ECE 2704
Signals and Systems
Spring 2006

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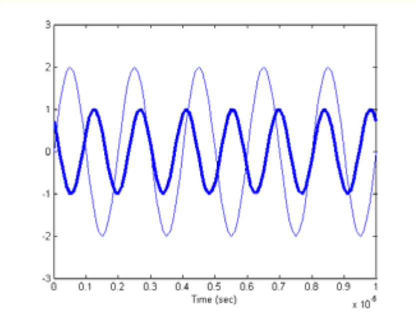
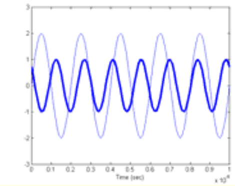
Lecture #8: An Introduction to the Fourier Series

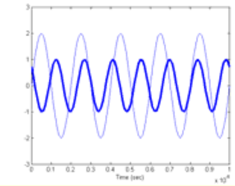


Overview



- Today we introduce the concept of the Continuous Time Fourier Series (CTFS)
- What to read – Section 4.1-4.2 in the text
- We will discuss the two versions of the CTFS
  - In terms of complex sinusoids
    - Applies to both real and complex signals
  - In terms of real sinusoids
    - Applies to real signals

Motivation

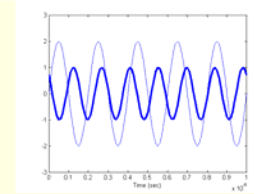


- *If* a system is linear and time invariant (LTI), *then* when the input is a weighted complex sinusoid, the output is also a complex sinusoid at the same frequency but (in general) with a different weighting.

$$Ae^{jax} \overset{\mathcal{H}}{\longrightarrow} Be^{jax}$$

- If we can represent a signal as a weighted sum complex sinusoids, then for an LTI system we can represent the output as the sum of weighted complex sinusoids.
  - Thus, we wish to show that signals can be represented as a sum of complex sinusoids

Complex Sinusoids

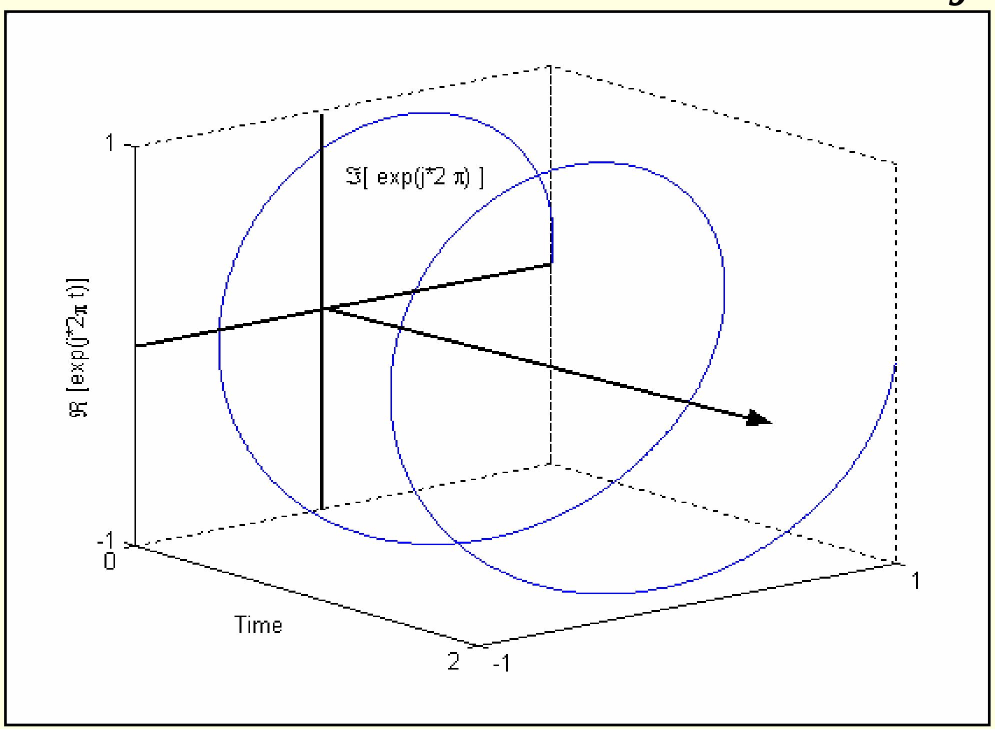


- Euler's identity defines a complex sinusoid as

$$e^{jt} = \cos t + j \sin t$$

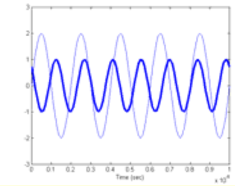
- Real sines and cosines can be written as

$$\cos t = \frac{e^{jt} + e^{-jt}}{2} \qquad \sin t = \frac{e^{jt} - e^{-jt}}{2j}$$



