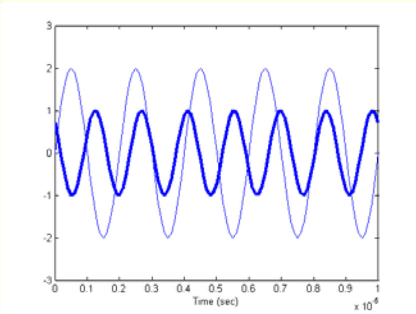
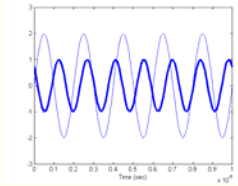
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer

Lecture #7: System Analysis

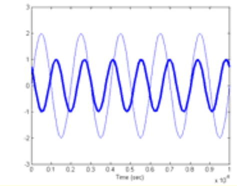




Overview

- What to read – Section 3.6 in the text
- In this lecture we continue to discuss the idea of the impulse response.
- First we will examine a third technique for finding the impulse response of a system
- Second we will examine a few more examples of finding system impulse responses

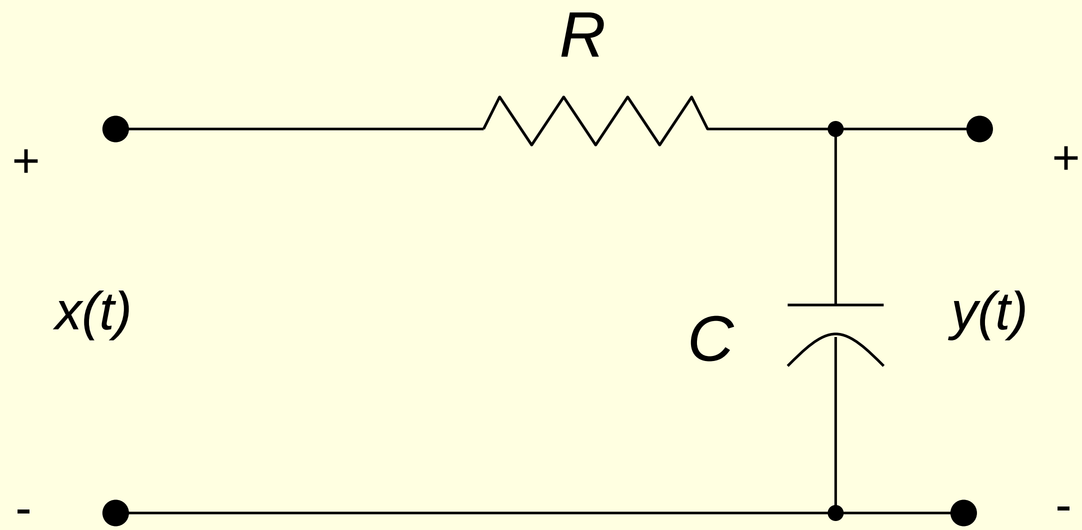
Finding the Impulse Response



- Method 3
- We reviewed two techniques previously, today we will review a third technique for finding the impulse response.
- The technique relies on an understanding of solving differential equations
  - Review your differential equations!

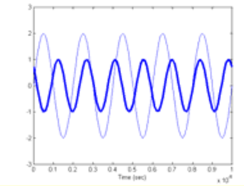
Example

- Consider the following RC circuit:



- Using Kirchoff's Voltage Law we can relate the input to the output

Example



- Using Kirchhoff's Voltage law, the sum of the voltages around the circuit is zero:

$$x(t) = Ri(t) + y(t)$$

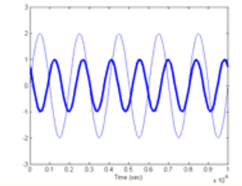
- The voltage across the capacitor is related to the current through the resistor

$$i(t) = C\frac{dy(t)}{dt}$$

- Eliminating *i(t)* from the previous equation we obtain an equation relating the input to the output:

$$x(t) = RC\frac{dy(t)}{dt} + y(t)$$

Impulse Response



- The system equation can be written as

$$RCy'(t) + y(t) = x(t)$$

- Now, let x(t) = δ(t)

$$RCy'(t) + y(t) = \delta(t)$$

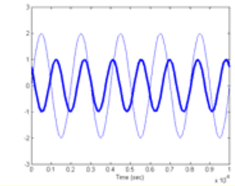
- 1. Since the input is zero for all t < 0, we know that

$$y(t) = 0 \quad t < 0$$

- 2. For t > 0, the input is also zero so that we can write

$$RCy'(t) + y(t) = 0$$

Homogeneous Linear Differential Equation



Again, this is a first order linear homogeneous differential equation

$$RC\frac{dy}{dt}+y=0$$

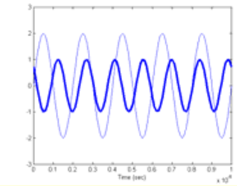
$$RC\frac{dy}{dt}=-y$$

$$\frac{dy}{y}=-\frac{1}{RC}dt$$

$$\int_{y_o}^{y(t)}\frac{dy}{y}=-\frac{1}{RC}\int_0^t dt$$

$$\ln\frac{y(t)}{y_o}=-\frac{t}{RC}$$

Homogeneous Linear Differential Equation



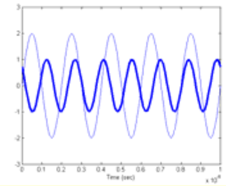
$$\ln \frac{y(t)}{y_o} = -\frac{t}{RC}$$

$$\frac{y(t)}{y_o} = e^{-\frac{t}{RC}}$$

$$y(t) = y_o e^{-\frac{t}{RC}}$$

This tells us the solution for t > 0, however, we need to determine y_o.
We also need to solve the equation for t=0.
How do we do this?

From the system equation, we can see that y(t) does not contain an impulse since the right hand of the equation does not have higher order impulse derivatives.



Example (cont.)

- To determine the constant y_o, we need to return to the original equation.
- By integrating through zero we can determine the constant.

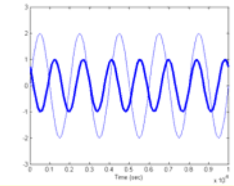
$$\int_{0^-}^{0^+} RCy'(t)dt + \int_{0^-}^{0^+} y(t)\,dt = \int_{0^-}^{0^+} \delta(t)\,dt$$

$$RC\left\{y\left(0^+\right) - y\left(0^-\right)\right\} + \int_{0^-}^{0^+} y(t)\,dt = 1$$

- Now, since there is no impulse or derivative of an impulse in the output at $t=0$.

$$\int_{0^-}^{0^+} y(t)\,dt = 0$$

Example (cont.)



