \right\} u(t) \\ &= \frac{1}{mL} \sqrt{\frac{L}{g}} \sin\left(\sqrt{\frac{g}{L}}t\right) u(t)\end{aligned}$$

Part (b)



- If the mass of the pendulum is 2kg and the length is 0.5m, at what frequency will the pendulum oscillate?
- The frequency in radians/sec is $\sqrt{\frac{g}{L}}$
- Thus, in Hz the oscillation frequency is

$$\begin{aligned}\frac{1}{2\pi} \sqrt{\frac{g}{L}} &= \frac{1}{2\pi} \sqrt{\frac{9.8m/s^2}{0.5m}} \\ &= 0.7Hz\end{aligned}$$

Note that the mass is irrelevant.