Plot of exp(j2πt)

Example



- Consider the waveform

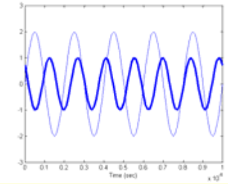
$$x(t) = rect\left(2t - \frac{1}{2}\right) - rect\left(2t - \frac{3}{2}\right)$$

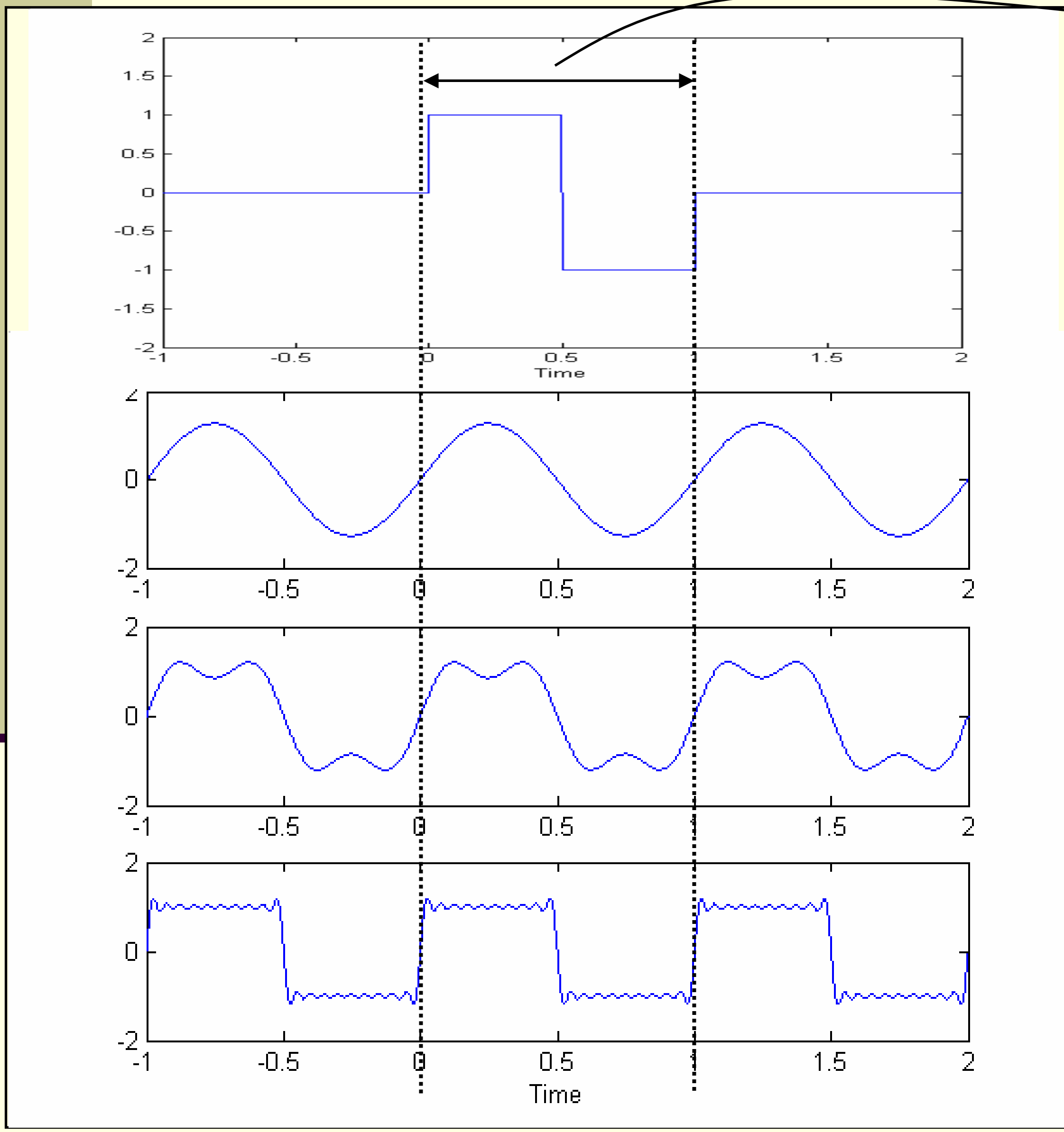
- Let us attempt to model this over the time frame 0<t<1 with sinusoids of period 1 and integer multiples of 1:

$$\hat{x}(t) = \frac{4}{\pi}\left(\sin(2\pi t) + \frac{1}{3}\sin(3*2\pi t) + \frac{1}{5}\sin(5*2\pi t) + \frac{1}{7}\sin(7*2\pi t)...\right)$$

