ECE 2704
Signals and Systems
Spring 2006

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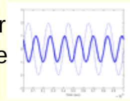
Lecture #8: An Introduction to the Fourier Series



Overview

- Today we introduce the concept of the Continuous Time Fourier Series (CTFS)
- What to read – Section 4.1-4.2 in the text
- We will discuss the two versions of the CTFS
  - In terms of complex sinusoids
    - Applies to both real and complex signals
  - In terms of real sinusoids
    - Applies to real signals

Motivation

- *If* a system is linear and time invariant (LTI), *then* when the input is a weighted complex sinusoid, the output is also a complex sinusoid at the same frequency but (in general) with a different weighting.

$$Ae^{jax} \xrightarrow{\mathcal{H}} Be^{jax}$$

- If we can represent a signal as a weighted sum complex sinusoids, then for an LTI system we can represent the output as the sum of weighted complex sinusoids.
  - Thus, we wish to show that signals can be represented as a sum of complex sinusoids

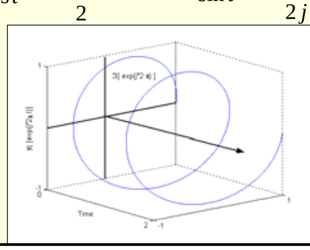
Complex Sinusoids

- Euler's identity defines a complex sinusoid as

$$e^{jt} = \cos t + j \sin t$$

- Real sines and cosines can be written as

$$\cos t = \frac{e^{jt} + e^{-jt}}{2} \qquad \sin t = \frac{e^{jt} - e^{-jt}}{2j}$$



Plot of exp(j2πt)

Example

- Consider the waveform

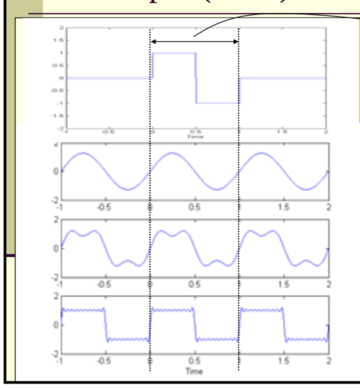
$$x(t) = rect\left(2t - \frac{1}{2}\right) - rect\left(2t - \frac{3}{2}\right)$$

- Let us attempt to model this over the time frame 0<t<1 with sinusoids of period 1 and integer multiples of 1:

$$\hat{x}(t) = \frac{4}{\pi}\left(\sin(2\pi t) + \frac{1}{3}\sin(3*2\pi t) + \frac{1}{5}\sin(5*2\pi t) + \frac{1}{7}\sin(7*2\pi t)...\right)$$

- Note that we only care about the approximation over the time interval of interest

Example (cont.)



Interval of interest

$$x(t) = rect\left(2t - \frac{1}{2}\right) - rect\left(2t - \frac{3}{2}\right)$$

$$\hat{x}(t) = \frac{4}{\pi}\sin(2\pi t)$$

$$\hat{x}(t) = \frac{4}{\pi}\left(\sin(2\pi t) + \frac{1}{3}\sin(3*2\pi t)\right)$$

$$\hat{x}(t) = \frac{4}{\pi}\left(\sum_{i=1}^{10} \frac{1}{2i-1}\sin(2\pi(2i-1)t)\right)$$

Example (cont.)

- This example shows us that for a specific time interval it appears possible to represent a signal using a sum of sinusoids.
- However, how do we determine the frequency of those sinusoids and how do we determine the weights of the sinusoids?
  - A reasonable guess for frequencies would be multiples of a *fundamental* frequency which is equal to $1/T_f$ where T_f is the length of the interval being examined
    - This will be particularly useful when we examine *periodic* signals
  - We still need to determine the weights
- Further we may want to use both sines and cosines

Signal Representation

- Let's first assume that a signal $x(t)$ can be represented over a time interval $t_o < t < t_o+T_f$ as a linear combination of complex sinusoids in the form

$$x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

where $f_F = 1/T_F$ is the fundamental frequency and

$$x(t) = x_F(t) \text{ for } t_o < t < t_o+T_f$$

- We will determine later when this can be safely assumed.
- Note that we are representing the signal $x(t)$ over a finite time interval, not over all time.
- This is defined as the *Continuous Time Fourier Series* or CTFS

Finding the coefficients

- The CTFS representation requires us to determine the coefficients $X[k]$
- Now, since $x(t) = x_F(t) \quad t_o < t < t_o + T_F$, then we can write

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} \quad t_o < t < t_o + T_F$$

- Now, let's multiply both sides by $exp(-j2\pi q f_F t)$ where q is an integer

$$x(t) e^{-j2\pi q f_F t} = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} e^{-j2\pi q f_F t} = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi (k-q) f_F t} \quad t_o < t < t_o + T_F$$

Finding the coefficients (cont.)

