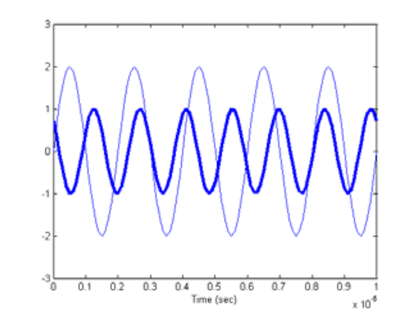
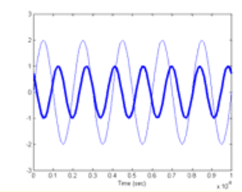
ECE 2704
Signals and Systems
Spring 2006

Instructor: Dr. R. Michael Buehrer

Lecture #9: Further Discussion of The Fourier Series

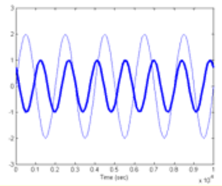




Overview

- Today we continue to discuss the concept of the Continuous Time Fourier Series (CTFS)
- What to read – Section 4.3 in the text
- Last class we introduced the concept, today we will discuss
  - Periodicity of the CTFS
  - Insights into the calculation of the CTFS
  - CTFS for periodic signals

Periodicity



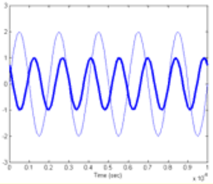
- The CTFS representation $x_F(t)$ of the signal $x(t)$ is equivalent to x(t) only over the interval

$$t_o < t < t_o + T_F$$

- The signal $x(t)$ is not, in general, periodic.
- The CTFS, however, is periodic. Furthermore, it is periodic with period T_F
- In other words

$$x_F\left(t\right) = x_F\left(t + qT_F\right)$$

Periodicity - Proof



Proof:

By definition –

$$x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

If we increment time by a multiple of T_F:

$$x_F(t+qT_F) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F (t+qT_F)}$$

$$= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} e^{j2\pi k q f_F T_F}$$

Now $f_F T_F = 1$. Thus,

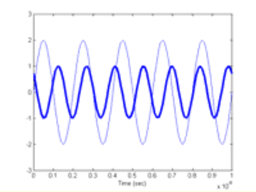
$$x_F(t+qT_F) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} \underbrace{e^{j2\pi k q}}_{=1}$$

$$= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t}$$

$$= x_F(t)$$

Q.E.D.

Calculating the CTFS



- Consider the function

$$x(t) = 2\cos(400\pi t)$$

over the interval $0 \le t \le 0.005$

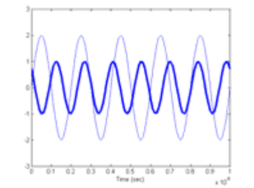
- Let's find the CTFS representation for this signal.
- Since it is an all real function, let's use the trigonometric form of the CTFS:

$$x_F(t) = X_C[0] + \sum_{k=1}^{\infty}\left\{X_C[k]\cos(2\pi k f_F t) + X_S[k]\sin(2\pi k f_F t)\right\}$$

where

$$X_C[k] = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\cos(2\pi k f_F t)\,dt \qquad X_S[k] = \frac{2}{T_F}\int_{t_o}^{t_o+T_F} x(t)\sin(2\pi k f_F t)\,dt$$

Calculating the coefficients



- Consider the first two coefficients

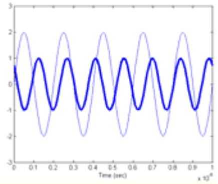
$$X_C[1]=\frac{2}{T_F}\int_0^{T_F} x(t)\cos(2\pi f_F t)\,dt \qquad X_S[1]=\frac{2}{T_F}\int_0^{T_F} x(t)\sin(2\pi f_F t)\,dt$$

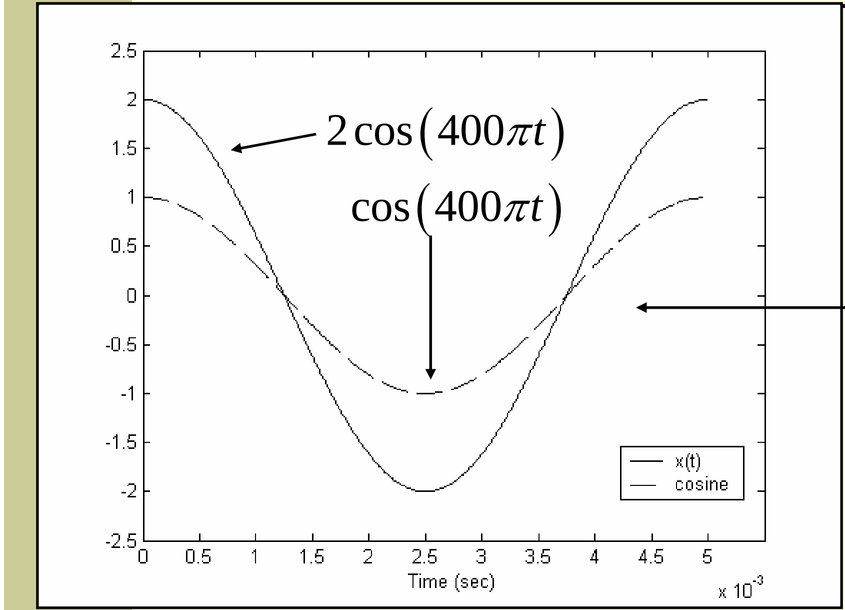
- Now, T_f = 0.005 and $f_F = 1/T_F$ = 200.
  - Note that we are investigating one period of the original signal
- Let's examine the following two integrals more carefully

$$X_C[1]=400\int_0^{0.005} 2\cos(400\pi t)\cos(400\pi t)\,dt$$

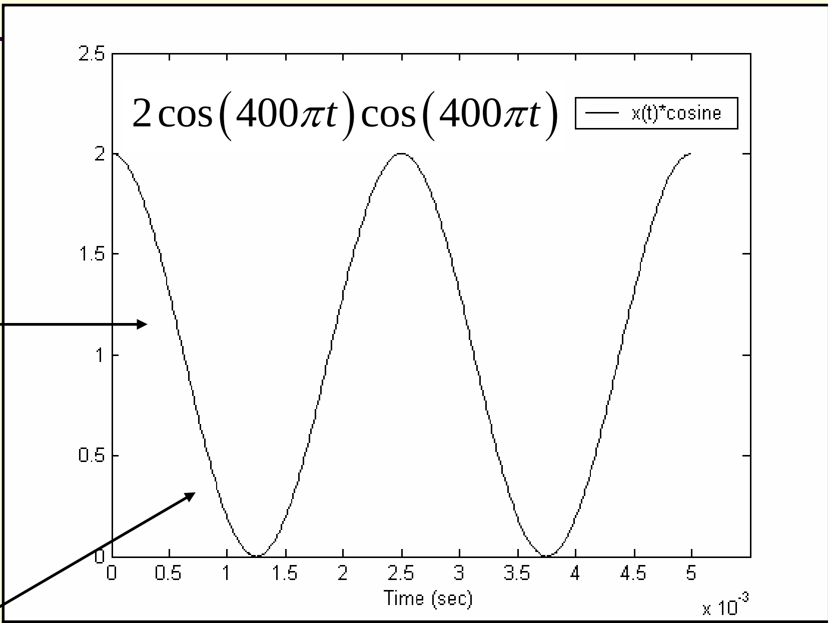
$$X_S[1]=400\int_0^{0.005} 2\cos(400\pi t)\sin(400\pi t)\,dt$$

Graphical Illustration of $X_C[1]$

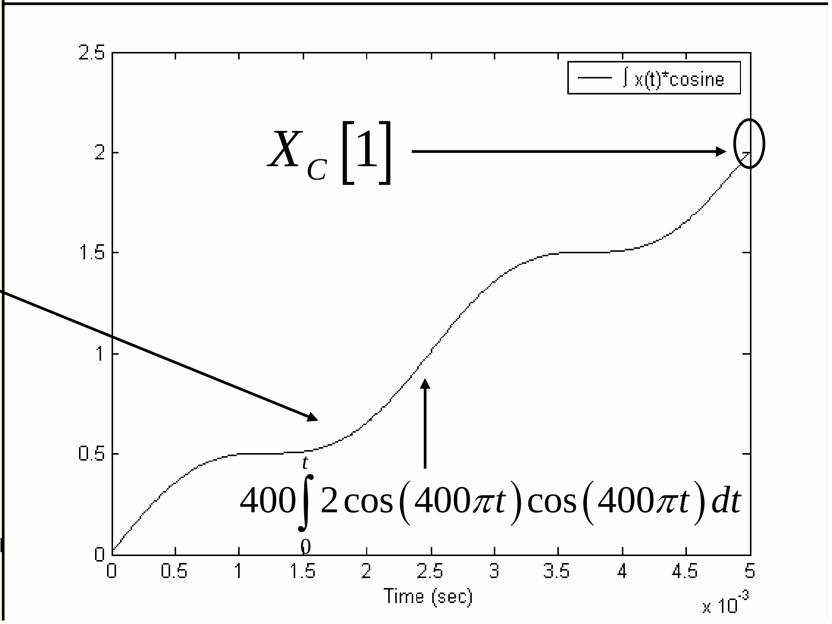




Since the functions rise and fall together, their product is positive

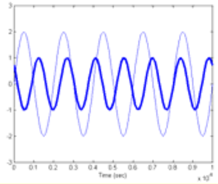


Area under this curve corresponds to the integral (both are shown versus time)

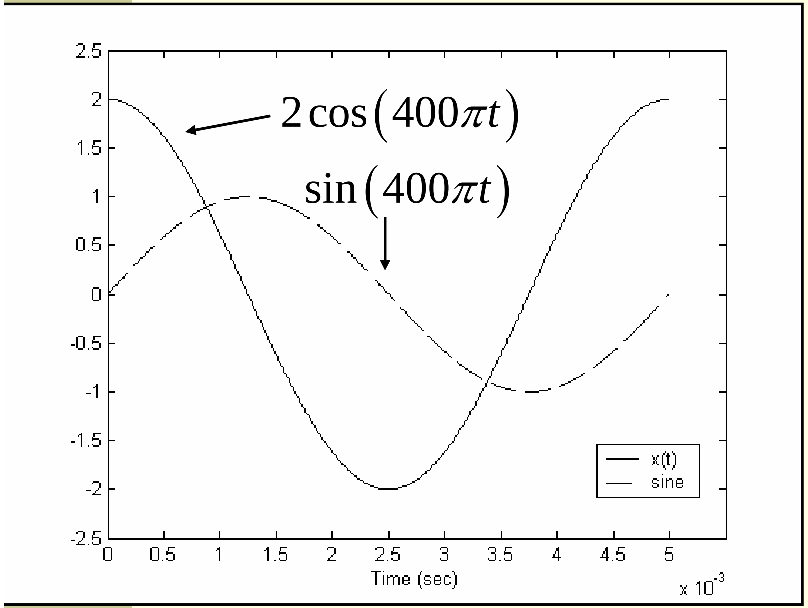


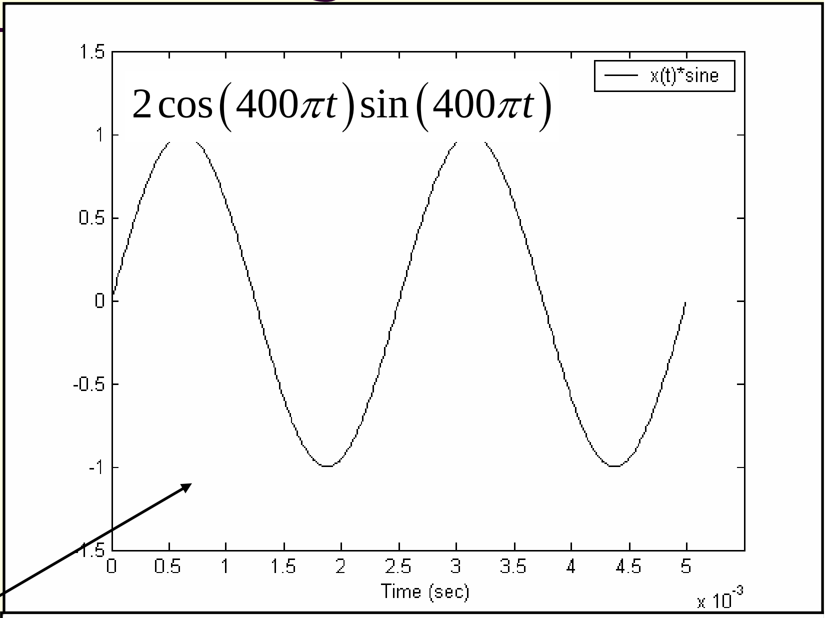
$$X_C[1] = 400\int_0^{0.005} 2\cos(400\pi t)\cos(400\pi t)\,dt$$