- Note that we only care about the approximation over the time interval of interest

Example (cont.)





Interval of interest

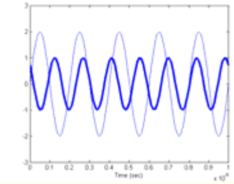
$$x(t) = rect\left(2t - \frac{1}{2}\right) - rect\left(2t - \frac{3}{2}\right)$$

$$\hat{x}(t) = \frac{4}{\pi}\sin(2\pi t)$$

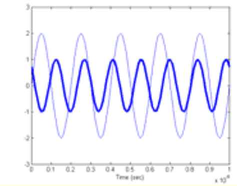
$$\hat{x}(t) = \frac{4}{\pi}\left(\sin(2\pi t) + \frac{1}{3}\sin(3*2\pi t)\right)$$

$$\hat{x}(t) = \frac{4}{\pi}\left(\sum_{i=1}^{10}\frac{1}{2i-1}\sin(2\pi(2i-1)t)\right)$$

Example (cont.)



- This example shows us that for a specific time interval it appears possible to represent a signal using a sum of sinusoids.
- However, how do we determine the frequency of those sinusoids and how do we determine the weights of the sinusoids?
  - A reasonable guess for frequencies would be multiples of a *fundamental* frequency which is equal to $1/T_f$ where T_f is the length of the interval being examined
    - This will be particularly useful when we examine *periodic* signals
  - We still need to determine the weights
- Further we may want to use both sines and cosines



Signal Representation

- Let's first assume that a signal *x(t)* can be represented over a time interval $t_o < t < t_o+T_f$ as a linear combination of complex sinusoids in the form

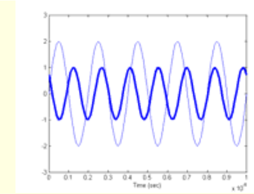
$$x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

where $f_F = 1/T_F$ is the fundamental frequency and

$$x(t) = x_F(t) \text{ for } t_o < t < t_o+T_f$$

- We will determine later when this can be safely assumed.
- Note that we are representing the signal *x(t)* over a finite time interval, not over all time.
- This is defined as the *Continuous Time Fourier Series* or CTFS

Finding the coefficients



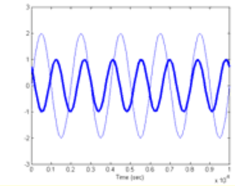
- The CTFS representation requires us to determine the coefficients *X[k]*
- Now, since $x(t) = x_F(t) \quad t_o < t < t_o + T_F$, then we can write

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} \quad t_o < t < t_o + T_F$$

- Now, let's multiply both sides by $exp(-j2\pi q f_F t)$ where *q* is an integer

$$\begin{aligned} x(t) e^{-j2\pi q f_F t} &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} e^{-j2\pi q f_F t} \\ &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi (k-q) f_F t} \end{aligned} \qquad t_o < t < t_o + T_F$$

Finding the coefficients (cont.)



- Now, integrating both sides over $t_o < t < t_o + T_f$

$$\int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi q f_F t} dt = \int_{t_o}^{t_o+T_F} \left[\sum_{k=-\infty}^{\infty} X[k] e^{j2\pi(k-q) f_F t} \right] dt$$

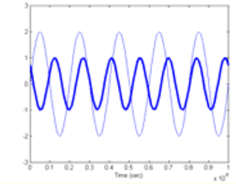
- Since k and t are independent we can interchange summation and integration

$$\int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi q f_F t} dt = \sum_{k=-\infty}^{\infty} X[k] \int_{t_o}^{t_o+T_F} e^{j2\pi(k-q) f_F t} dt \qquad \text{Equation A}$$

- Using Euler's identity

$$\int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi q f_F t} dt = \sum_{k=-\infty}^{\infty} X[k] \int_{t_o}^{t_o+T_F} \left[\cos\left(2\pi (k-q) f_F t\right) + j \sin\left(2\pi (k-q) f_F t\right) \right] dt$$