- Now, integrating both sides over $t_o < t < t_o + T_f$

$$\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi q f_F t}dt = \int_{t_o}^{t_o+T_F}\left[\sum_{k=-\infty}^{\infty} X[k]e^{j2\pi(k-q)f_F t}\right]dt$$

- Since k and t are independent we can interchange summation and integration

$$\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi q f_F t}dt = \sum_{k=-\infty}^{\infty} X[k]\int_{t_o}^{t_o+T_F} e^{j2\pi(k-q)f_F t}dt \qquad \text{Equation A}$$

- Using Euler's identity

$$\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi q f_F t}dt = \sum_{k=-\infty}^{\infty} X[k]\int_{t_o}^{t_o+T_F}\left[\cos(2\pi(k-q)f_F t) + j\sin(2\pi(k-q)f_F t)\right]dt$$

Continuing…

$$\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi q f_F t}dt = \sum_{k=-\infty}^{\infty} X[k]\int_{t_o}^{t_o+T_F}\left[\cos(2\pi(k-q)f_F t) + j\sin(2\pi(k-q)f_F t)\right]dt$$

- *k-q* is an integer. For $k \neq q$ we are integrating $cos(2\pi[k-q]f_F t)$ and $sin(2\pi[k-q]f_F t)$ over exactly *k-q* periods which results in zero.
- For $k \neq q$

$$\int_{t_o}^{t_o+T_F}\left[\cos(2\pi(k-q)f_F t) + j\sin(2\pi(k-q)f_F t)\right]dt = 0$$

- For $k = q$

$$\int_{t_o}^{t_o+T_F}\left[\cos(0) + j\sin(0)\right]dt = T_F$$

Continuing…

- Therefore,

$$\sum_{k=-\infty}^{\infty} X[k]\int_{t_o}^{t_o+T_F} e^{-j2\pi(k-q)f_F t}dt = X[q]T_F$$

- Making this substitution into *Equation A*

$$\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi q f_F t}dt = X[q]T_F$$

- Solving for *X[q]:*

$$X[q] = \frac{1}{T_F}\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi q f_F t}dt$$

- Since this is true for *X[q]*, then *X[k]* is

$$X[k] = \frac{1}{T_F}\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi k f_F t}dt$$

Continuous Time Fourier Transform

- Thus, if the integral converges, we can represent the signal x(t) exactly, on the time interval $t_o < t < t_o + T_f$ by

$$x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

where

$$X[k] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt$$

- Note that if the above integral does not converge, the CTFS cannot be found over the region of interest.
- We can represent the relationship between *x(t)* and *X[k]* as

$$x(t) \overset{FS}{\longleftrightarrow} X[k]$$

Dirichlet Conditions

- If the integral

$$X[k] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt$$

diverges, the CTFS cannot be found for the signal over the region of interest.
- This condition is necessary but not sufficient.
- Sufficient conditions on the existence of the CTFS are known as the *Dirichlet Conditions:*
  1. The signal must be absolutely integrable over the time $t_o < t < t_o + T_f$
     $$\int_{t_o}^{t_o+T_F} |x(t)| dt < \infty$$
  2. The signal must have a finite number of maxima and minima in the time $t_o < t < t_o + T_f$
  3. The signal must have a finite number of discontinuities in the time $t_o < t < t_o + T_f$

Complex Conjugate of *x(t)*

- The CTFS derived previously holds for any signal which satisfies the Dirichlet conditions
  - $x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$ applies to both real signals and complex signals
- Consider the complex conjugate of a signal $x_F(t)$
- By conjugating both sides of our expression:

$$x_F^*(t) = \sum_{k=-\infty}^{\infty} X^*[k] e^{-j2\pi k f_F t}$$

Complex Conjugate (cont.)

- Simplifying:

$$x_F^*(t) = \sum_{k=-\infty}^{\infty} X^*[k] e^{-j2\pi k f_F t}$$

$$= \sum_{k=\infty}^{-\infty} X^*[-k] e^{j2\pi k f_F t}$$

$$= \sum_{k=-\infty}^{\infty} X^*[-k] e^{j2\pi k f_F t}$$

- Thus, we can say that if

$$x(t) \overset{FS}{\leftrightarrow} X[k]$$

then

$$x^*(t) \overset{FS}{\leftrightarrow} X^*[-k]$$

Real Signals

- For the important case where the signal is all real

$$x(t) = x^*(t)$$

which means that

$$x_F(t) = x_F^*(t)$$

- In this case

$$x_F^*(t) = \sum_{k=-\infty}^{\infty} X^*[-k] e^{j2\pi k f_F t} = x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

which leads to

$$\sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} = \sum_{k=-\infty}^{\infty} X^*[-k] e^{-j2\pi(-k) f_F t}$$

$$= \sum_{k=-\infty}^{\infty} X^*[-k] e^{j2\pi k f_F t}$$

meaning that

$$X[k] = X^*[-k]$$

Real Signals (cont.)