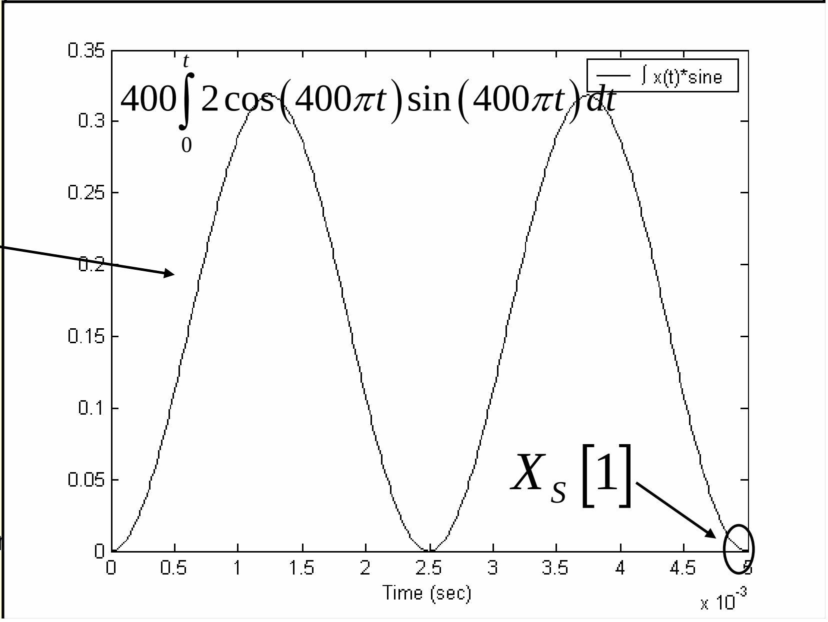
$$= 2$$



Graphical Illustration of $X_S[1]$



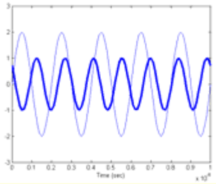




Area under this curve corresponds to the integral (both are shown versus time)

$$X_S[1] = 400 \int_0^{0.005} 2\cos(400\pi t)\sin(400\pi t)\,dt$$

$$= 0$$



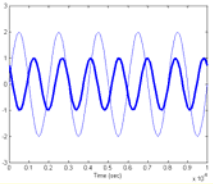
Definitions

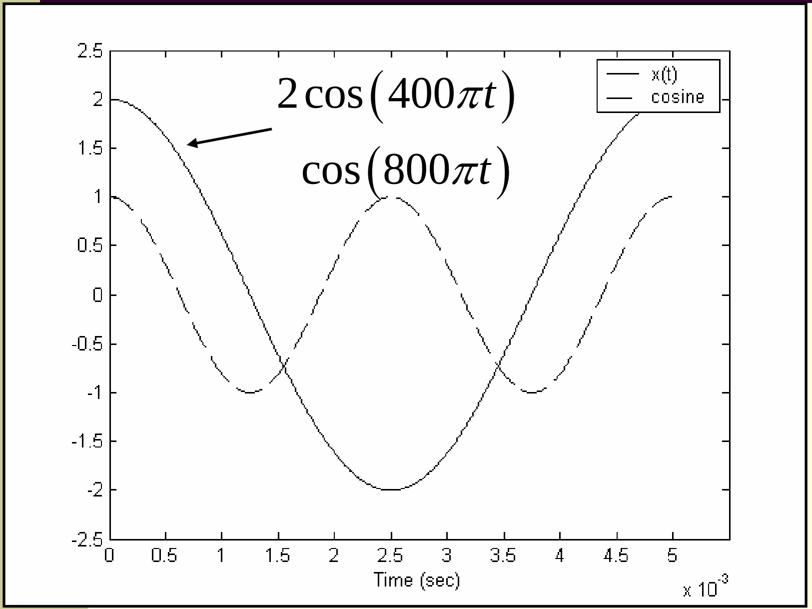
- We define the operation

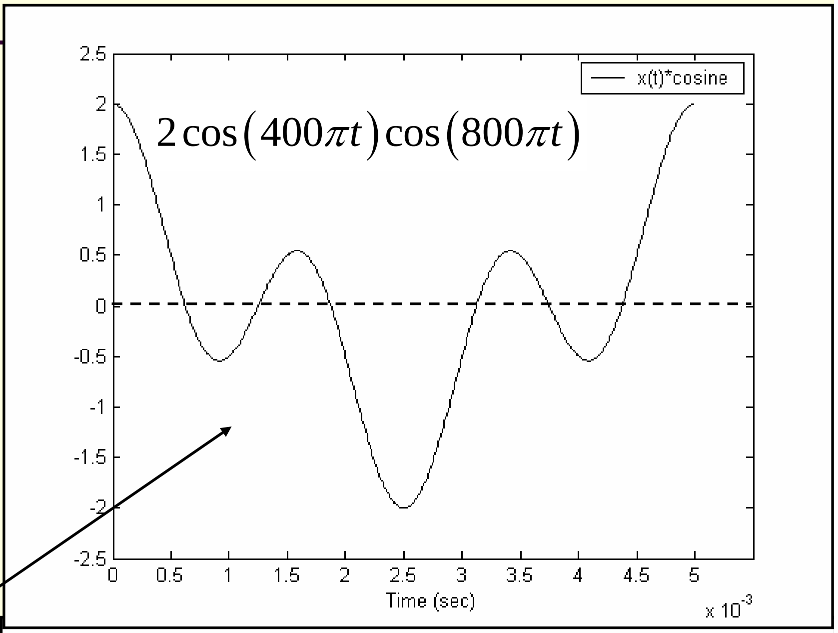
  $$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x(t)\,y(t)\,dt$$

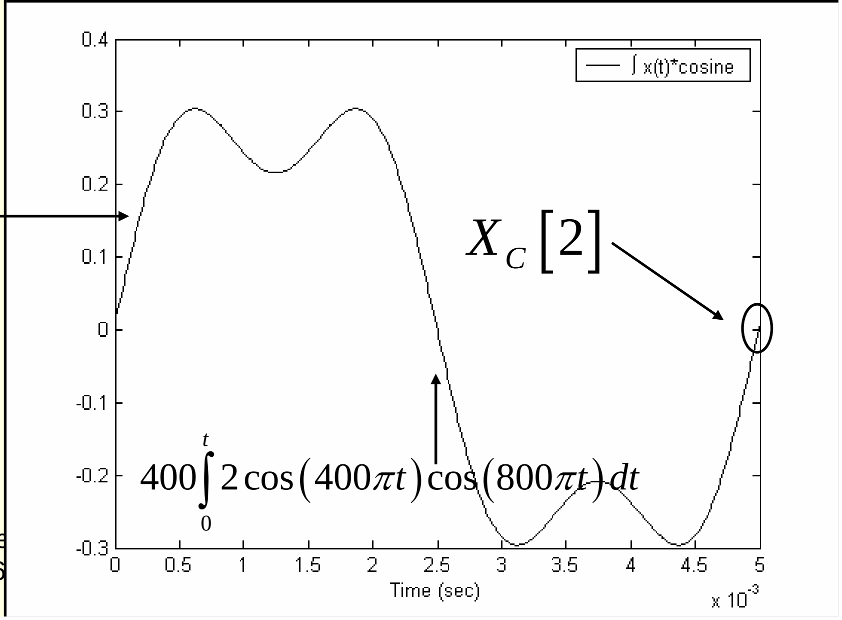
  the ***correlation*** between the functions *x(t)* and *y(t)* over the interval $t_1 \le t \le t_2$
- If the correlation between two functions is zero, we say that over this interval the two functions are ***orthogonal***

Second Harmonic $X_C[2]$







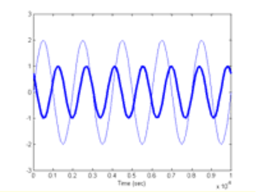


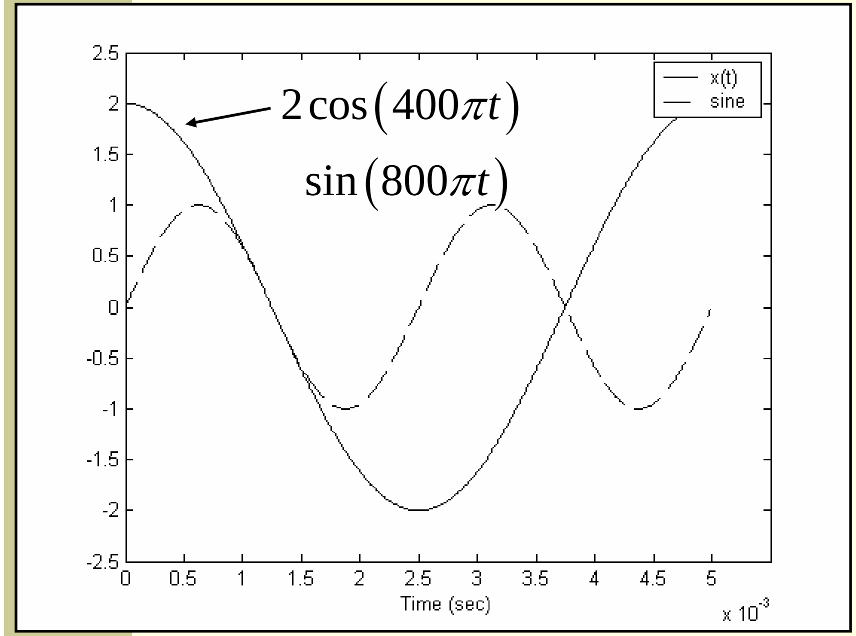
Area under this curve
corresponds to the integral
(both are shown versus time)