Continuing…



$$\int_{t_o}^{t_o+T_F} x(t)e^{-j2\pi q f_F t}dt = \sum_{k=-\infty}^{\infty} X[k] \int_{t_o}^{t_o+T_F} \left[\cos\left(2\pi(k-q) f_F t\right) + j\sin\left(2\pi(k-q) f_F t\right)\right]dt$$

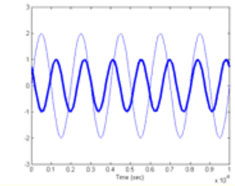
- *k-q* is an integer. For $k \neq q$ we are integrating *cos(2π[k-q]f*$_F$*t)* and *sin(2π[k-q]f*$_F$*t)* over exactly *k-q* periods which results in zero.
- For $k \neq q$

$$\int_{t_o}^{t_o+T_F} \left[\cos\left(2\pi(k-q) f_F t\right) + j\sin\left(2\pi(k-q) f_F t\right)\right]dt = 0$$

- For *k = q*

$$\int_{t_o}^{t_o+T_F} \left[\cos(0) + j\sin(0)\right]dt = T_F$$

Continuing…



- Therefore,

$$\sum_{k=-\infty}^{\infty} X[k] \int_{t_o}^{t_o+T_F} e^{-j2\pi(k-q)f_F t}\, dt = X[q]T_F$$

- Making this substitution into *Equation A*

$$\int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi q f_F t} dt = X[q]T_F$$

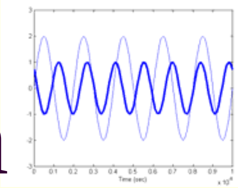
- Solving for *X[q]:*

$$X[q] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi q f_F t} dt$$

- Since this is true for *X[q]*, then *X[k]* is

$$X[k] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt$$

Continuous Time Fourier Transform



- Thus, if the integral converges, we can represent the signal x(t) exactly, on the time interval $t_o < t < t_o + T_f$ by

$$x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

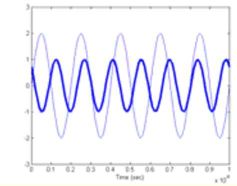
where

$$X[k] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt$$

- Note that if the above integral does not converge, the CTFS cannot be found over the region of interest.
- We can represent the relationship between *x(t)* and *X[k]* as

$$x(t) \overset{FS}{\leftrightarrow} X[k]$$

Dirichlet Conditions



- If the integral

$$X[k] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt$$

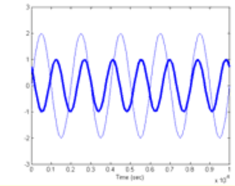
diverges, the CTFS cannot be found for the signal over the region of interest.

- This condition is necessary but not sufficient.
- Sufficient conditions on the existence of the CTFS are known as the *Dirichlet Conditions:*
  1. The signal must be absolutely integrable over the time $t_o < t < t_o + T_f$

$$\int_{t_o}^{t_o+T_F} |x(t)| dt < \infty$$

  2. The signal must have a finite number of maxima and minima in the time $t_o < t < t_o + T_f$
  3. The signal must have a finite number of discontinuities in the time $t_o < t < t_o + T_f$

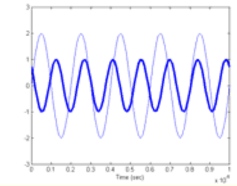
Complex Conjugate of *x(t)*



- The CTFS derived previously holds for any signal which satisfies the Dirichlet conditions
  - $x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$ applies to both real signals and complex signals
- Consider the complex conjugate of a signal $x_F(t)$
- By conjugating both sides of our expression:

$$x_F^*(t) = \sum_{k=-\infty}^{\infty} X^*[k] e^{-j2\pi k f_F t}$$

Complex Conjugate (cont.)