- Continuing

$$RC\left\{y\left(0^{+}\right)-y\left(0^{-}\right)\right\}=1$$

$$y_{o}RC=1$$

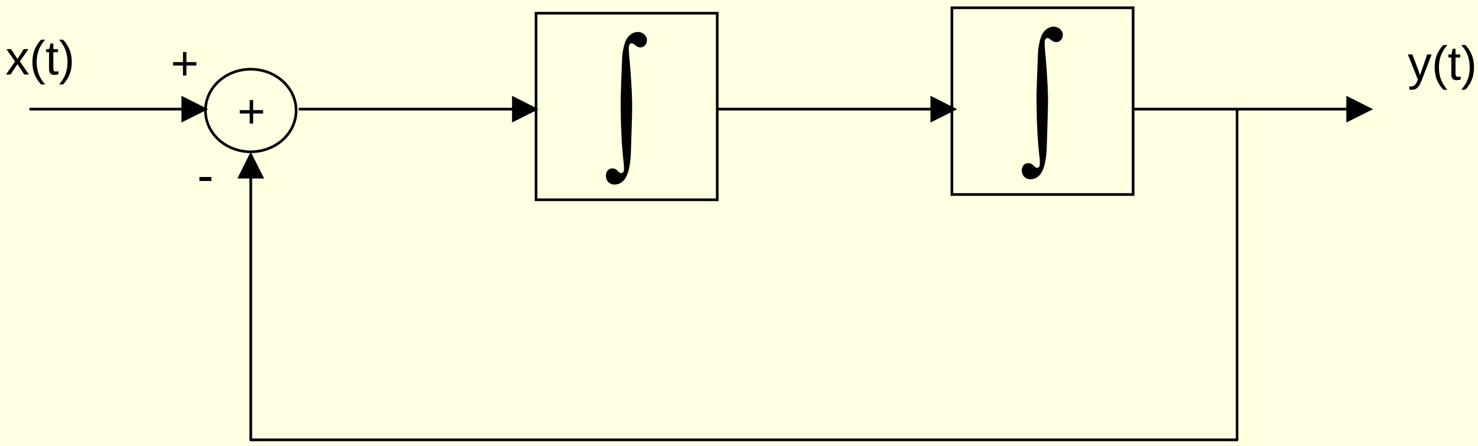
$$y_{o}=\frac{1}{RC}$$

- Thus,

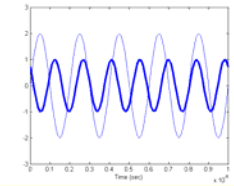
$$y\left(t\right)=\frac{1}{RC}e^{-\frac{t}{RC}}u\left(t\right)$$

Example B

- Find the impulse response for the following system



- How do we write the system equation?



Example B (cont.)

- System equation

$$x(t) - y(t) = y''(t)$$

$$y''(t) + y(t) = x(t)$$

- If the input is an impulse

$$y''(t) + y(t) = \delta(t)$$

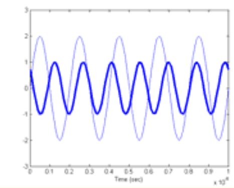
- For time less than zero

$$y(t) = 0 \quad t < 0$$

- For time greater than zero

$$y''(t) + y(t) = 0 \quad t > 0$$

2^{nd} Order Linear Homogeneous Differential Equation



- Define a general second order linear differential equation as

$$ay''(t)+by'(t)+cy(t)=0$$

- Consider a solution of the form

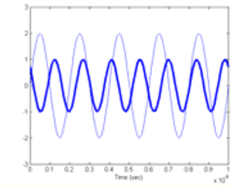
$$y(t)=e^{rt}$$

- Then

$$ar^2e^{rt}+bre^{rt}+ce^{rt}=0$$

$$\left(ar^2+br+c\right)e^{rt}=0$$

Continued…



- The equation

$$\left(ar^2 + br + c\right)e^{rt} = 0$$

- will hold when

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

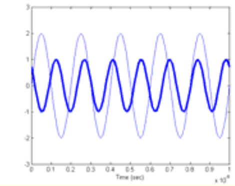
- Thus, if we define

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- Then all solutions will be of the form

$$y\left(t\right) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Proof



- The general solution is of the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

- Substituting into the original equation

$$ay''(t) + by'(t) + cy(t) = 0$$

$$a\frac{d^2}{dt^2}\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) + b\frac{d}{dt}\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) + c\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0$$

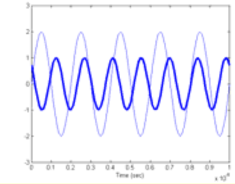
$$a\left(c_1 r_1^2 e^{r_1 t} + c_1 r_2^2 e^{r_2 t}\right) + b\left(c_1 r_1 e^{r_1 t} + c_1 r_2 e^{r_2 t}\right) + c\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0$$

$$c_1\left(ar_1^2 + br_1 + c\right)e^{r_1 t} + c_2\left(ar_2^2 + br_2 + c\right)e^{r_2 t} = 0$$

- Which can only be true if

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Example (cont.)