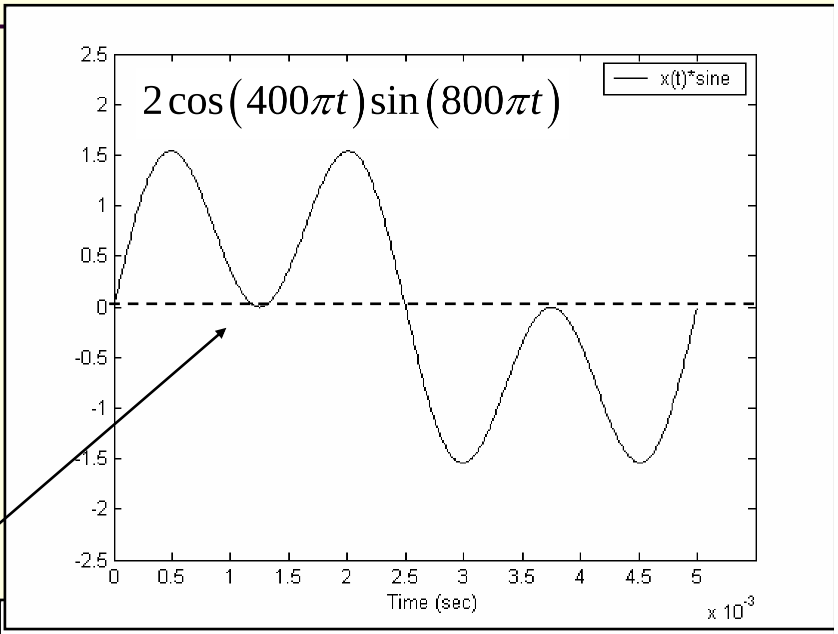
$$X_C[2] = 400\int_0^{0.005} 2\cos(400\pi t)\cos(800\pi t)\,dt$$

$$= 0$$

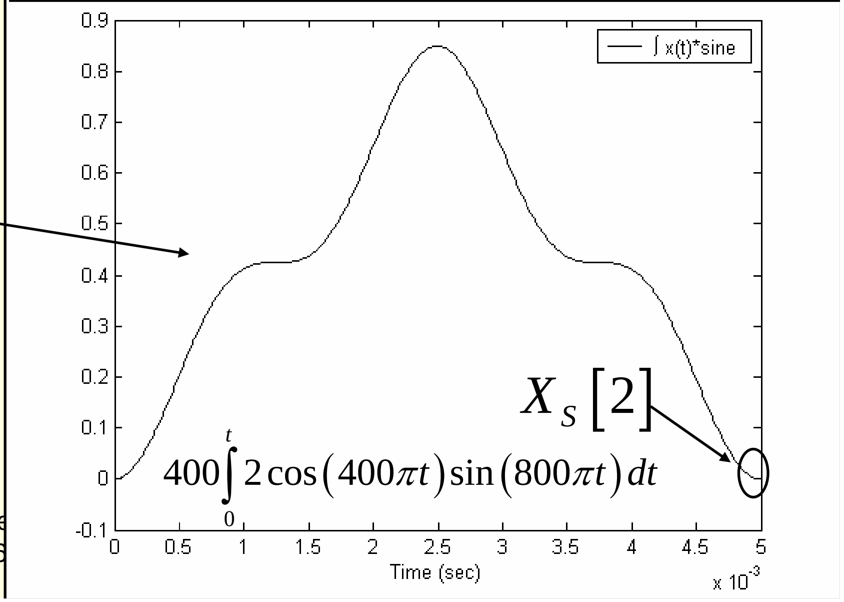
Second Harmonic $X_S[2]$







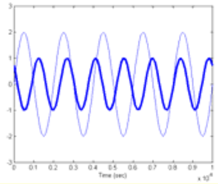
Area under this curve corresponds to the integral (both are shown versus time)



$$X_S[2] = 400\int_0^{0.005} 2\cos(400\pi t)\sin(800\pi t)\,dt$$

$$= 0$$

Further Harmonics

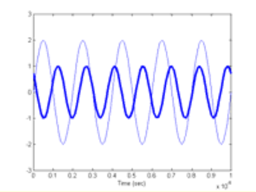


- The second harmonics are both zero
- We can also show that all of the higher harmonics (*k > 2*) are also zero:

$$
\begin{aligned}
X_C[k] &= 400\int_0^{T_F} 2\cos(2\pi f_F t)\cos(2\pi k f_F t)\,dt \\
&= 400\int_0^{T_F} \left\{\cos\left(2\pi(k+1)f_F t\right)+\cos\left(2\pi(k-1)f_F t\right)\right\}dt \\
&= \frac{400}{2\pi f_F}\left\{\frac{1}{k+1}\sin\left(2\pi(k+1)f_F t\right)+\frac{1}{k-1}\sin\left(2\pi(k-1)f_F t\right)\right\}\Bigg|_0^{T_F} \\
&= \frac{400}{2\pi f_F}\left\{\frac{1}{k+1}\sin\left(2\pi(k+1)f_F T_F\right)-0+\frac{1}{k-1}\sin\left(2\pi(k-1)f_F T_F\right)-0\right\} \\
&= \frac{400}{2\pi f_F}\left\{\frac{1}{k+1}\underbrace{\sin\left(2\pi(k+1)\right)}_{=0}+\frac{1}{k-1}\underbrace{\sin\left(2\pi(k-1)\right)}_{=0}\right\} \\
&= 0
\end{aligned}
$$

Notes: $f_F T_F=1$
$\sin(2\pi k)=0$ when k = integer

Further Harmonics

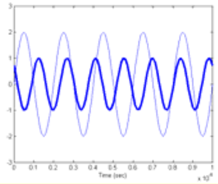


- For $X_S[k]$ $(k>2)$

$$
\begin{aligned}
X_S[k] &= 400\int_0^{T_F} 2\cos(2\pi f_F t)\sin(2\pi k f_F t)\,dt \\
&= 400\int_0^{T_F} \left\{\cos\left(2\pi(k-1) f_F t\right)-\cos\left(2\pi(k+1) f_F t\right)\right\} dt \\
&= \frac{400}{2\pi f_F}\left\{\frac{1}{k-1}\sin\left(2\pi(k-1) f_F t\right)-\frac{1}{k+1}\sin\left(2\pi(k+1) f_F t\right)\right\}\Bigg|_0^{T_F} \\
&= \frac{400}{2\pi f_F}\left\{\frac{1}{k-1}\sin\left(2\pi(k-1) f_F T_F\right)-0-\frac{1}{k+1}\sin\left(2\pi(k+1) f_F T_F\right)+0\right\} \\
&= \frac{400}{2\pi f_F}\left\{\frac{1}{k-1}\underbrace{\sin\left(2\pi(k-1)\right)}_{=0}-\frac{1}{k+1}\underbrace{\sin\left(2\pi(k+1)\right)}_{=0}\right\} \\
&= 0
\end{aligned}
$$

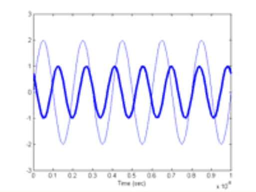
Notes: $f_F T_F=1$
$\sin(2\pi k)=0$ when k = integer

$X_C[0]$



- The remaining term of the Fourier Series is $X_C[0]$.
- We could calculate this term as before, but recall its meaning:
  - $X_C[0]$ is the average value of the function over the interval of interest.
  - Since we are examining one period of a sinusoidal function, the average value is zero.
  - Thus,

$$X_C[0] = 0$$



Final

- Thus, the signal $x(t)=2\cos(400\pi t)$ over the interval $0 \le t \le 0.005$ has only one non-zero Fourier Series term,

$$X_C[k]=\begin{cases}2 & k=1\\ 0 & k\neq 1\end{cases}$$

$$X_S[k]=0 \qquad \forall k$$

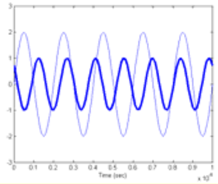
- This leads to

$$x_F(t)=X_C[0]+\sum_{k=1}^{\infty}\left\{X_C[k]\cos(2\pi k f_F t)+X_S[k]\sin(2\pi k f_F t)\right\}$$

$$=2\cos(2\pi f_F t)$$

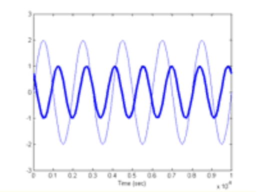
$$=2\cos(400\pi t)$$

as expected.



Interpretation

- The trigonometric form of the CTFS represents the signal by using only cosines and sines.
- The actual Fourier Series components $X_C[k]$ and $X_S[k]$ represent the ***correlation*** between the function of interest and the sinusoids at the **fundamental frequency** ($f_F=1/T_F$) and its **harmonics** (kf_F)
- The correlation tells us how much of that frequency term is contained in the signal

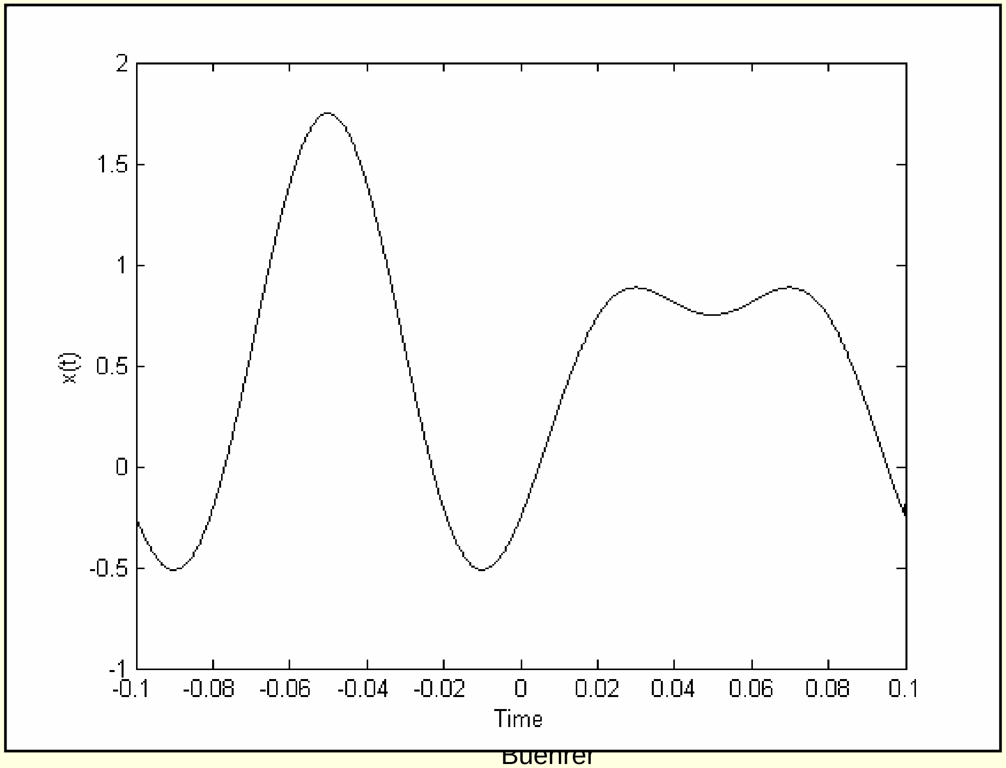


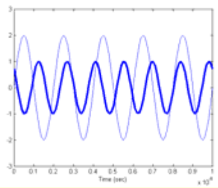
Example

- The previous example was trivial, so let’s look at a *slightly* more complicated signal

$$x(t) = \frac{1}{2} - \frac{3}{4}\cos(20\pi t) + \frac{1}{2}\sin(30\pi t)$$

over the interval $-0.1 \le t \le 0.1$





Example (cont.)

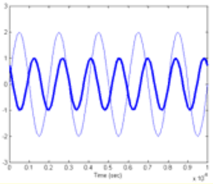
$$x(t)=\frac{1}{2}-\frac{3}{4}\cos(20\pi t)+\frac{1}{2}\sin(30\pi t)$$

- What would you guess the solution to be?

- First, T_F = 200ms, f_F = 5Hz
- Second, $X_C[0]$ = average value = ?

$$X_C[0]=\frac{1}{2}$$

$$X_C[k]=\frac{2}{T_F}\int_0^{T_F}x(t)\cos(2\pi k f_F t)\,dt \qquad X_S[k]=\frac{2}{T_F}\int_0^{T_F}x(t)\sin(2\pi k f_F t)\,dt$$



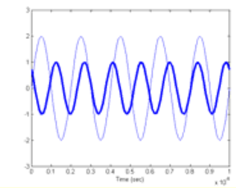
$X_C[k]$

$$\begin{aligned}
X_C[k] &= \frac{2}{T_F}\int_0^{T_F}\left\{\frac{1}{2}-\frac{3}{4}\cos(20\pi t)+\frac{1}{2}\sin(30\pi t)\right\}\cos(2\pi k f_F t)\,dt \\
&= \frac{2}{T_F}\int_0^{T_F}\left\{\frac{1}{2}-\frac{3}{4}\cos(2\pi(2f_F)t)+\frac{1}{2}\sin(2\pi(3f_F)t)\right\}\cos(2\pi k f_F t)\,dt \\
&= \frac{1}{2}\frac{2}{T_F}\int_0^{T_F}\cos(2\pi k f_F t)\,dt-\frac{3}{4}\frac{2}{T_F}\int_0^{T_F}\cos(2\pi(2f_F)t)\cos(2\pi k f_F t)\,dt+\ldots \\
&\quad \frac{1}{2}\frac{2}{T_F}\int_0^{T_F}\sin(2\pi(3f_F)t)\cos(2\pi k f_F t)\,dt
\end{aligned}$$

Looking at the first term:

$$\begin{aligned}
\frac{1}{2}\frac{2}{T_F}\int_0^{T_F}\cos(2\pi k f_F t)\,dt &= \frac{1}{2\pi k f_F T_F}\sin(2\pi k f_F t)\Big|_0^{T_F} \\
&= \frac{1}{2\pi k f_F T_F}\left[\sin(2\pi k f_F T_f)-0\right] \\
&= \frac{1}{2\pi k f_F T_F}\sin(2\pi k) \\
&= 0
\end{aligned}$$

$k>0$

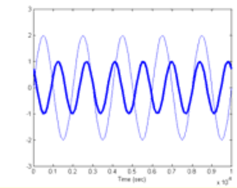


$X_C[k]$ (cont.)