- Simplifying:

$$x_F^*(t) = \sum_{k=-\infty}^{\infty} X^*[k] e^{-j2\pi k f_F t}$$

$$= \sum_{k=\infty}^{-\infty} X^*[-k] e^{j2\pi k f_F t}$$

$$= \sum_{k=-\infty}^{\infty} X^*[-k] e^{j2\pi k f_F t}$$

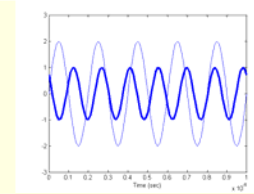
- Thus, we can say that if

$$x(t) \overset{FS}{\leftrightarrow} X[k]$$

then

$$x^*(t) \overset{FS}{\leftrightarrow} X^*[-k]$$

Real Signals



- For the important case where the signal is all real

$$x(t) = x^{*}(t)$$

which means that

$$x_F(t) = x_F^{*}(t)$$

- In this case

$$x_F^{*}(t) = \sum_{k=-\infty}^{\infty} X^{*}[-k] e^{j2\pi k f_F t} = x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

which leads to

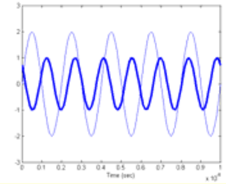
$$\sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} = \sum_{k=-\infty}^{\infty} X^{*}[-k] e^{-j2\pi(-k) f_F t}$$

$$= \sum_{k=-\infty}^{\infty} X^{*}[-k] e^{j2\pi k f_F t}$$

meaning that

$$X[k] = X^{*}[-k]$$

Real Signals (cont.)



- For real signals then we can write the CTFS as

$$x_F(t) = X[0] + \sum_{k=1}^{\infty}\left[X[k] e^{j2\pi k f_F t} + X^*[k] e^{-j2\pi k f_F t} \right]$$

which can also be written as

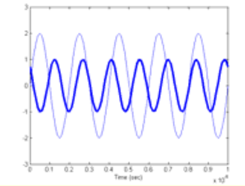
$$x_F(t) = X[0] + \sum_{k=1}^{\infty}\Big\{ \mathrm{Re}\big(X[k]\big) e^{j2\pi k f_F t} + \mathrm{Re}\big(X[k]\big) e^{-j2\pi k f_F t}$$
$$+ j\,\mathrm{Im}\big(X[k]\big) e^{j2\pi k f_F t} - j\,\mathrm{Im}\big(X[k]\big) e^{-j2\pi k f_F t} \Big\}$$

Examining the first term:

$$X[0] = \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t)\,dt$$

which is simply the average value of the function (note that it is all real).

Trigonometric CTFS



- Recall that $\cos t = \dfrac{e^{jt} + e^{-jt}}{2} \quad \sin t = \dfrac{e^{jt} - e^{-jt}}{2j}$

- Substituting these relationships into the previous equation we have

$$x_F(t) = X[0] + \sum_{k=1}^{\infty} \left\{ 2\,\mathrm{Re}\left(X[k]\right)\cos\left(2\pi k f_F t\right) + 2\,\mathrm{Im}\left(X[k]\right)\sin\left(2\pi k f_F t\right) \right\}$$

which can be re-written as

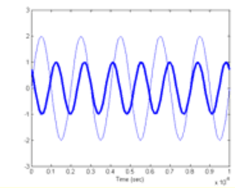
$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty} \left\{ X_C[k]\cos\left(2\pi k f_F t\right) + X_S[k]\sin\left(2\pi k f_F t\right) \right\}$$

with

$$X_C[k] = 2\,\mathrm{Re}\left\{X[k]\right\} = 2\,\mathrm{Re}\left\{ \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt \right\} = \frac{2}{T_F} \int_{t_o}^{t_o+T_F} x(t) \cos\left(2\pi k f_F t\right) dt$$

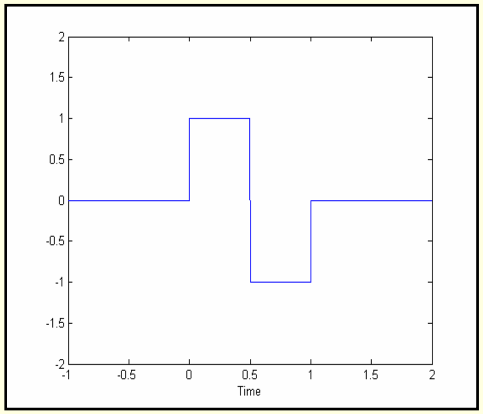
$$X_S[k] = -2\,\mathrm{Im}\left\{X[k]\right\} = -2\,\mathrm{Im}\left\{ \frac{1}{T_F} \int_{t_o}^{t_o+T_F} x(t) e^{-j2\pi k f_F t} dt \right\} = \frac{2}{T_F} \int_{t_o}^{t_o+T_F} x(t) \sin\left(2\pi k f_F t\right) dt$$