- For real signals then we can write the CTFS as

$$x_F(t) = X[0] + \sum_{k=1}^{\infty} \left[X[k] e^{j2\pi k f_F t} + X^*[k] e^{-j2\pi k f_F t} \right]$$

which can also be written as

$$x_F(t) = X[0] + \sum_{k=1}^{\infty} \Big\{ \mathrm{Re}\big(X[k]\big) e^{j2\pi k f_F t} + \mathrm{Re}\big(X[k]\big) e^{-j2\pi k f_F t} + j\,\mathrm{Im}\big(X[k]\big) e^{j2\pi k f_F t} - j\,\mathrm{Im}\big(X[k]\big) e^{-j2\pi k f_F t} \Big\}$$

Examining the first term:

$$X[0] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t)\,dt$$

which is simply the average value of the function (note that it is all real).

Trigonometric CTFS

- Recall that $\cos t = \frac{e^{jt} + e^{-jt}}{2} \quad \sin t = \frac{e^{jt} - e^{-jt}}{2j}$
- Substituting these relationships into the previous equation we have

$$x_F(t) = X[0] + \sum_{k=1}^{\infty} \left\{ 2\,\mathrm{Re}\left(X[k]\right)\cos\left(2\pi k f_F t\right) + 2\,\mathrm{Im}\left(X[k]\right)\sin\left(2\pi k f_F t\right) \right\}$$

which can be re-written as

$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty} \left\{ X_C[k]\cos\left(2\pi k f_F t\right) + X_S[k]\sin\left(2\pi k f_F t\right) \right\}$$

with

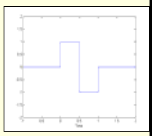
$$X_C[k] = 2\,\mathrm{Re}\left\{X[k]\right\} = 2\,\mathrm{Re}\left\{\frac{1}{T_F}\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi k f_F t}dt\right\} = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\cos\left(2\pi k f_F t\right)dt$$

$$X_S[k] = -2\,\mathrm{Im}\left\{X[k]\right\} = -2\,\mathrm{Im}\left\{\frac{1}{T_F}\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi k f_F t}dt\right\} = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\sin\left(2\pi k f_F t\right)dt$$

Example

- Let's return to our previous example of

$$x(t) = rect\left(2t - \frac{1}{2}\right) - rect\left(2t - \frac{3}{2}\right)$$



- Define the time interval to be 0<t<1

$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty} \left\{ X_C[k]\cos\left(2\pi k f_F t\right) + X_S[k]\sin\left(2\pi k f_F t\right) \right\}$$

$$\begin{aligned} X_C[0] &= \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\cos(0)\,dt \\ &= \frac{2}{T_F}\int_0^{0.5} dt - \frac{2}{T_F}\int_{0.5}^{1} dt \\ &= 0 \end{aligned}$$

Example (cont.)

- First note that *$f_F = 1/T_F = 1$*
- Thus, for the cosine terms we have

$$\begin{aligned} X_C[k] &= 2\int_0^{0.5} \cos(2\pi k t)\,dt - 2\int_{0.5}^{1} \cos(2\pi k t)\,dt \\ &= \frac{2}{2\pi k}\sin(2\pi k t)\Big|_0^{0.5} - \frac{2}{2\pi k}\sin(2\pi k t)\Big|_{0.5}^{1} \\ &= \frac{2}{2\pi k}\left(\sin(k\pi) - \sin(0) - \sin(2k\pi) + \sin(k\pi)\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} X_S[k] &= 2\int_0^{0.5} \sin(2\pi k t)\,dt - 2\int_{0.5}^{1} \sin(2\pi k t)\,dt \\ &= -\frac{2}{2\pi k}\left(\cos(2\pi k t)\right)\Big|_0^{0.5} + \frac{2}{2\pi k}\left(\cos(2\pi k t)\right)\Big|_{0.5}^{1} \\ &= \frac{1}{\pi k}\left(-\cos(\pi k) + \cos(0) + \cos(2\pi k) - \cos(\pi k)\right) \\ &= \frac{1}{\pi k}\left(2 - 2\cos(\pi k)\right) \end{aligned}$$

Example (cont.)