- Looking at the 2nd term:

$$\begin{aligned}
X_C[k] &= -\frac{3}{4}\frac{2}{T_F}\int_0^{T_F}\cos(20\pi t)\cos(2\pi k f_F t)\,dt \\
&= -\frac{3}{4}\frac{1}{T_F}\int_0^{T_F}\left\{\cos([4-2k]\pi f_F t)+\cos([4+2k]\pi f_F t)\right\}dt \\
&= -\frac{3}{4}\frac{1}{T_F}\left\{\frac{1}{[4-2k]\pi f_F}\sin([4-2k]\pi f_F t)+\frac{1}{[4+2k]\pi f_F}\sin([4+2k]\pi f_F t)\right\}\Bigg|_o^{T_F} \\
&= -\frac{3}{4}\frac{1}{T_F}\left\{\frac{1}{[4-2k]\pi f_F}\sin\left([4-2k]\pi\underbrace{f_F T_F}_{=1}\right)-0+\frac{1}{[4+2k]\pi f_F}\sin\left([4+2k]\pi\underbrace{f_F T_F}_{=1}\right)-0\right\} \\
&= \begin{cases} 0 & k \neq 2 \\ \dfrac{0}{0} & k = 2 \end{cases}
\end{aligned}$$

$X_C[k]$ (cont.)

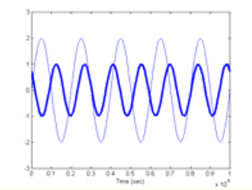


- Using L'Hopital's Rule

$$-\frac{3}{4}\frac{1}{T_F}\left\{\frac{\frac{d}{dk}\left[\sin\left(\left[4-2k\right]\pi\right)\right]}{\frac{d}{dk}\left[4-2k\right]\pi f_F}\right\} = -\frac{3}{4}\frac{1}{T_F}\frac{-2\pi\cos\left(\left[4-2k\right]\pi\right)}{-2\pi f_F}$$

$$= -\frac{3}{4}$$

$$-\frac{3}{4}\frac{2}{T_F}\int_0^{T_F}\cos\left(20\pi t\right)\cos\left(k2\pi f_F \pi t\right)dt = \begin{cases}-\frac{3}{4} & k=2\\ 0 & k\neq 2\end{cases}$$



$X_C[k]$ (cont.)

- Examining the third term

$$\frac{1}{2}\frac{2}{T_F}\int_0^{T_F}\sin(2\pi(3f_F)t)\cos(2\pi k f_F t)\,dt = \frac{1}{2T_F}\int_0^{T_F}\{\sin((6-2k)\pi f_F t)+\sin((6+2k)\pi f_F t)\}\,dt$$

$$= \frac{1}{2T_F}\left\{-\frac{1}{(6-2k)\pi f_F}\cos((6-2k)\pi f_F t) - \dots\right.$$

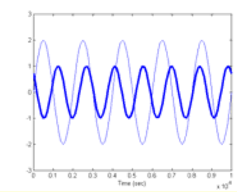
$$\left.\frac{1}{(6+2k)\pi f_F}\cos((6+2k)\pi f_F t)\right\}\Bigg|_0^{T_F}$$

$$= -\frac{1}{2T_F}\frac{1}{(6-2k)\pi f_F}\left[\underbrace{\cos\left(\underbrace{(6-2k)\pi\underbrace{f_F T_f}_{1}}_{q*2\pi}\right)}_{1}-1\right]+\dots$$

$$-\frac{1}{2T_F}\frac{1}{(6+2k)\pi f_F}\left[\underbrace{\cos\left(\underbrace{(6+2k)\pi\underbrace{f_F T_F}_{1}}_{n*2\pi}\right)}_{1}-1\right]$$

$$=0$$

Note: We must use L'Hopital's rule to evaluate the case where k=3. It still evaluates to zero.



$X_C[k]$ (cont.) and $X_S[k]$

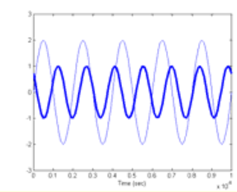
- Putting together all three terms

$$X_C[k]=\begin{cases}0 & k\neq 2\\ -\dfrac{3}{4} & k=2\end{cases}$$

- Turning to $X_S[k]$

$$X_S[k]=\frac{2}{T_F}\int_0^{T_F} x(t)\sin(2\pi k f_F t)\,dt$$

$$=\frac{2}{T_F}\int_0^{T_F}\left\{\frac{1}{2}-\frac{3}{4}\cos(20\pi t)+\frac{1}{2}\sin(30\pi t)\right\}\sin(2\pi k f_F t)\,dt$$



$X_S[k]$

$$X_S[k] = \frac{1}{T_F}\int_0^{T_F} \sin(2\pi k f_F t)\,dt - \frac{3}{2T_F}\int_0^{T_F} \frac{1}{2}\cos(20\pi t)\sin(2\pi k f_F t)\,dt + \ldots$$

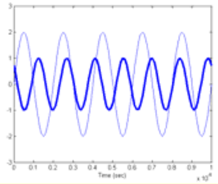
$$\frac{1}{T_F}\int_0^{T_F} \sin(30\pi t)\sin(2\pi k f_F t)\,dt$$

$$= 0 + 0 + \frac{1}{T_F}\int_0^{T_F} \sin(6\pi f_F t)\sin(2\pi k f_F t)\,dt$$

$$= \frac{1}{2T_F}\int_0^{T_F} \left\{\cos\left([6-2k]\pi f_F t\right) - \cos\left([6+2k]\pi f_F t\right)\right\} dt$$

$$\vdots$$

$$\vdots$$



$X_S[k]$ (cont.)

$$X_S[k] = \frac{1}{2[6-2k]\pi f_F T_F}\sin\left([6-2k]\pi f_F t\right) - \ldots$$

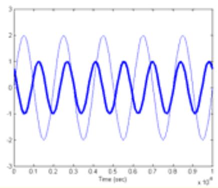
$$\left.\frac{1}{2[6+2k]\pi f_F T_F}\sin\left([6+2k]\pi f_F t\right)\right|_0^{T_F}$$

$$= \frac{1}{2[6-2k]\pi f_F T_F}\left[\sin\left([6-2k]\pi f_F T_F\right) - 0\right]$$

$$- \frac{1}{2[6+2k]\pi f_F T_F}\left[\sin\left([6+2k]\pi f_F t\right) - 0\right]$$

$$= \begin{cases} 0 & k \neq 3 \\ \dfrac{0}{0} & k = 3 \end{cases}$$

$X_S[k]$ (cont.)



- Using L'Hopital's Rule

$$\frac{\frac{d}{dk}\left[\sin\left(\left[6-2k\right]\pi\right)\right]}{\frac{d}{dk}\left\{2\left[6-2k\right]\pi\right\}}=\frac{-2\pi\left[\cos\left(\left[6-2k\right]\pi\right)\right]}{-4\pi}$$

$$=\frac{1}{2}$$

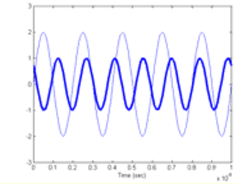
$$X_S\left[k\right]=\begin{cases}\frac{1}{2} & k=3\\ 0 & k\neq 3\end{cases}$$

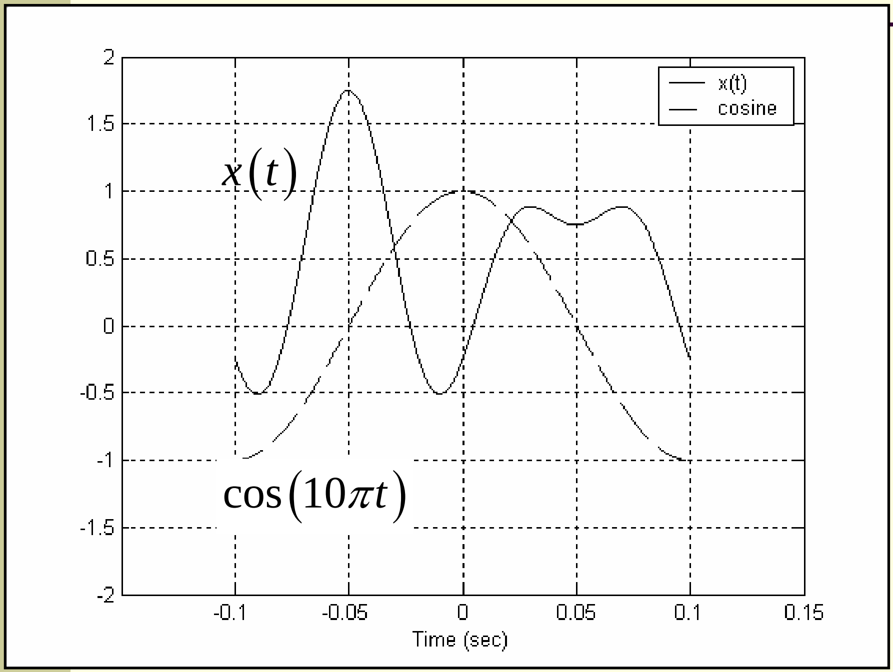
- Putting together all the pieces

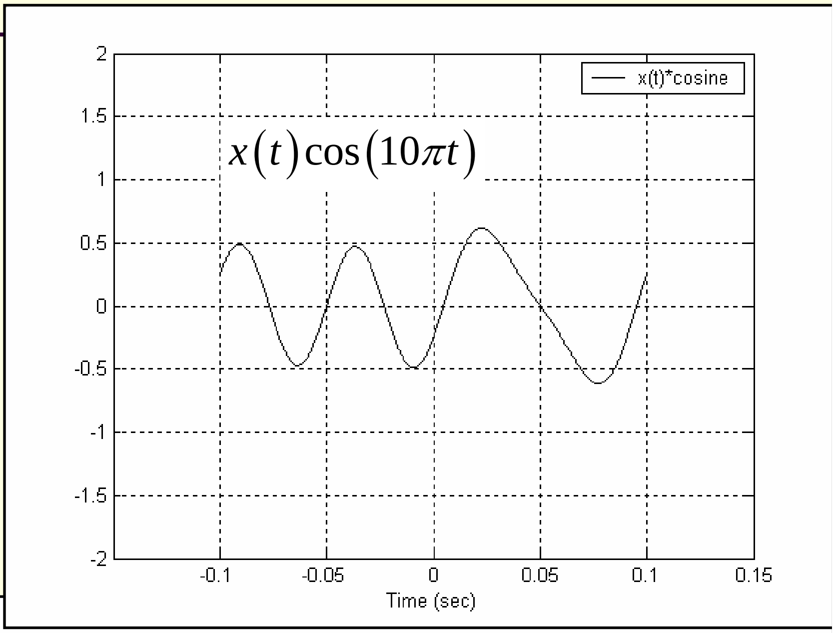
$$x_F\left(t\right)=X_C\left[0\right]+\sum_{k=1}^{\infty}\left\{X_C\left[k\right]\cos\left(2\pi k f_F t\right)+X_S\left[k\right]\sin\left(2\pi k f_F t\right)\right\}$$