Example



- Let's return to our previous example of

$$x(t) = rect\left(2t - \frac{1}{2}\right) - rect\left(2t - \frac{3}{2}\right)$$

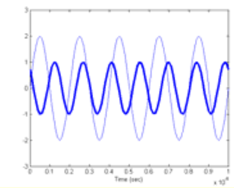


- Define the time interval to be 0<t<1

$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty}\left\{X_C[k]\cos(2\pi k f_F t) + X_S[k]\sin(2\pi k f_F t)\right\}$$

$$X_C[0] = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\cos(0)\,dt$$
$$= \frac{2}{T_F}\int_{0}^{0.5} dt - \frac{2}{T_F}\int_{0.5}^{1} dt$$
$$= 0$$

Example (cont.)

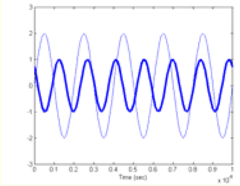


- First note that $f_F = 1/T_F = 1$
- Thus, for the cosine terms we have

$$
\begin{aligned}
X_C[k] &= 2\int_0^{0.5}\cos(2\pi kt)\,dt - 2\int_{0.5}^{1}\cos(2\pi kt)\,dt \\
&= \frac{2}{2\pi k}\sin(2\pi kt)\Big|_0^{0.5} - \frac{2}{2\pi k}\sin(2\pi kt)\Big|_{0.5}^{1} \\
&= \frac{2}{2\pi k}\left(\sin(k\pi) - \sin(0) - \sin(2k\pi) + \sin(k\pi)\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
X_S[k] &= 2\int_0^{0.5}\sin(2\pi kt)\,dt - 2\int_{0.5}^{1}\sin(2\pi kt)\,dt \\
&= -\frac{2}{2\pi k}\left(\cos(2\pi kt)\right)\Big|_0^{0.5} + \frac{2}{2\pi k}\left(\cos(2\pi kt)\right)\Big|_{0.5}^{1} \\
&= \frac{1}{\pi k}\left(-\cos(\pi k) + \cos(0) + \cos(2\pi k) - \cos(\pi k)\right) \\
&= \frac{1}{\pi k}\left(2 - 2\cos(\pi k)\right)
\end{aligned}
$$

Example (cont.)



- Thus, we can write the sine terms as

$$X_s[k] = \frac{1}{\pi k}\left(2 - 2\cos(\pi k)\right)$$
$$= \begin{cases} 0 & k = even \\ \dfrac{4}{\pi k} & k = odd \end{cases}$$

- Thus we have

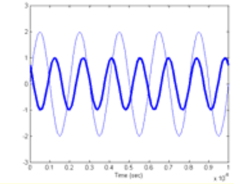
$$x_F(t) = \sum_{k=1}^{\infty} \frac{2 - 2\cos(\pi k)}{\pi k} \sin(2\pi k f_F t)$$

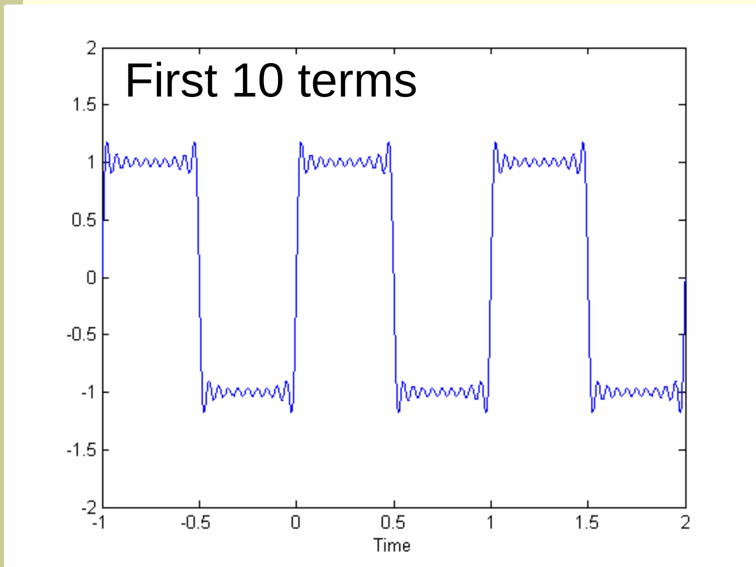
- Since the even terms are zero we can rewrite this as

