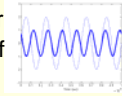


ECE 2704  
Signals and Systems  
Spring 2006

Instructor: Dr. R. Michael Buehrer  
Lecture #9: Further Discussion of  
The Fourier Series



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### Overview



- Today we continue to discuss the concept of the Continuous Time Fourier Series (CTFS)
- What to read – Section 4.3 in the text
- Last class we introduced the concept, today we will discuss
  - Periodicity of the CTFS
  - Insights into the calculation of the CTFS
  - CTFS for periodic signals

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### Periodicity



- The CTFS representation  $x_F(t)$  of the signal  $x(t)$  is equivalent to  $x(t)$  only over the interval

$$t_o < t < t_o + T_F$$

- The signal  $x(t)$  is not, in general, periodic.
- The CTFS, however, is periodic. Furthermore, it is periodic with period  $T_F$
- In other words

$$x_F(t) = x_F(t + qT_F)$$

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## Periodicity - Proof



### Proof:

By definition –  $x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$

If we increment time by a multiple of  $T_F$ :

$$\begin{aligned} x_F(t + qT_F) &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F (t + qT_F)} \\ &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} e^{j2\pi k q f_F T_F} \end{aligned}$$

Now  $f_F T_F = 1$ . Thus,

$$\begin{aligned} x_F(t + qT_F) &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} \underbrace{e^{j2\pi k q}}_{=1} \\ &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} \\ &= x_F(t) \quad \text{Q.E.D.} \end{aligned}$$

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## Calculating the CTFS



- Consider the function

$$x(t) = 2 \cos(400\pi t)$$

over the interval  $0 \leq t \leq 0.005$

- Let's find the CTFS representation for this signal.
- Since it is an all real function, let's use the trigonometric form of the CTFS:

$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty} \{X_C[k] \cos(2\pi k f_F t) + X_S[k] \sin(2\pi k f_F t)\}$$

where

$$X_C[k] = \frac{2}{T_F} \int_{t_c}^{t_c + T_F} x(t) \cos(2\pi k f_F t) dt \quad X_S[k] = \frac{2}{T_F} \int_{t_c}^{t_c + T_F} x(t) \sin(2\pi k f_F t) dt$$

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## Calculating the coefficients



- Consider the first two coefficients

$$X_C[1] = \frac{2}{T_F} \int_0^{T_F} x(t) \cos(2\pi f_F t) dt \quad X_S[1] = \frac{2}{T_F} \int_0^{T_F} x(t) \sin(2\pi f_F t) dt$$

- Now,  $T_F = 0.005$  and  $f_F = 1/T_F = 200$ .

- Note that we are investigating one period of the original signal

- Let's examine the following two integrals more carefully

$$X_C[1] = 400 \int_0^{0.005} 2 \cos(400\pi t) \cos(400\pi t) dt$$

$$X_S[1] = 400 \int_0^{0.005} 2 \cos(400\pi t) \sin(400\pi t) dt$$

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## Further Harmonics



- For  $X_S[k]$  ( $k > 2$ )

$$\begin{aligned}
 X_S[k] &= 400 \int_0^{T_f} 2 \cos(2\pi f_f t) \sin(2\pi k f_f t) dt \\
 &= 400 \int_0^{T_f} \{\cos(2\pi(k-1)f_f t) - \cos(2\pi(k+1)f_f t)\} dt \\
 &= \frac{400}{2\pi f_f} \left\{ \frac{1}{k-1} \sin(2\pi(k-1)f_f t) - \frac{1}{k+1} \sin(2\pi(k+1)f_f t) \right\} \Big|_0^{T_f} \\
 &= \frac{400}{2\pi f_f} \left\{ \frac{1}{k-1} \sin(2\pi(k-1)f_f T_f) - 0 - \frac{1}{k+1} \sin(2\pi(k+1)f_f T_f) + 0 \right\} \\
 &= \frac{400}{2\pi f_f} \left\{ \frac{1}{k-1} \underbrace{\sin(2\pi(k-1))}_{=0} - \frac{1}{k+1} \underbrace{\sin(2\pi(k+1))}_{=0} \right\} \\
 &= 0
 \end{aligned}$$

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Notes:  $f_f T_f = 1$   
 $\sin(2\pi k) = 0$  when  $k = \text{integer}$

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## $X_C[0]$



- The remaining term of the Fourier Series is  $X_C[0]$ .
- We could calculate this term as before, but recall its meaning:
  - $X_C[0]$  is the average value of the function over the interval of interest.
  - Since we are examining one period of a sinusoidal function, the average value is zero.
  - Thus,  $X_C[0] = 0$