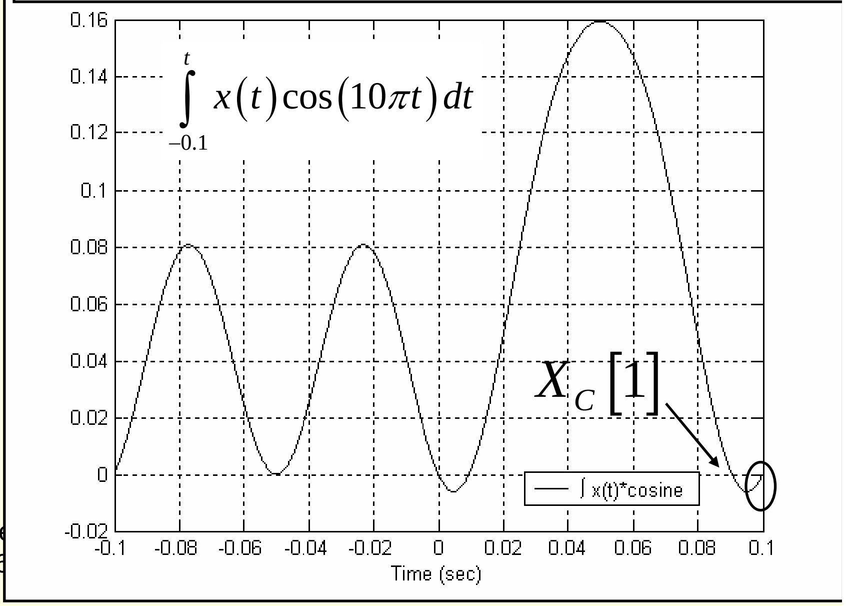
$$=\frac{1}{2}-\frac{3}{4}\cos\left(4\pi f_F t\right)+\frac{1}{2}\sin\left(6\pi f_F t\right)$$

Graphical Illustration of $X_C[1]$





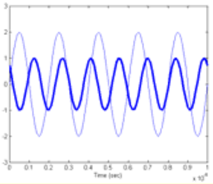


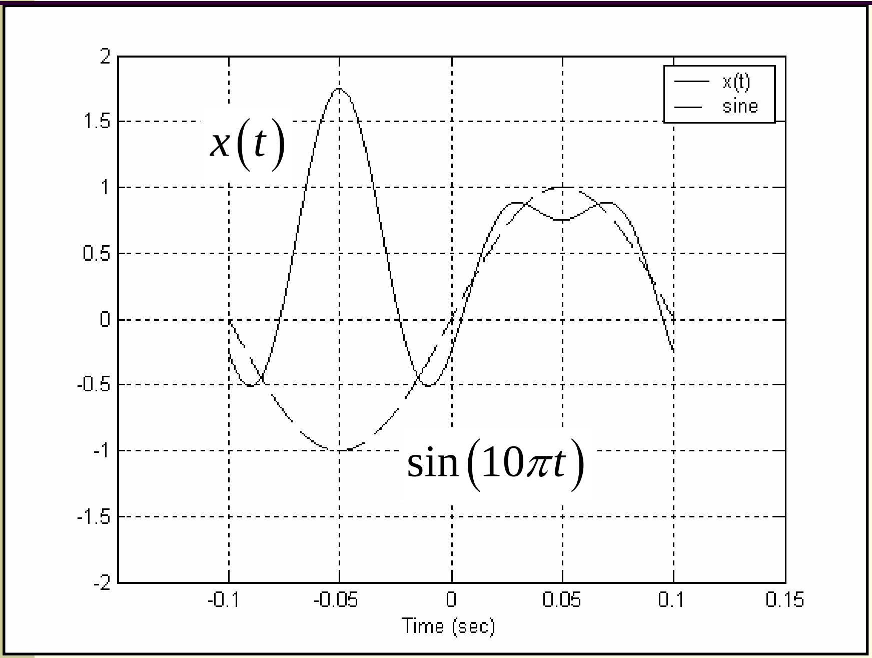


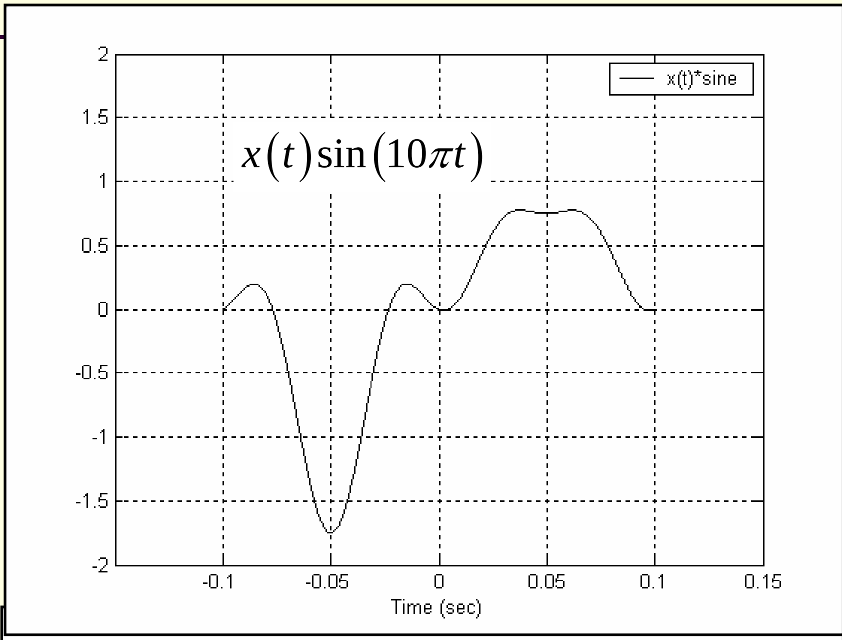
$$X_C[1] = \frac{2}{T_F}\int_{-0.1}^{0.1} x(t)\cos(10\pi t)\,dt$$

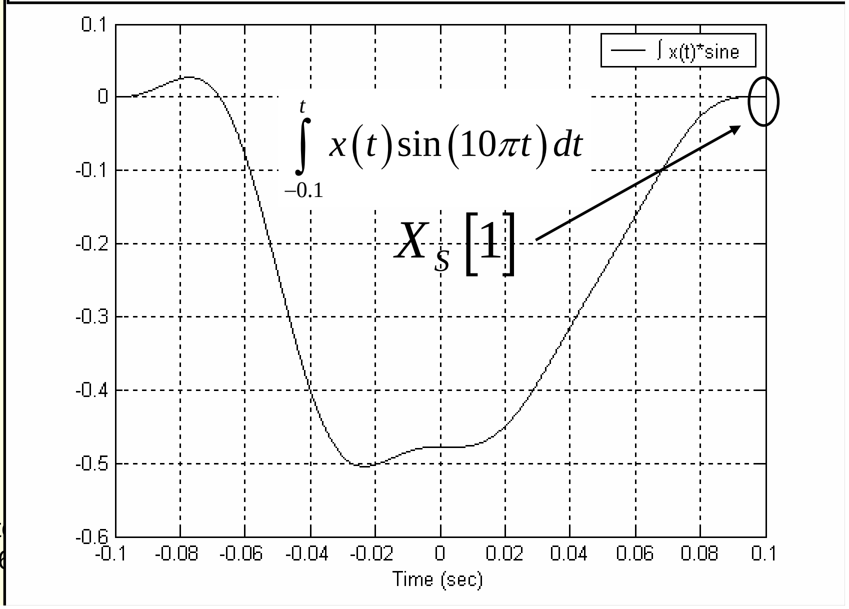
$$= 0$$

Graphical Illustration of $X_S[1]$





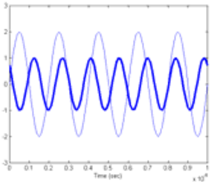


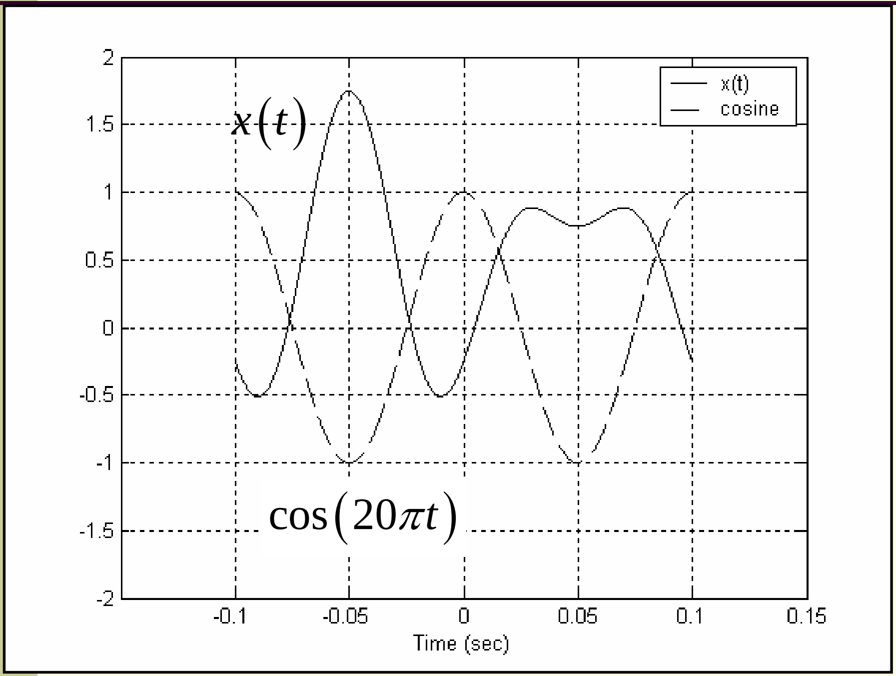


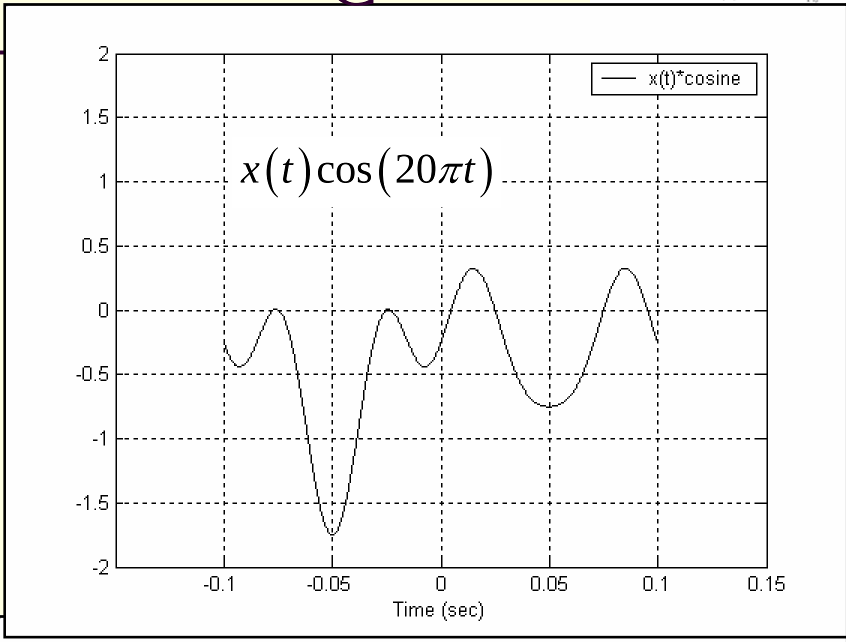
$$X_S[1] = \frac{2}{T_F}\int_{-0.1}^{0.1} x(t)\sin(10\pi t)\,dt$$