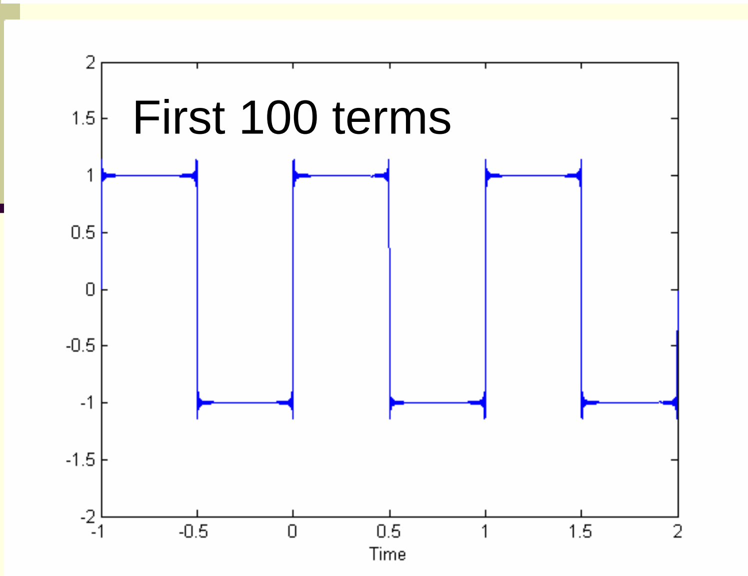
$$x_F(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin\left(2\pi(2k-1) f_F t\right)$$

- This is the representation that we examined earlier!

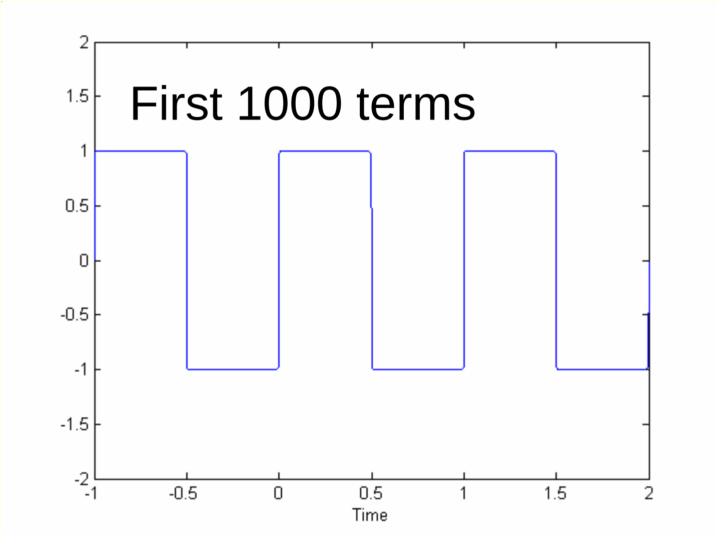
Example - Plots



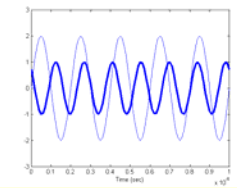




- As we continue to add the sinusoidal terms our representation is closer and closer to the original signal *over the interval of interest.*
- When the number of terms goes to infinity the representation is *exact*

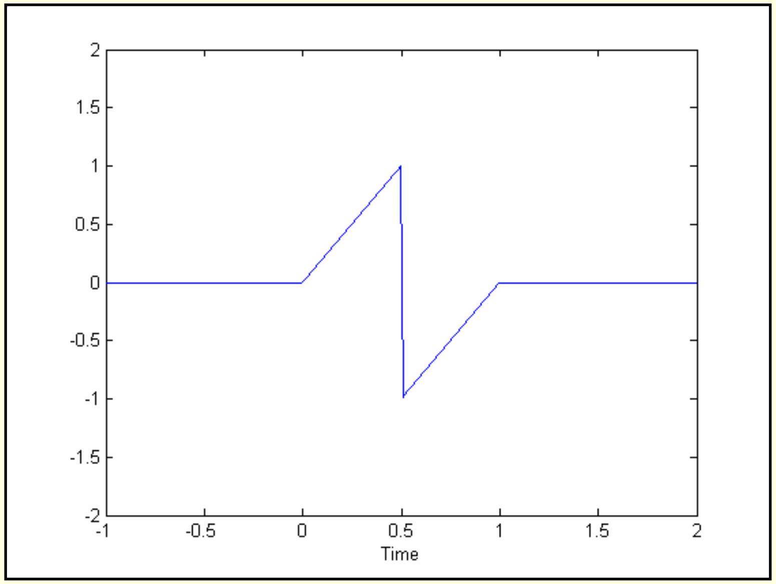


Example B

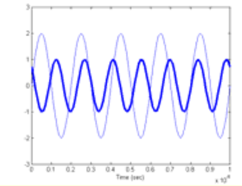


- Let's consider a sawtooth waveform

$$x(t) = 2 * ramp(t)\left\{u(t) - u\left(t - \frac{1}{2}\right)\right\} + \left(2 * ramp\left(t - \frac{1}{2}\right) - 2\right)\left\{u\left(t - \frac{1}{2}\right) - u(t-1)\right\}$$