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## Final



- Thus, the signal  $x(t) = 2\cos(400\pi t)$  over the interval  $0 \leq t \leq 0.005$  has only one non-zero Fourier Series term,

$$\begin{aligned}
 X_C[k] &= \begin{cases} 2 & k = 1 \\ 0 & k \neq 1 \end{cases} \\
 X_S[k] &= 0 \quad \forall k
 \end{aligned}$$

- This leads to

$$\begin{aligned}
 x_f(t) &= X_C[0] + \sum_{k=1}^{\infty} \{X_C[k] \cos(2\pi k f_f t) + X_S[k] \sin(2\pi k f_f t)\} \\
 &= 2 \cos(2\pi f_f t) \\
 &= 2 \cos(400\pi t)
 \end{aligned}$$

as expected.

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## Interpretation



- The trigonometric form of the CTFS represents the signal by using only cosines and sines.
- The actual Fourier Series components  $X_C[k]$  and  $X_S[k]$  represent the **correlation** between the function of interest and the sinusoids at the **fundamental frequency** ( $f_F = 1/T_F$ ) and its **harmonics** ( $kf_F$ )
- The correlation tells us how much of that frequency term is contained in the signal

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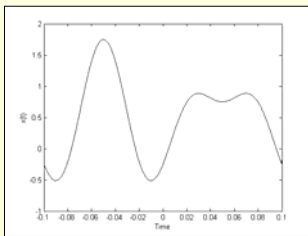
## Example



- The previous example was trivial, so let's look at a *slightly* more complicated signal

$$x(t) = \frac{1}{2} - \frac{3}{4} \cos(20\pi t) + \frac{1}{2} \sin(30\pi t)$$

over the interval  $-0.1 \leq t \leq 0.1$




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## Example (cont.)



$$x(t) = \frac{1}{2} - \frac{3}{4} \cos(20\pi t) + \frac{1}{2} \sin(30\pi t)$$

- What would you guess the solution to be?
- First,  $T_F = 200\text{ms}$ ,  $f_F = 5\text{Hz}$
- Second,  $X_C[0] = \text{average value} = ?$

$$X_C[0] = \frac{1}{2}$$

$$X_C[k] = \frac{2}{T_F} \int_0^{T_F} x(t) \cos(2\pi k f_F t) dt \quad X_S[k] = \frac{2}{T_F} \int_0^{T_F} x(t) \sin(2\pi k f_F t) dt$$

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## $X_C[k]$



$$\begin{aligned}
 X_C[k] &= \frac{2}{T_f} \int_0^{T_f} \left\{ \frac{1}{2} - \frac{3}{4} \cos(20\pi t) + \frac{1}{2} \sin(30\pi t) \right\} \cos(2\pi k f_f t) dt \\
 &= \frac{2}{T_f} \int_0^{T_f} \left\{ \frac{1}{2} - \frac{3}{4} \cos(2\pi(2f_f)t) + \frac{1}{2} \sin(2\pi(3f_f)t) \right\} \cos(2\pi k f_f t) dt \\
 &= \frac{1}{2T_f} \int_0^{T_f} \cos(2\pi k f_f t) dt - \frac{3}{4T_f} \int_0^{T_f} \cos(2\pi(2f_f)t) \cos(2\pi k f_f t) dt + \dots \\
 &\quad \frac{1}{2T_f} \int_0^{T_f} \sin(2\pi(3f_f)t) \cos(2\pi k f_f t) dt
 \end{aligned}$$

Looking at the first term:

$$\begin{aligned}
 \frac{1}{2T_f} \int_0^{T_f} \cos(2\pi k f_f t) dt &= \frac{1}{2\pi k f_f T_f} \sin(2\pi k f_f t) \Big|_0^{T_f} \\
 &= \frac{1}{2\pi k f_f T_f} [\sin(2\pi k f_f T_f) - 0] \\
 &= \frac{1}{2\pi k f_f T_f} \sin(2\pi k) \\
 &= 0 \qquad \qquad \qquad k > 0
 \end{aligned}$$

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## $X_C[k]$ (cont.)