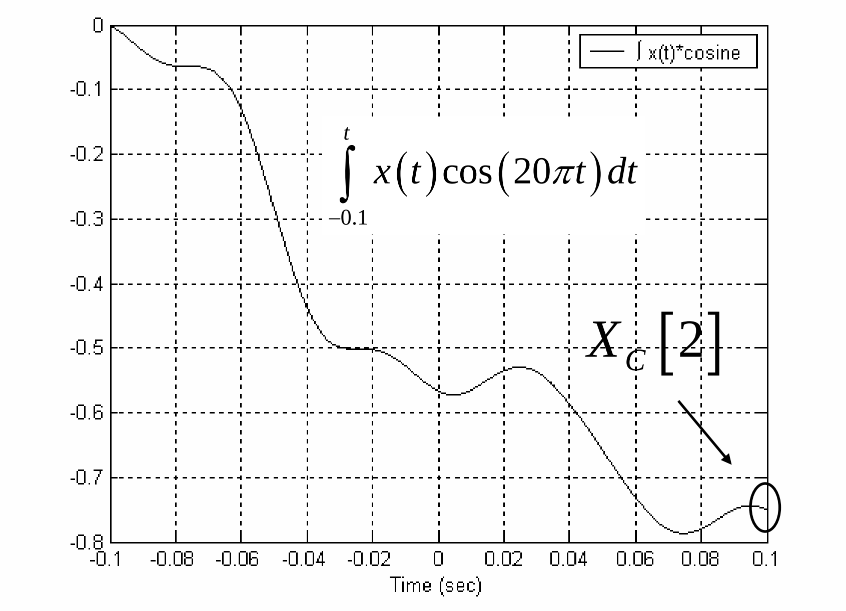
$$= 0$$

Graphical Illustration of $X_C[2]$



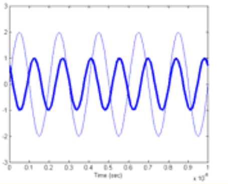




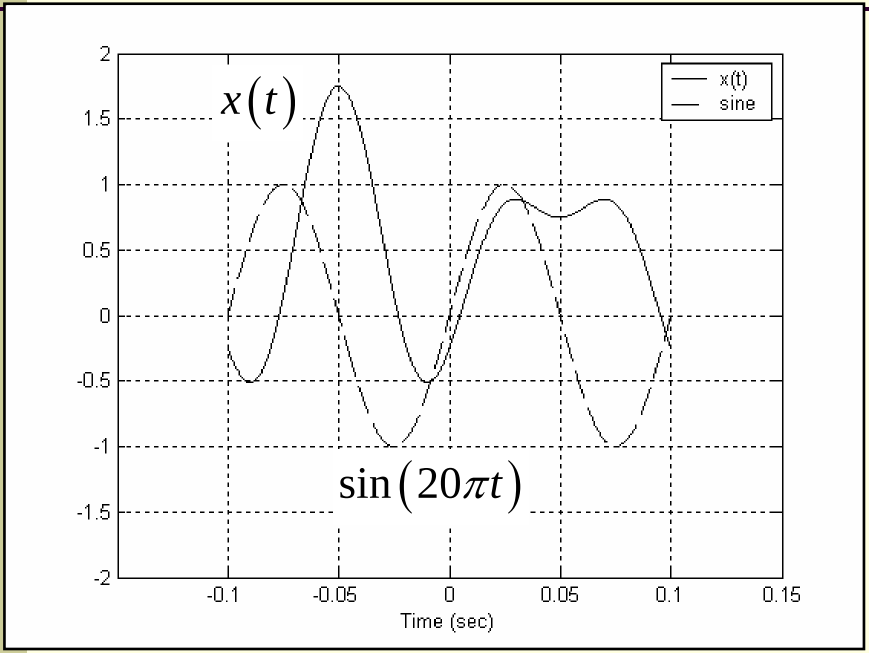


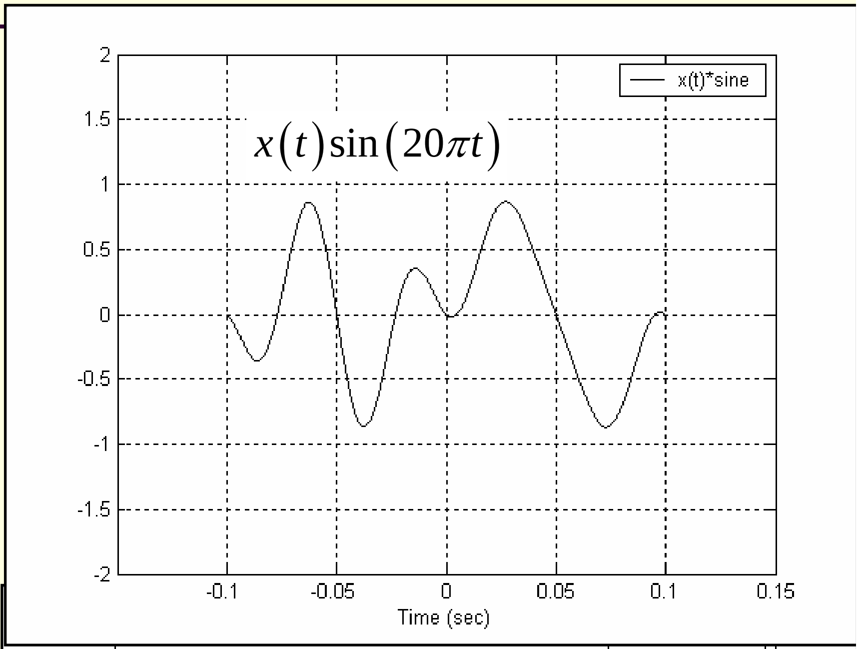
$$X_C[2]=\frac{2}{T_F}\int_{-0.1}^{0.1}x(t)\cos(20\pi t)\,dt$$

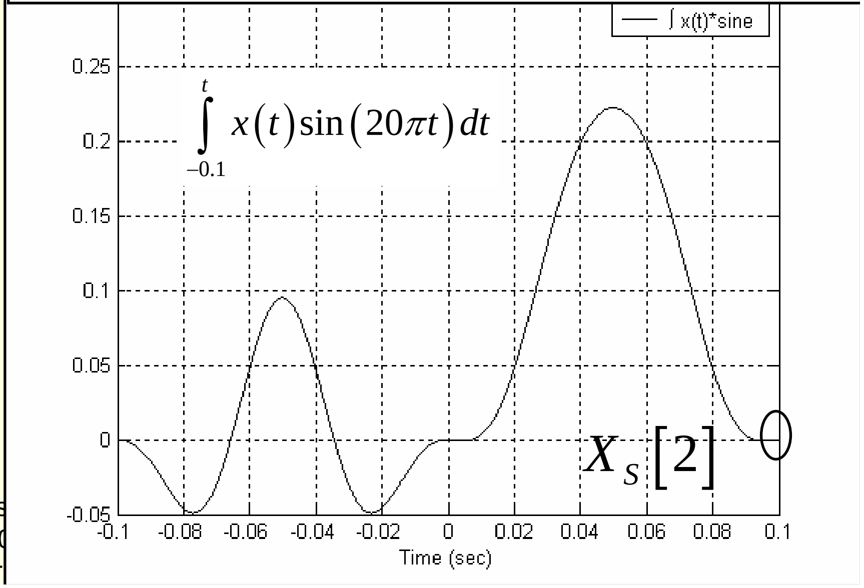
$$=-0.75$$



Graphical Illustration of $X_S[2]$



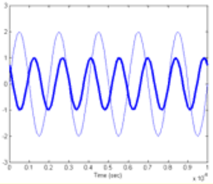


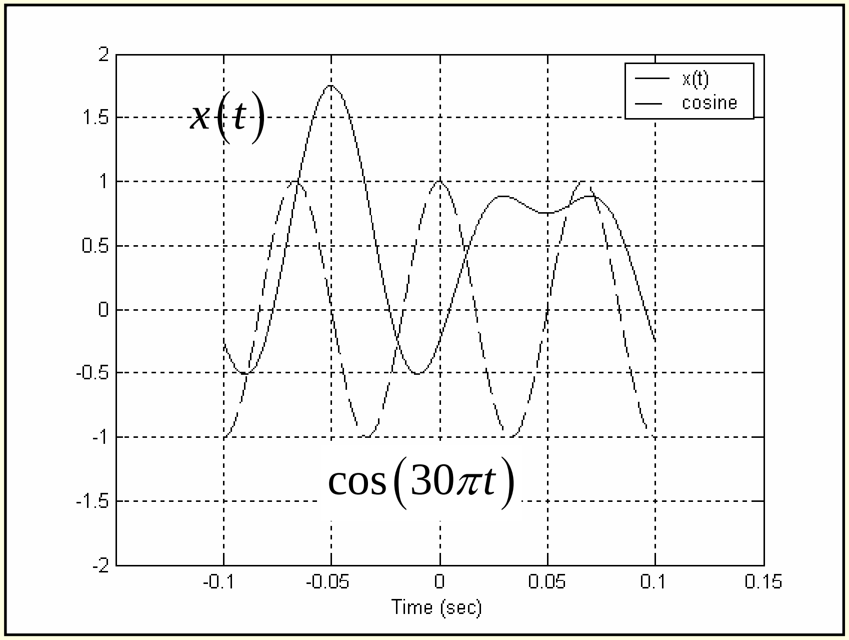


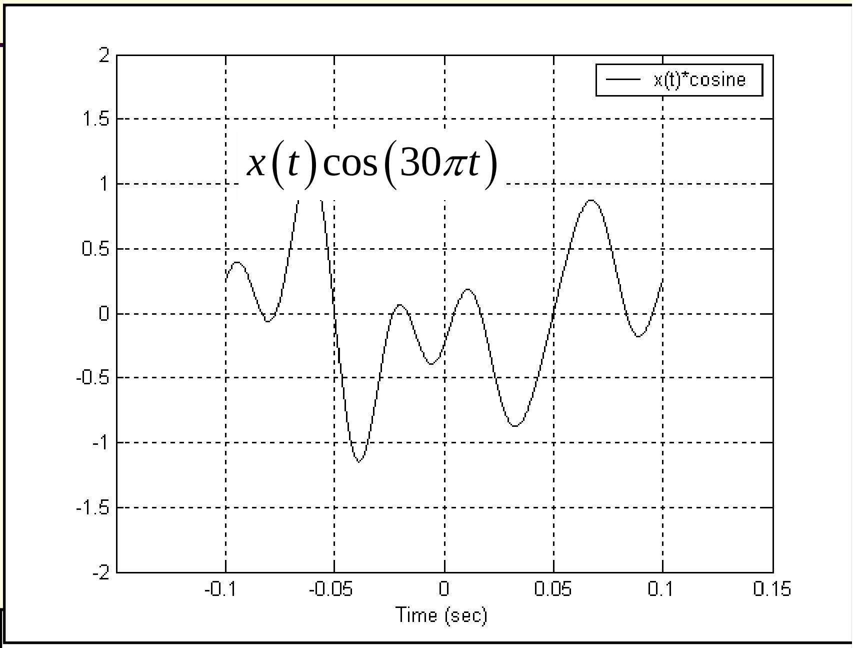
$$X_S[2] = \frac{2}{T_F}\int_{-0.1}^{0.1} x(t)\sin(20\pi t)\,dt$$