- Returning to our example for time > 0

$$a = 1 \quad b = 0 \quad c = 1$$

$$r_1 = \frac{\sqrt{-4}}{2} = j, \quad r_2 = \frac{-\sqrt{-4}}{2} = -j$$

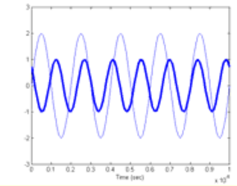
$$y(t) = \left\{ c_1 e^{jt} + c_2 e^{-jt} \right\} u(t)$$

- To determine the constants we can integrate the system over time = 0

$$\int_{0^-}^{0^+} y''(t) dt + \int_{0^-}^{0^+} y(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt$$

$$\left\{ y'(0^+) - y'(0^-) \right\} + \int_{0^-}^{0^+} y(t)\, dt = 1$$

$$y'(0^+) = 1$$



Example (cont.)

- Taking the derivative of the solution

$$y'(t) = \left\{ jc_1 e^{jt} - jc_2 e^{-jt} \right\} u(t) + \left\{ c_1 e^{jt} + c_2 e^{-jt} \right\} \delta(t)$$

$$y'(0^+) = \left\{ jc_1 e^{jt} - jc_2 e^{-jt} \right\}$$

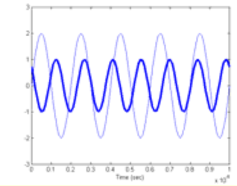
$$= jc_1 - jc_2$$

$$c_1 - c_2 = \frac{1}{j}$$

- We can also plug the solution into the original equation

$$y''(t) + y(t) = \delta(t)$$

Example B (cont.)



- Substituting:

$$\frac{d^2}{dt^2}\left\{\left(c_1e^{jt}+c_2e^{-jt}\right)u(t)\right\}+\left(c_1e^{jt}+c_2e^{-jt}\right)u(t)=\delta(t)$$

$$\frac{d}{dt}\left\{\left(c_1e^{jt}+c_2e^{-jt}\right)\delta(t)+\left(c_1je^{jt}-c_2je^{-jt}\right)u(t)\right\}+\left(c_1e^{jt}+c_2e^{-jt}\right)u(t)=\delta(t)$$

$$\left(c_1e^{jt}+c_2e^{-jt}\right)\delta'(t)+\left(jc_1e^{jt}-jc_2e^{-jt}\right)\delta(t)+\left(c_1je^{jt}-c_2je^{-jt}\right)\delta(t)\ldots\ldots$$

$$+\left(-c_1e^{jt}-c_2e^{-jt}\right)u(t)+\left(c_1e^{jt}+c_2e^{-jt}\right)u(t)=\delta(t)$$

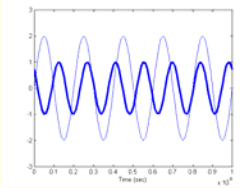
$$\left(c_1+c_2\right)\delta'(t)-\left(c_1j-c_2j\right)\delta(t)+\left(2c_1je^{jt}-2c_2je^{-jt}\right)\delta(t)=\delta(t)$$

$$\left(c_1+c_2\right)\delta'(t)+\left(c_1j-c_2j\right)\delta(t)=\delta(t)$$

Note: x(t)δ'(t) = x(0)δ'(t)-x'(0)δ(t)
x(t)δ(t) = x(0)δ(t)

- Thus, we know that $c_1=-c_2$ and $jc_1-jc_2=1$

$$c_1-c_2=\frac{1}{j}$$



Solution

- Substituting $c_2 = -c_1$

$$c_1 + c_1 = \frac{1}{j}$$

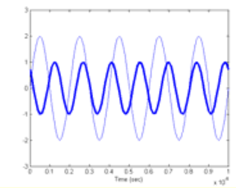
$$c_1 = \frac{1}{2j}$$

- Thus,

$$y(t) = \left\{ \frac{1}{2j} e^{jt} - \frac{1}{2j} e^{-jt} \right\} u(t)$$

$$= \sin(t) u(t)$$

Checking



- Returning to the system equation

$$y''(t) + y(t) = \delta(t)$$

$$\frac{d^2}{dt^2}\{\sin(t)u(t)\} + \sin(t)u(t) = \delta(t)$$

$$\frac{d}{dt}\{\cos(t)u(t) + \sin(t)\delta(t)\} + \sin(t)u(t) = \delta(t)$$

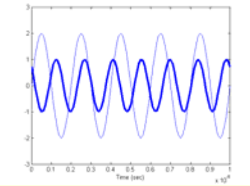
$$-\sin(t)u(t) + \cos(t)\delta(t) + \ldots.$$