- Thus, we can write the sine terms as

$$X_S[k] = \frac{1}{\pi k}\left(2 - 2\cos(\pi k)\right)$$
$$= \begin{cases} 0 & k = even \\ \frac{4}{\pi k} & k = odd \end{cases}$$

- Thus we have

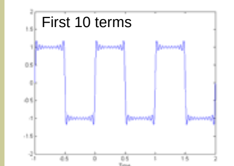
$$x_F(t) = \sum_{k=1}^{\infty} \frac{2 - 2\cos(\pi k)}{\pi k} \sin(2\pi k f_F t)$$

- Since the even terms are zero we can rewrite this as

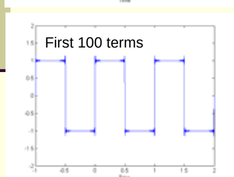
$$x_F(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin(2\pi(2k-1) f_F t)$$

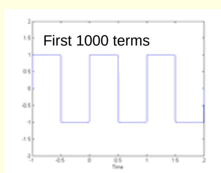
- This is the representation that we examined earlier!

Example - Plots



- As we continue to add the sinusoidal terms our representation is closer and closer to the original signal *over the interval of interest.*
- When the number of terms goes to infinity the representation is *exact*

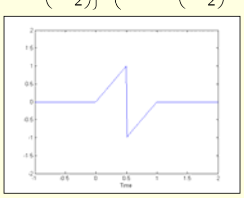




Example B

- Let's consider a sawtooth waveform

$$x(t) = 2*ramp(t)\left\{u(t) - u\left(t - \frac{1}{2}\right)\right\} + \left(2*ramp\left(t - \frac{1}{2}\right) - 2\right)\left\{u\left(t - \frac{1}{2}\right) - u(t-1)\right\}$$



- Define the time interval to be 0<t<1
- $T_F = 1$, $f_F = 1$

Example B

- The trigonometric Fourier Series is

$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty}\left\{X_C[k]\cos(2\pi k f_F t) + X_S[k]\sin(2\pi k f_F t)\right\}$$

- Looking at the cosine terms

$$X_C[k] = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\cos(2\pi k f_F t)\,dt$$
$$= 2\int_0^{0.5} 2t\cos(2\pi kt)\,dt + 2\int_{0.5}^{1}(2t-2)\cos(2\pi kt)\,dt$$
$$= \frac{4}{(2\pi k)^2}\left[\cos(2\pi kt) + 2\pi kt\sin(2\pi kt)\right]\Big|_0^{0.5} + \frac{4}{(2\pi k)^2}\left[\cos(2\pi kt) + 2\pi kt\sin(2\pi kt)\right]\Big|_{0.5}^{1}$$
$$-\frac{4}{2\pi k}\sin(2\pi kt)\Big|_{0.5}^{1}$$
$$= \frac{4}{(2\pi k)^2}\left[\cos(2\pi k) + 2\pi k\underbrace{\sin(2\pi k)}_{0} - \cos(0) + \underbrace{\sin(0)}_{0}\right] - \frac{4}{2\pi k}\left(\underbrace{\sin(2\pi k)}_{0} - \underbrace{\sin(\pi k)}_{0}\right)$$
$$= 0$$

Example B (cont.)

- Looking at the sine terms

$$X_S[k] = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\sin(2\pi k f_F t)\,dt$$
$$= 2\int_0^{0.5} 2t\sin(2\pi kt)\,dt + 2\int_{0.5}^{1}(2t-2)\sin(2\pi kt)\,dt$$
$$= \frac{4}{(2\pi k)^2}\left[\sin(2\pi kt) - 2\pi kt\cos(2\pi kt)\right]\Big|_0^{0.5} + \frac{4}{(2\pi k)^2}\left[\sin(2\pi kt) - 2\pi kt\cos(2\pi kt)\right]\Big|_{0.5}^{1}$$
$$+\frac{4}{2\pi k}\cos(2\pi kt)\Big|_{0.5}^{1}$$
$$= \frac{4}{(2\pi k)^2}\left[\sin(2\pi k) - 2\pi k\cos(2\pi k) - \sin(0) + \cos(0)\right] + \frac{4}{2\pi k}\left(\cos(2\pi k) - \cos(\pi k)\right)$$
$$= -\frac{2}{\pi k}\cos(2\pi k) + \frac{2}{\pi k}\cos(2\pi k) - \frac{2}{\pi k}\cos(\pi k)$$
$$= -\frac{2}{\pi k}\cos(\pi k)$$
$$= \begin{cases} -\frac{2}{\pi k} & k = even \\ \frac{2}{\pi k} & k = odd \end{cases}$$

Example B (cont.)

- Summarizing we can write our Continuous Time Fourier Series as

$$x_F(t) = \sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{\pi k}\sin(2\pi kt)$$

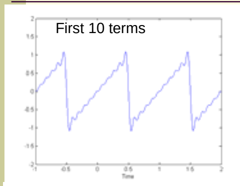
recalling that

$$x(t) = x_F(t) \quad 0 \le t \le 1$$

We can then write

$$x(t) = \sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{\pi k}\sin(2\pi kt) \quad 0 \le t \le 1$$

Example B (final)



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