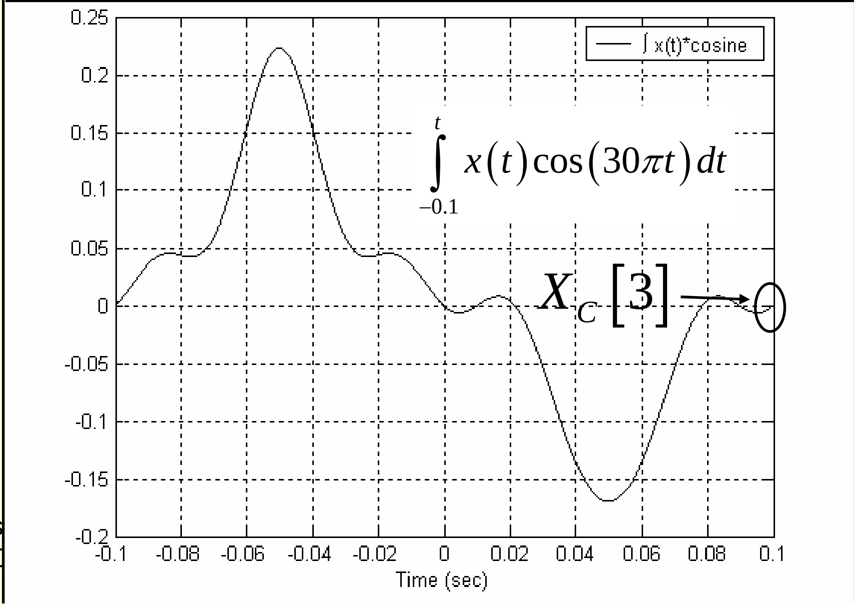
$$= 0$$

Graphical Illustration of $X_C[3]$





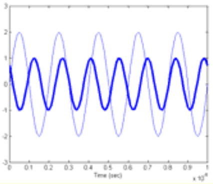


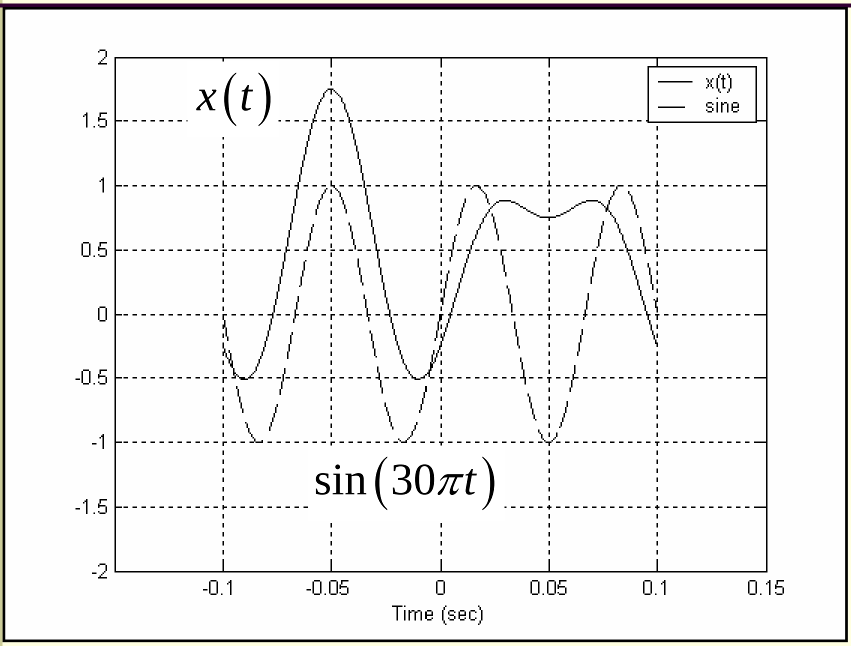


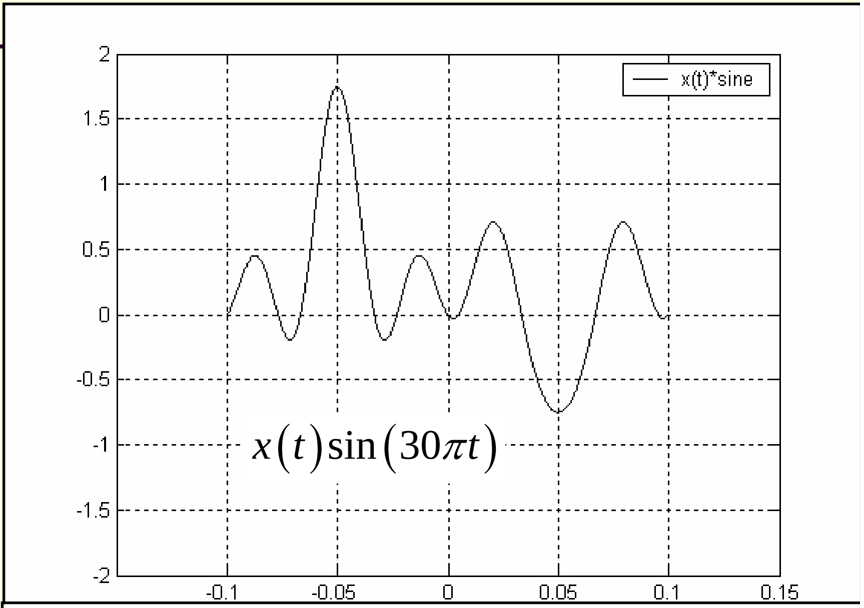
$$X_C[3] = \frac{2}{T_F} \int_{-0.1}^{0.1} x(t)\cos(30\pi t)\,dt$$

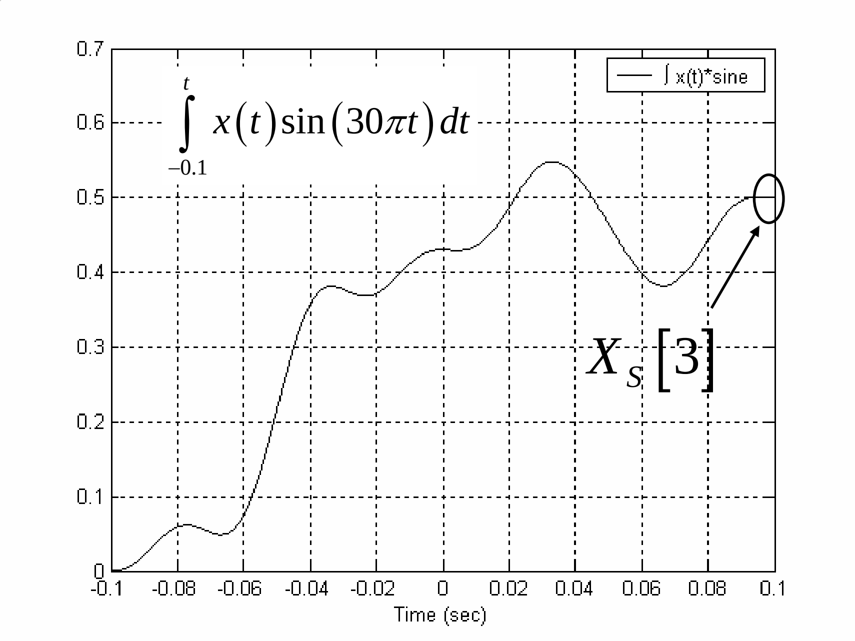
$$= 0$$

Graphical Illustration of $X_S[3]$





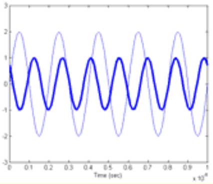


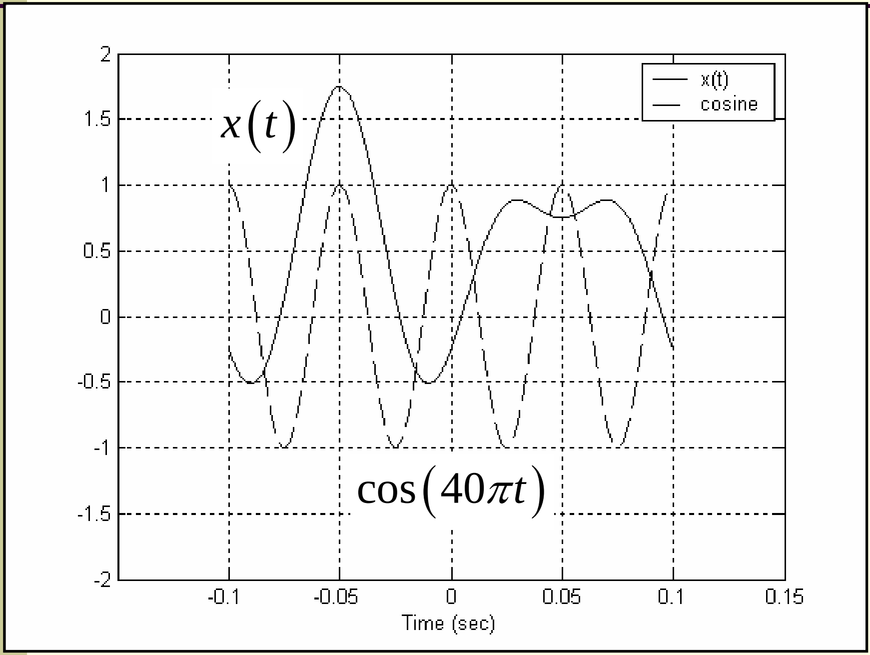


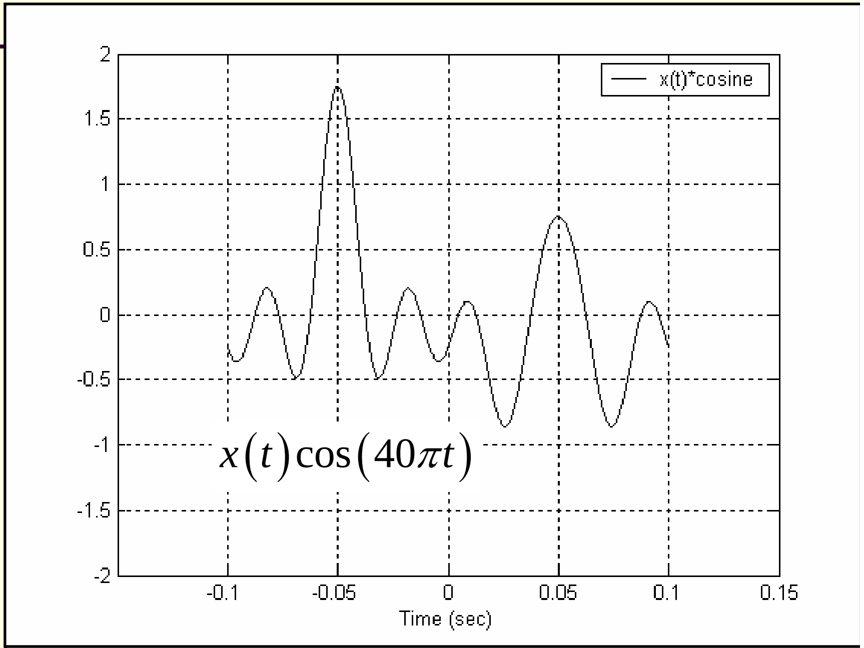
$$X_S[3] = \frac{2}{T_F}\int_{-0.1}^{0.1} x(t)\sin(30\pi t)\,dt$$