- Define the time interval to be 0<t<1
- $T_F = 1$, $f_F = 1$



Example B

- The trigonometric Fourier Series is

$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty}\left\{X_C[k]\cos(2\pi k f_F t) + X_S[k]\sin(2\pi k f_F t)\right\}$$

- Looking at the cosine terms

$$X_C[k] = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\cos(2\pi k f_F t)\,dt$$

$$= 2\int_0^{0.5} 2t\cos(2\pi kt)\,dt + 2\int_{0.5}^{1}(2t-2)\cos(2\pi kt)\,dt$$

$$= \frac{4}{(2\pi k)^2}\Big[\cos(2\pi kt) + 2\pi kt\sin(2\pi kt)\Big]\Big|_0^{0.5} + \frac{4}{(2\pi k)^2}\Big[\cos(2\pi kt) + 2\pi kt\sin(2\pi kt)\Big]\Big|_{0.5}^{1}$$

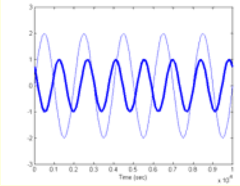
$$-\frac{4}{2\pi k}\sin(2\pi kt)\Big|_{0.5}^{1}$$

0 0 0 0

$$= \frac{4}{(2\pi k)^2}\Big[\cos(2\pi k) + 2\pi k\sin(2\pi k) - \cos(0) + \sin(0)\Big] - \frac{4}{2\pi k}\big(\sin(2\pi k) - \sin(\pi k)\big)$$

$$= 0$$

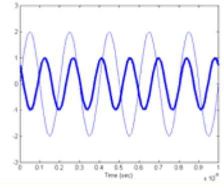
Example B (cont.)



- Looking at the sine terms

$$
\begin{aligned}
X_S[k] &= \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\sin(2\pi k f_F t)\,dt \\
&= 2\int_0^{0.5} 2t\sin(2\pi kt)\,dt + 2\int_{0.5}^{1}(2t-2)\sin(2\pi kt)\,dt \\
&= \frac{4}{(2\pi k)^2}\left[\sin(2\pi kt) - 2\pi kt\cos(2\pi kt)\right]\Big|_0^{0.5} + \frac{4}{(2\pi k)^2}\left[\sin(2\pi kt) - 2\pi kt\cos(2\pi kt)\right]\Big|_{0.5}^{1} \\
&\quad + \frac{4}{2\pi k}\cos(2\pi kt)\Big|_{0.5}^{1} \\
&= \frac{4}{(2\pi k)^2}\left[\sin(2\pi k) - 2\pi k\cos(2\pi k) - \sin(0) + \cos(0)\right] + \frac{4}{2\pi k}\left(\cos(2\pi k) - \cos(\pi k)\right) \\
&= -\frac{2}{\pi k}\cos(2\pi k) + \frac{2}{\pi k}\cos(2\pi k) - \frac{2}{\pi k}\cos(\pi k) \\
&= -\frac{2}{\pi k}\cos(\pi k) \\
&= \begin{cases} -\dfrac{2}{\pi k} & k = even \\ \dfrac{2}{\pi k} & k = odd \end{cases}
\end{aligned}
$$

Example B (cont.)



- Summarizing we can write our Continuous Time Fourier Series as

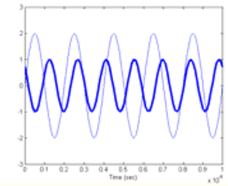
$$x_F(t) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi k} \sin(2\pi k t)$$

recalling that

$$x(t) = x_F(t) \quad 0 \le t \le 1$$

We can then write

$$x(t) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi k} \sin(2\pi k t) \quad 0 \le t \le 1$$



Example B (final)

- As we continue to add the sinusoidal terms our representation is closer and closer to the original signal *over the interval of interest.*
- When the number of terms goes to infinity the representation is *exact*