3614: Introduction to Communication Systems

Midterm Exam I
September 21, 2006

SOLUTION

I pledge that I have neither given nor received any assistance on this exam.

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(signed)

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Name (print) Student Number

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1. (20 points) Multiple Choice – Choose the answer which best completes the sentence

1.1 [5 points] The Fourier Transform of a periodic time-domain signal is

(a) periodic in the frequency domain
(b) discrete in the frequency domain
(c) continuous in the frequency domain
(d) None of the above

1.2 [5 points] The Fourier Transform of a signal which is of finite duration in the time domain is

(a) periodic in the frequency domain
(b) of infinite duration in the frequency domain
(c) discrete in the frequency domain
(d) None of the above

1.3 [5 points] $\delta(t-10)\sin\left(\frac{\pi}{20}t\right)=$

(a) $\delta(t-10)$
(b) 1
(c) $\delta(t)\sin\left(\frac{\pi}{20}t\right)$
(d) $\sin\left(\frac{\pi}{20}t\right)$
(e) None of the above

1.4 [5 points] Which of the following represents the relationship between the autocorrelation functions of the input, *x(t)*, and output, *y(t)*, of a linear time-invariant system?

(a) $R_y(\tau)=h^2(\tau)\otimes R_x(\tau)$
(b) $R_y(\tau)=h(\tau)\otimes h(-\tau)\otimes R_x(\tau)$
(c) $R_y(\tau)=h(\tau)h(-\tau)R_x(\tau)$
(d) $R_y(\tau)=h(\tau)h(\tau)R_x(\tau)$

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2. (25 points) Fourier Transform

(a) [10 points] Determine the Fourier Transform of the signal

$$x(t) = rect\left(\frac{t-1}{10}\right)$$

First, we can write $x(t) = y(t-1)$ *where* $y(t) = rect\left(\frac{t}{10}\right)$. *From the time-delay property, we know that* $X(f) = Y(f)e^{j2\pi f}$. *From our FT table, we know that the Fourier Transform of y(t) is*

$$Y(f) = 10\text{sinc}(10f)$$

Thus, we have

$$X(f) = 10\text{sinc}(10f)e^{-j2\pi f}$$

(b) [10 points] Determine the Fourier Transform of

$$x(t) = rect\left(\frac{t}{10}\right)\cos(2000\pi t)$$

First, let $x(t) = y(t)\cos(2000\pi t)$. *Secondly, from the modulation property we have*

$$X(f) = \frac{1}{2}Y(f-1000) + \frac{1}{2}Y(f+1000)$$

From the FT of the rectangular pulse we have

$$\begin{aligned} X(f) &= \frac{10}{2}\text{sinc}(10(f-1000)) + \frac{10}{2}\text{sinc}(10(f+1000)) \\ &= 5\text{sinc}(10(f-1000)) + 5\text{sinc}(10(f+1000)) \end{aligned}$$

(c) [5 points] Determine the Fourier Transform of

$$x(t) = rect(t) \otimes \sum_{k=-\infty}^{\infty} \delta(t - 10k)$$

where $\otimes$ is the convolution operation.

From the properties of the impulse we have:

$$x(t) = \sum_{k=-\infty}^{\infty} rect(t - 10k)$$

Now, this is a periodic signal with period $T_o = 10$. Thus, using the Fourier Series coefficients we can write the FT directly as

$$X(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{10}\right)$$

Now, the FS coefficients are (NOTE: you can either calculate these from the coefficient equation or use a formula at the back)

$$c_k = \frac{1}{10}\text{sinc}\left(\frac{k}{10}\right)$$

Thus,

$$X(f) = \frac{1}{10}\sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{10}\right)\delta\left(f - \frac{k}{10}\right)$$

Alternatively, since convolution in the time domain is multiplication in the frequency domain we can write:

$$\begin{aligned} X(f) &= F\{rect(t)\} F\left\{\sum_{k=-\infty}^{\infty} \delta(t - 10k)\right\} \\ &= \text{sinc}(f)\frac{1}{10}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{10}\right) \\ &= \frac{1}{10}\sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{10}\right)\delta\left(f - \frac{k}{10}\right) \end{aligned}$$

Note that this uses the FT pair

$$\sum_{k=-\infty}^{\infty} \delta(t - 10k) \Longleftrightarrow \frac{1}{10}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{10}\right)$$

We could have also used the pair

$$\sum_{k=-\infty}^{\infty} \delta(t-10k) \rightleftharpoons \sum_{k=-\infty}^{\infty} e^{-j20\pi kf}$$

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3. (25 points) System Response

Consider the signal $x(t) = rect(10t)\exp(j2000\pi t)$.

(a) [10 points] Determine the Energy Spectral Density.