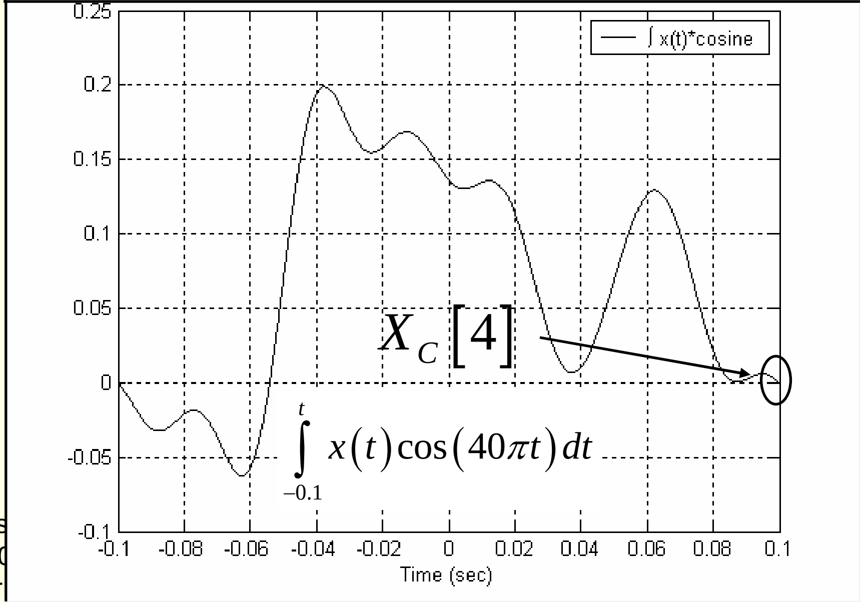
$$= 0.5$$

Graphical Illustration of $X_C[4]$





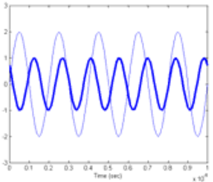


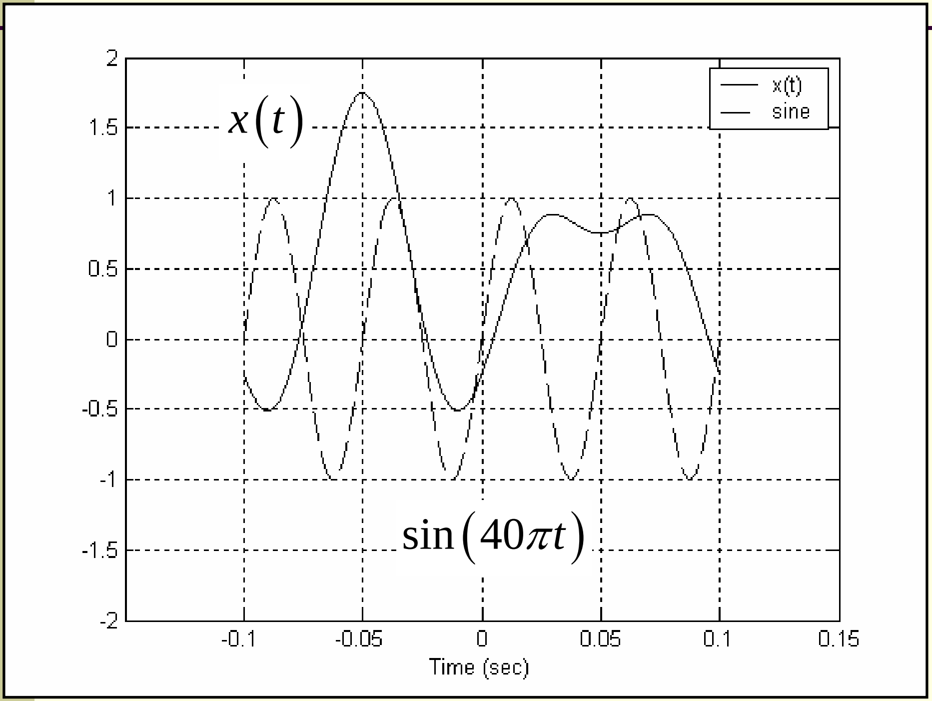


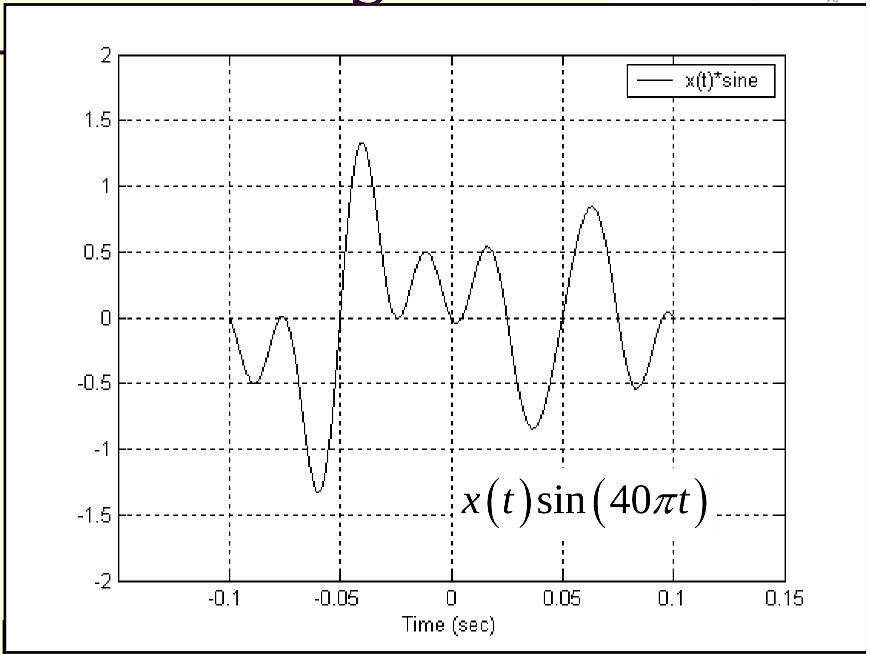
$$X_C[4] = \frac{2}{T_F}\int_{-0.1}^{0.1} x(t)\cos(40\pi t)\,dt$$

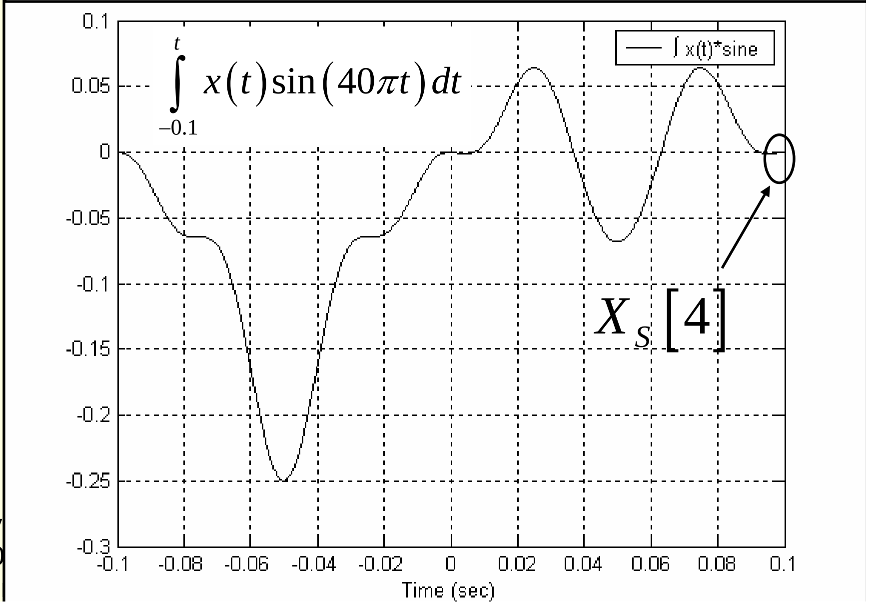
$$= 0$$

Graphical Illustration of $X_S[4]$

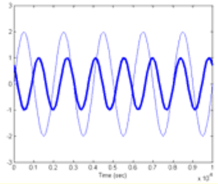








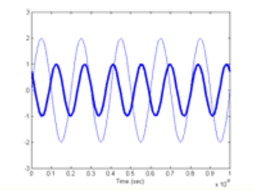
$$X_S[4] = \frac{2}{T_F}\int_{-0.1}^{0.1} x(t)\sin(40\pi t)\,dt$$
$$= 0$$



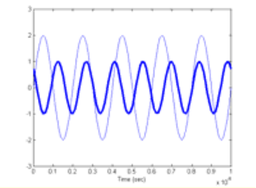
Summary

- The Fourier Series is a sum of sinusoids (cosines and sines in the case of the trigonometric version)
- The weights of these sinusoids depend on the ***correlation*** between the signal of interest and each sinusoid in the series.
- When the signal of interest is a sum of sinusoids
  - The Fourier series terms which correspond to the frequencies in the signal will have weights corresponding to the amplitudes of the sinusoids
    - In other words there is perfect correlation between sines and cosines of the same frequency
    - sine and cosine of the same frequency are ***orthogonal***
  - Provided that the interval of interest is an integer number of periods for all sinusoids in the sum, all other Fourier Series components will be zero.
    - This is because sinusoids whose frequencies are integer multiples or factors of each other are ***orthogonal*** when we integrate over an integer number of periods
    - This will not be true if we choose an arbitrary interval.

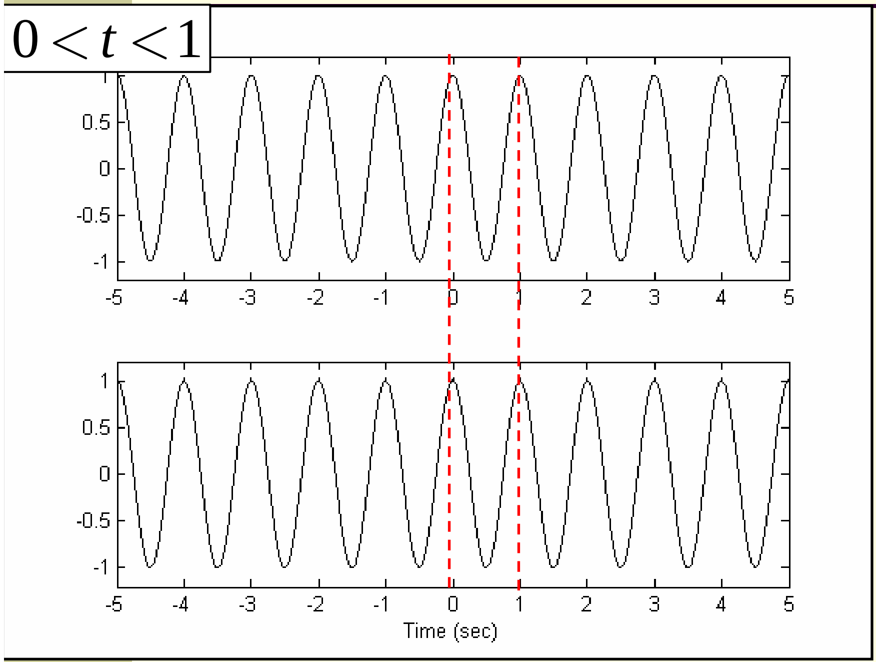
Periodic Signals

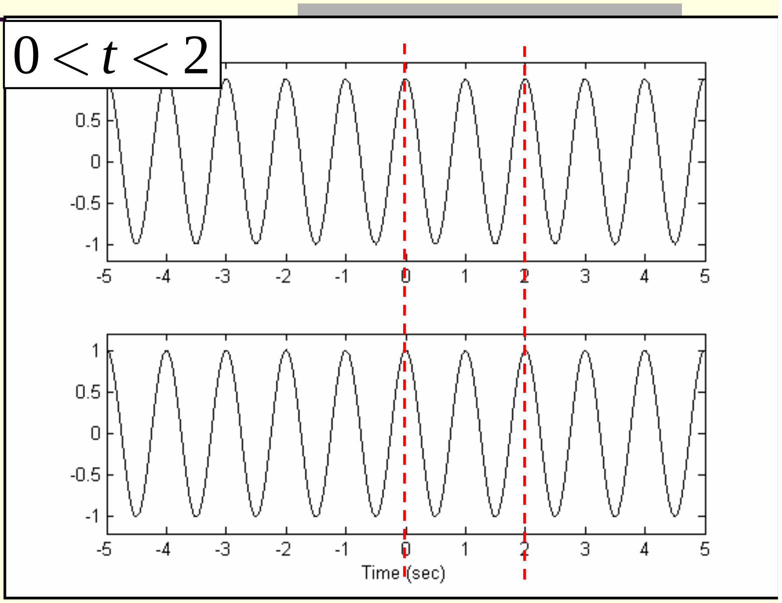


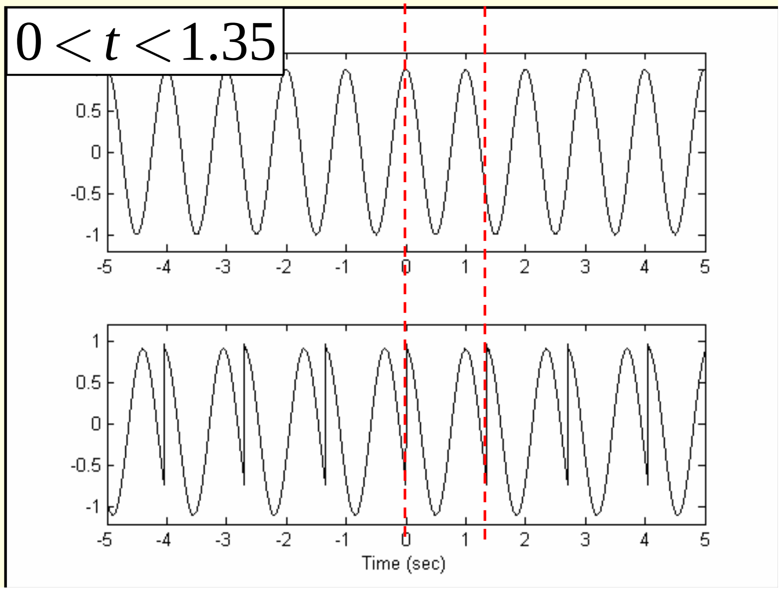
- The continuous time Fourier series is defined such that it is equivalent to the desired signal only over the interval of interest
- The continuous time Fourier series is periodic with period T_F.
- For a periodic signal the Fourier series representation is identical to the original signal over all time *if we choose the interval of interest to be one period (or an integer number of periods) of the original signal*



Impact of Chosen Interval


$0 < t < 1$


$0 < t < 2$


$0 < t < 1.35$

- The CTFS representation always equals the signal over the region of interest
- Since both signals are periodic, if the period of the two signals is the same, they will be equal for *all time.*
- The periods will be the same when the interval of interest is equal to one period or an integer number of periods of the original signal