■ Looking at the 2<sup>nd</sup> term:

$$\begin{aligned}
 X_C[k] &= -\frac{3}{4T_f} \int_0^{T_f} \cos(20\pi t) \cos(2\pi k f_f t) dt \\
 &= -\frac{3}{4T_f} \int_0^{T_f} \left\{ \cos([4-2k]\pi f_f t) + \cos([4+2k]\pi f_f t) \right\} dt \\
 &= -\frac{3}{4T_f} \left\{ \frac{1}{[4-2k]\pi f_f} \sin([4-2k]\pi f_f t) + \frac{1}{[4+2k]\pi f_f} \sin([4+2k]\pi f_f t) \right\} \Big|_0^{T_f} \\
 &= -\frac{3}{4T_f} \left\{ \frac{1}{[4-2k]\pi f_f} \sin\left([4-2k]\pi f_f \frac{T_f}{1}\right) - 0 + \frac{1}{[4+2k]\pi f_f} \sin\left([4+2k]\pi f_f \frac{T_f}{1}\right) - 0 \right\} \\
 &= \begin{cases} 0 & k \neq 2 \\ 0 & k = 2 \end{cases}
 \end{aligned}$$

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## $X_C[k]$ (cont.)



■ Using L'Hopital's Rule

$$\begin{aligned}
 \frac{3}{4T_f} \frac{1}{\frac{d}{dk} \left[ \frac{\sin([4-2k]\pi)}{[4-2k]\pi f_f} \right]}{\frac{d}{dk} [4-2k]\pi f_f} &= \frac{3}{4T_f} \frac{1}{-2\pi} \frac{-2\pi \cos([4-2k]\pi)}{-2\pi f_f} \\
 &= -\frac{3}{4}
 \end{aligned}$$

$$-\frac{3}{4T_f} \int_0^{T_f} \cos(20\pi t) \cos(k2\pi f_f \pi t) dt = \begin{cases} \frac{3}{4} & k = 2 \\ 0 & k \neq 2 \end{cases}$$

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## $X_C[k]$ (cont.)



### ■ Examining the third term

$$\begin{aligned} \frac{1}{2T_F} \int_0^{T_F} \sin(2\pi(3f_f)t) \cos(2\pi kf_f t) dt &= \frac{1}{2T_F} \int_0^{T_F} \left\{ \sin((6-2k)\pi f_f t) + \sin((6+2k)\pi f_f t) \right\} dt \\ &= \frac{1}{2T_F} \left[ -\frac{1}{(6-2k)\pi f_f} \cos((6-2k)\pi f_f t) - \dots \right. \\ &\quad \left. -\frac{1}{(6+2k)\pi f_f} \cos((6+2k)\pi f_f t) \right]_0^{T_F} \\ &= -\frac{1}{2T_F} \frac{1}{(6-2k)\pi f_f} \left[ \underbrace{\cos((6-2k)\pi f_f T_F)}_{\cos(2\pi)} - 1 \right] + \dots \\ &= -\frac{1}{2T_F} \frac{1}{(6+2k)\pi f_f} \left[ \underbrace{\cos((6+2k)\pi f_f T_F)}_{\cos(2\pi)} - 1 \right] \\ &= 0 \end{aligned}$$

Note: We must use L'Hopital's rule to evaluate the case where  $k=3$ . It still evaluates to zero.

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## $X_C[k]$ (cont.) and $X_S[k]$



### ■ Putting together all three terms

$$X_C[k] = \begin{cases} 0 & k \neq 2 \\ -\frac{3}{4} & k = 2 \end{cases}$$

### ■ Turning to $X_S[k]$

$$\begin{aligned} X_S[k] &= \frac{2}{T_F} \int_0^{T_F} x(t) \sin(2\pi kf_f t) dt \\ &= \frac{2}{T_F} \int_0^{T_F} \left\{ \frac{1}{2} - \frac{3}{4} \cos(20\pi t) + \frac{1}{2} \sin(30\pi t) \right\} \sin(2\pi kf_f t) dt \end{aligned}$$

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## $X_S[k]$



$$\begin{aligned} X_S[k] &= \frac{1}{T_F} \int_0^{T_F} \sin(2\pi kf_f t) dt - \frac{3}{2T_F} \int_0^{T_F} \cos(20\pi t) \sin(2\pi kf_f t) dt + \dots \\ &\quad + \frac{1}{T_F} \int_0^{T_F} \sin(30\pi t) \sin(2\pi kf_f t) dt \\ &= 0 + 0 + \frac{1}{T_F} \int_0^{T_F} \sin(6\pi f_f t) \sin(2\pi kf_f t) dt \\ &= \frac{1}{2T_F} \int_0^{T_F} \left\{ \cos([6-2k]\pi f_f t) - \cos([6+2k]\pi f_f t) \right\} dt \\ &\quad \vdots \\ &\quad \vdots \end{aligned}$$

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### $X_S[k]$ (cont.)