$$\cos(t)\delta(t) + \sin(t)\delta'(t) + \sin(t)u(t) = \delta(t)$$

$$\cos(0)\delta(t) + \cos(0)\delta(t) + \sin(0)\delta'(t) - \cos(0)\delta(t) = \delta(t)$$

$$\delta(t) = \delta(t)$$

Note: $x(t)\delta'(t) = x(0)\delta'(t) - x'(0)\delta(t)$
$x(t)\delta(t) = x(0)\delta(t)$

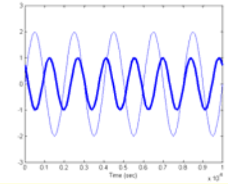


Example C

- Consider a pendulum of length *L* and mass *m*. For small changes in the angle $\theta(t)$, the angle can be related to the force applied to the mass tangential to the direction of motion as

$$mL\,\theta''(t) + mg\,\theta(t) \approx x(t)$$

- (a) Determine the impulse response of this system
- (b) If the mass of the pendulum is 2kg and the length is 0.5m, at what frequency will the pendulum oscillate?



Solution

- We again have a second order linear differential equation. As before, we can solve this by examining the solution for t< 0, t> 0 and then t=0.
- For t < 0

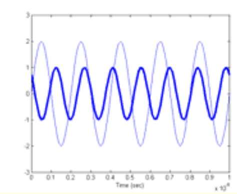
$$y(t) = 0 \quad t < 0$$

- For t > 0

$$mL\,\theta''(t) + mg\,\theta(t) = 0$$

- We can determine the solution from our previous problem using $a = mL \quad b = 0 \quad c = mg$

Solution (cont.)



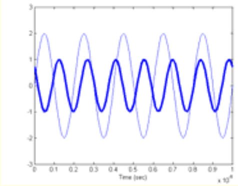
- This results in

$$a = mL \quad b = 0 \quad c = mg$$

$$r_1 = \frac{\sqrt{-4mLmg}}{2mL} = j\sqrt{\frac{g}{L}}, \quad r_2 = \frac{-\sqrt{-4mLmg}}{2mL} = -j\sqrt{\frac{g}{L}}$$

$$y(t) = \left\{ c_1 e^{j\sqrt{\frac{g}{L}}t} + c_2 e^{-j\sqrt{\frac{g}{L}}t} \right\} u(t)$$

- Thus, we must solve for c_1 and c_2. Recalling our results from the last problem, we can integrate over zero.



Finding the constants

- Integrating over t=0

$$mL\int_{0^-}^{0^+}\theta''(t)dt+mg\int_{0^-}^{0^+}\theta(t)dt=\int_{0^-}^{0^+}\delta(t)dt$$

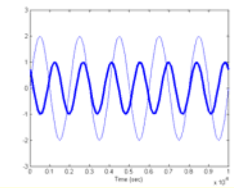
$$mL\left[\theta'\left(0^+\right)-\theta'\left(0^-\right)\right]+0=1$$

$$\theta'\left(0^+\right)=\frac{1}{mL}$$

$$\theta'(t)=\frac{d}{dt}\left\{\left[c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right]u(t)\right\}$$

$$=\left[c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right]\delta(t)+\left[j\sqrt{\frac{g}{L}}c_1e^{j\sqrt{\frac{g}{L}}t}+-j\sqrt{\frac{g}{L}}c_2e^{-j\sqrt{\frac{g}{L}}t}\right]u(t)$$

Finding the constants (cont.)