We know that the ESD is related to the FT as

$$\psi_x(f) = |X(f)|^2$$

Further, let's define $x(t) = y(t)\exp(j2000\pi t)$. *In this case X(f)=Y(f-1000). Further,*

$$Y(f) = \frac{1}{10}\text{sinc}(f/10)$$

Thus,

$$\begin{aligned}\psi_x(f) &= |X(f)|^2 \\ &= \left|\frac{1}{10}\text{sinc}\left(\frac{(f-1000)}{10}\right)\right|^2 \\ &= \frac{1}{100}\text{sinc}^2\left(\frac{(f-1000)}{10}\right)\end{aligned}$$

(b) [10 points] Determine the auto-correlation function.

The auto-correlation function is simply the inverse FT of the ESD:

$$\begin{aligned}R_x(\tau) &= F^{-1}\{\psi_x(f)\} \\ &= F^{-1}\left\{\frac{1}{100}\text{sinc}^2\left(\frac{(f-1000)}{10}\right)\right\} \\ &= \frac{1}{10} tri(10\tau)e^{j2000\pi\tau}\end{aligned}$$

where $tri(t) = \begin{cases} 1-|t| & |t|<1 \\ 0 & else \end{cases}$

(c) [5 points] What is the bandwidth of the signal if bandwidth is defined according as the location of the first null after the center frequency minus the location of the first null before the center frequency ? (This is termed the first-null-to-null bandwidth).

The first null of the sinc function occurs at integer multiples of its argument. Thus, the nulls occur at 1010Hz and 990Hz and the null-to-null BW is 20Hz.

4. (30 points) Filtering

Consider an ideal low pass filter with a gain of one and a bandwidth of *B* Hz

(a) [15 points] If a power signal with autocorrelation function $R_x(\tau) = \delta(\tau)$ is passed through this filter, what is the autocorrelation function of the output?

Since the PSD and the auto-correlation function are FT pairs, we have:

$$\begin{aligned} S_x(f) &= F\{\delta(\tau)\} \\ &= 1 \end{aligned}$$

Now, at the output of a LTI system, the PSD is

$$S_y(f) = |H(f)|^2 S_x(f)$$

The ideal LPF with unit (note that the delay is irrelevant) has a transfer function

$$H(f) = rect\left(\frac{f}{2B}\right)$$

Thus, we have

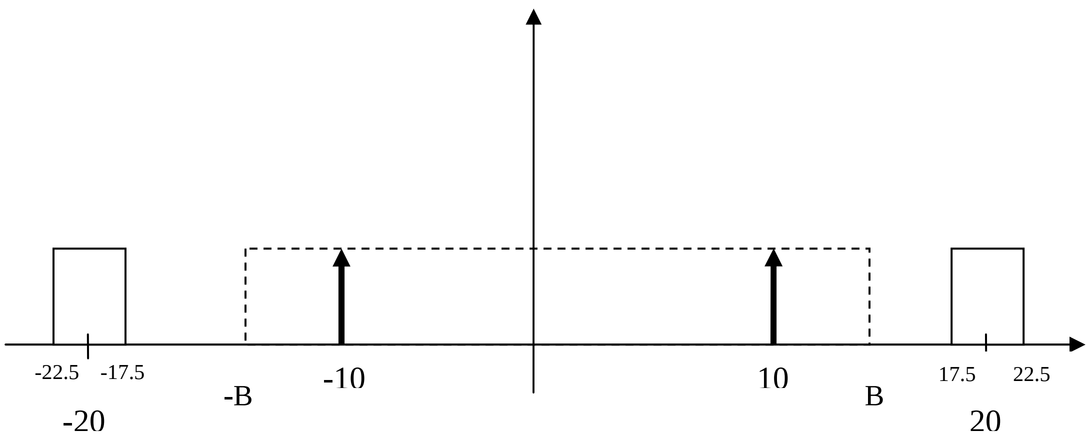
$$\begin{aligned} S_y(f) &= \left|rect\left(\frac{f}{2B}\right)\right|^2 1 \\ &= rect\left(\frac{f}{2B}\right) \end{aligned}$$

The autocorrelation of the output is then

$$R_y(\tau) = F^{-1}\{S_y(f)\} = 2B\text{sinc}(2B\tau)$$

(b) [15 points] Consider a signal *x(t)* that has a power spectral density equal to $S_x(f) = \frac{A_c^2}{4}\delta(f-10) + \frac{A_c^2}{4}\delta(f+10)$. Further, consider a second signal *y(t)* which has a power spectral density equal to $S_y(f) = \frac{1}{2} rect\left(\frac{f-20}{5}\right) + \frac{1}{2} rect\left(\frac{f+20}{5}\right)$. Determine an appropriate value for *B*, the bandwidth of the ideal LPF, such that *x(t)* is within the passband of the ideal filter, while *y(t)* is rejected (i.e., *y(t)* is within the stopband of the ideal filter).

Plotting the two PSD's along with the filter we have:



From the plot, we can see that we desire the bandwidth of the filter B*, to be*

$$10 < B < 17.5$$

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Useful Info:

Exponential Fourier Series coefficients of a square pulse train with amplitude A, pulse duration T and period T_o:

$$c_n = \frac{AT}{T_o}\,\mathrm{sinc}\left(\frac{nT}{T_o}\right)$$