$$\begin{aligned}
 X_S[k] &= \frac{1}{2[6-2k]\pi f_r T_F} \sin([6-2k]\pi f_r t) - \dots \\
 &\quad - \frac{1}{2[6+2k]\pi f_r T_F} \sin([6+2k]\pi f_r t) \Big|_0^{T_F} \\
 &= \frac{1}{2[6-2k]\pi f_r T_F} [\sin([6-2k]\pi f_r T_F) - 0] \\
 &\quad - \frac{1}{2[6+2k]\pi f_r T_F} [\sin([6+2k]\pi f_r T_F) - 0] \\
 &= \begin{cases} 0 & k \neq 3 \\ 0 & k = 3 \end{cases}
 \end{aligned}$$




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### $X_S[k]$ (cont.)

- Using L'Hopital's Rule

$$\frac{\frac{d}{dk} [\sin([6-2k]\pi)]}{\frac{d}{dk} [2[6-2k]\pi]} = \frac{-2\pi [\cos([6-2k]\pi)]}{-4\pi}$$

$$\begin{aligned}
 &= \frac{1}{2} \\
 X_S[k] &= \begin{cases} \frac{1}{2} & k = 3 \\ 0 & k \neq 3 \end{cases}
 \end{aligned}$$

- Putting together all the pieces

$$\begin{aligned}
 x_r(t) &= X_C[0] + \sum_{k=1}^{\infty} \{X_C[k] \cos(2\pi k f_r t) + X_S[k] \sin(2\pi k f_r t)\} \\
 &= \frac{1}{2} - \frac{3}{4} \cos(4\pi f_r t) + \frac{1}{2} \sin(6\pi f_r t)
 \end{aligned}$$




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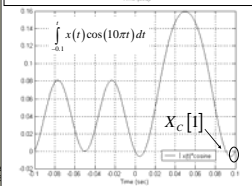
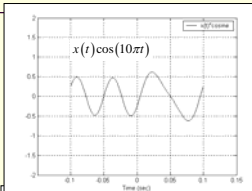
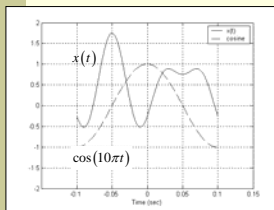
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### Graphical Illustration of $X_C[1]$



$$\begin{aligned}
 X_C[1] &= \frac{2}{T_F} \int_{-0.1}^{0.1} x(t) \cos(10\pi t) dt \\
 &= 0
 \end{aligned}$$




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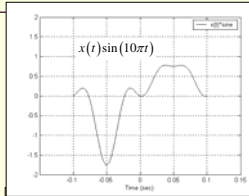
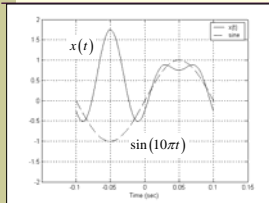
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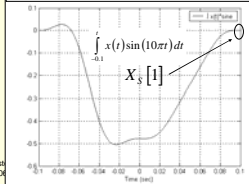
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## Graphical Illustration of $X_S[1]$



$$X_S[1] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t)\sin(10\pi t) dt = 0$$

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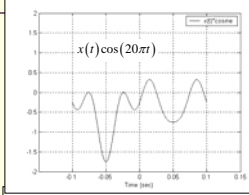
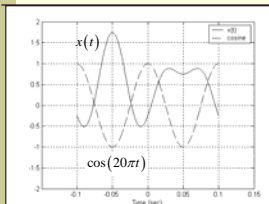
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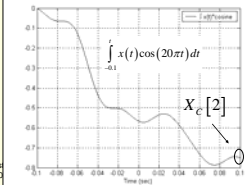
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## Graphical Illustration of $X_C[2]$



$$X_C[2] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t)\cos(20\pi t) dt = -0.75$$

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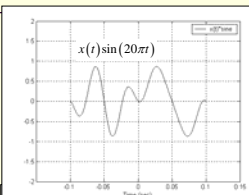
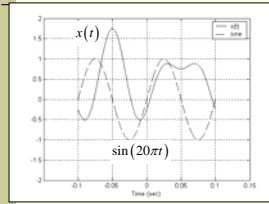
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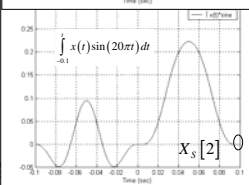
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## Graphical Illustration of $X_S[2]$



$$X_S[2] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t)\sin(20\pi t) dt = 0$$