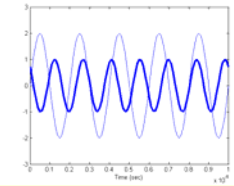
- Substituting for $t = 0^+$

$$\theta'\left(0^+\right) = \left[j\sqrt{\frac{g}{L}}c_1 - j\sqrt{\frac{g}{L}}c_2 \right] = \frac{1}{mL}$$

$$c_1 - c_2 = \frac{1}{jmL}\sqrt{\frac{L}{g}}$$

- Next we can substitute the solution into the system equation

Returning to the System Equation



$$mL\frac{d^2}{dt^2}\left\{\left(c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right)u(t)\right\}+mg\left(c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right)u(t)=\delta(t)$$

$$mL\frac{d}{dt}\left\{\left(c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right)\delta(t)+j\sqrt{\frac{g}{L}}\left(c_1e^{j\sqrt{\frac{g}{L}}t}-c_2e^{-j\sqrt{\frac{g}{L}}t}\right)u(t)\right\}+....$$

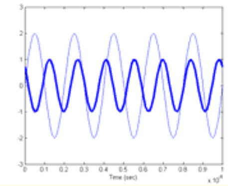
$$mg\left(c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right)u(t)=\delta(t)$$

$$mL\left(c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right)\delta'(t)+jmL\sqrt{\frac{g}{L}}\left(c_1e^{j\sqrt{\frac{g}{L}}t}-c_2e^{-j\sqrt{\frac{g}{L}}t}\right)\delta(t)+......$$

$$jmL\sqrt{\frac{g}{L}}\left(c_1e^{j\sqrt{\frac{g}{L}}t}-c_2e^{-j\sqrt{\frac{g}{L}}t}\right)\delta(t)-.....$$

$$mL\frac{g}{L}\left(c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right)u(t)+mg\left(c_1e^{j\sqrt{\frac{g}{L}}t}+c_2e^{-j\sqrt{\frac{g}{L}}t}\right)u(t)=\delta(t)$$

At time t=0$^+$



$$mL\frac{g}{L}\left(c_1+c_2\right)+mg\left(c_1+c_2\right)=0$$

$$\left\{mL\frac{g}{L}+mg\right\}\left(c_1+c_2\right)=0$$

$$c_1=-c_2$$

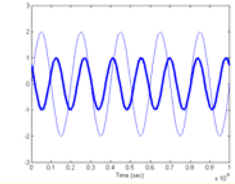
- From the previous calculation

$$c_1-c_2=\frac{1}{jmL}\sqrt{\frac{L}{g}}$$

$$2c_1=\frac{1}{jmL}\sqrt{\frac{L}{g}}$$

$$c_1=\frac{1}{2\,jmL}\sqrt{\frac{L}{g}}$$

Solution



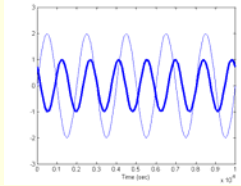
- Substituting the constants

$$y(t)=\left\{\frac{1}{2jmL}\sqrt{\frac{L}{g}}e^{j\sqrt{\frac{g}{L}}t}-\frac{1}{2jmL}\sqrt{\frac{L}{g}}e^{-j\sqrt{\frac{g}{L}}t}\right\}u(t)$$

$$=\frac{1}{mL}\sqrt{\frac{L}{g}}\left\{\frac{1}{2j}e^{j\sqrt{\frac{g}{L}}t}-\frac{1}{2j}e^{-j\sqrt{\frac{g}{L}}t}\right\}u(t)$$

$$=\frac{1}{mL}\sqrt{\frac{L}{g}}\sin\left(\sqrt{\frac{g}{L}}t\right)u(t)$$

Part (b)



- If the mass of the pendulum is 2kg and the length is 0.5m, at what frequency will the pendulum oscillate?

- The frequency in radians/sec is $\sqrt{\frac{g}{L}}$

- Thus, in Hz the oscillation frequency is

$$\frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.8m/s^2}{0.5m}}$$
$$= 0.7Hz$$

Note that the mass is irrelevant.