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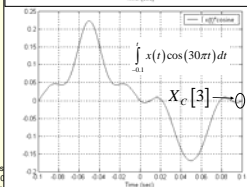
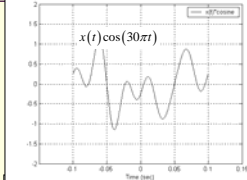
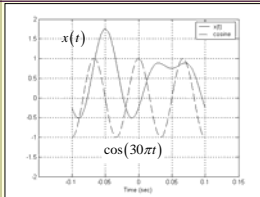
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## Graphical Illustration of $X_C[3]$



$$X_C[3] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t) \cos(30\pi t) dt = 0$$

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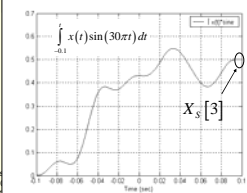
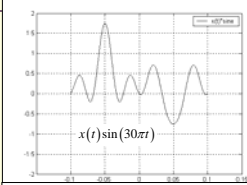
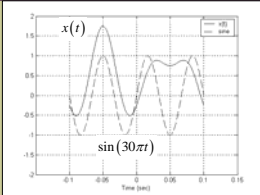
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## Graphical Illustration of $X_S[3]$



$$X_S[3] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t) \sin(30\pi t) dt = 0.5$$

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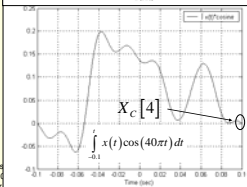
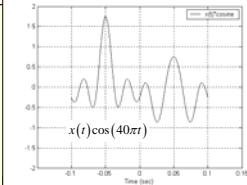
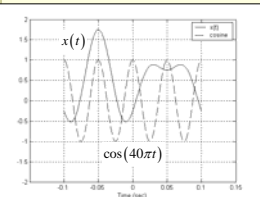
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## Graphical Illustration of $X_C[4]$



$$X_C[4] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t) \cos(40\pi t) dt = 0$$

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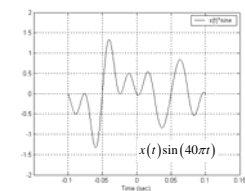
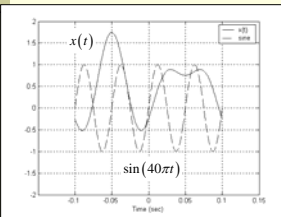
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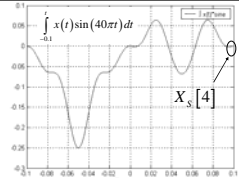
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## Graphical Illustration of $X_s[4]$



$$X_s[4] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t) \sin(40\pi t) dt = 0$$

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## Summary

- The Fourier Series is a sum of sinusoids (cosines and sines in the case of the trigonometric version)
- The weights of these sinusoids depend on the **correlation** between the signal of interest and each sinusoid in the series.
- When the signal of interest is a sum of sinusoids
  - The Fourier series terms which correspond to the frequencies in the signal will have weights corresponding to the amplitudes of the sinusoids
    - In other words there is perfect correlation between sines and cosines of the same frequency
    - sine and cosine of the same frequency are **orthogonal**
  - Provided that the interval of interest is an integer number of periods for all sinusoids in the sum, all other Fourier Series components will be zero.
    - This is because sinusoids whose frequencies are integer multiples or factors of each other are **orthogonal** when we integrate over an integer number of periods
    - This will not be true if we choose an arbitrary interval.

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## Periodic Signals

- The continuous time Fourier series is defined such that it is equivalent to the desired signal only over the interval of interest
- The continuous time Fourier series is periodic with period  $T_F$ .
- For a periodic signal the Fourier series representation is identical to the original signal over all time *if we choose the interval of interest to be one period (or an integer number of periods) of the original signal*

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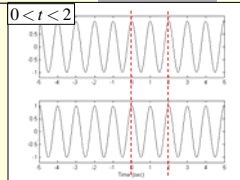
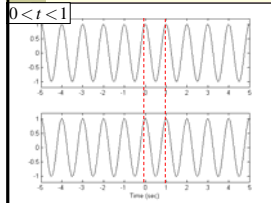
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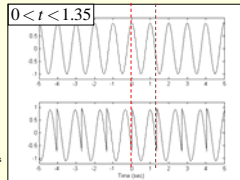
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## Impact of Chosen Interval



- The CTFS representation always equals the signal over the region of interest
- Since both signals are periodic, if the period of the two signals is the same, they will be equal for *all time*.
- The periods will be the same when the interval of interest is equal to one period or an integer number of periods of the original signal



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