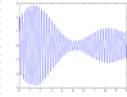


ECE3614
Introduction to
Communications Systems
Fall 2007

Instructor: Dr. R. Michael Buehrer
Lecture #14: Frequency Modulation



Overview

- Today we are going to examine two versions of Frequency Modulation
 - Narrowband FM
 - Wideband FM
- Traditionally, due to the non-linear nature of FM, it is not straightforward to analyze
- Thus, typically FM is taught using a simple sinusoidal message
 - We will follow that approach here
- Reading
 - 4.4 – 4.5

Angle Modulation

□ Phase Modulation: $s_{PM}(t) = A_c \cos(2\pi f_c t + k_p m(t))$

□ Frequency Modulation: $s_{FM}(t) = A_c \cos\left(2\pi f_c t + k_f \int_{-\infty}^t m(\lambda) d\lambda\right)$

where

- $m(t)$ - message signal
- A_c - signal amplitude
- f_c - carrier frequency
- k_p - phase sensitivity constant (radians/volt)
- k_f - frequency deviation constant (radians/volt-second)

FM - Sinusoidal Message Signal

- Consider a sinusoidal message signal

$$m(t) = A_m \cos(2\pi f_m t)$$

- The instantaneous frequency of the resulting FM wave can be written as

$$\begin{aligned} f(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned}$$

where $\Delta f = k_f A_m$ is the maximum difference between the instantaneous frequency of the carrier and the nominal carrier frequency f_c .

Sinusoidal Message Signal– cont.

- The instantaneous phase is simply 2π times the integral of the instantaneous frequency:

$$\begin{aligned} \theta(t) &= 2\pi \int_0^t (f_c + \Delta f \cos(2\pi f_m \tau)) d\tau \\ &= 2\pi f_c t + 2\pi \Delta f \frac{1}{2\pi f_m} \sin(2\pi f_m t) \\ &= 2\pi f_c t + \beta \sin(2\pi f_m t) \end{aligned}$$

$$\beta = \frac{\Delta f}{f_m}$$

where β is termed the *modulation index* and represents the maximum phase deviation from $2\pi f_c t$.

- Thus, the FM wave can be written as

$$s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$$

Narrowband FM

- Using trig identities we can write

$$\begin{aligned} s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= A_c \cos(2\pi f_c t) \cos(\beta \sin(2\pi f_m t)) - A_c \sin(2\pi f_c t) \sin(\beta \sin(2\pi f_m t)) \end{aligned}$$

- Now, if $\beta \ll 1$ (i.e., $k_f A_m \ll f_m$)

$$\cos(\beta \sin(2\pi f_m t)) \approx 1$$

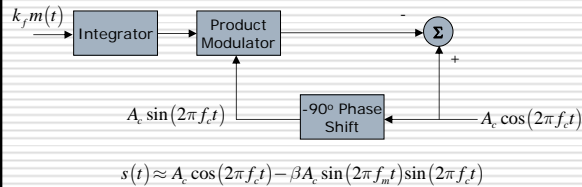
$$\sin(\beta \sin(2\pi f_m t)) \approx \beta \sin(2\pi f_m t)$$

- Which leads to

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_m t) \sin(2\pi f_c t)$$

Narrowband FM

- Thus, for small modulation indices – (small β values which can also correspond to having a small frequency sensitivity factor k_f) we can create the following circuit for FM modulation:



$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_m t) \sin(2\pi f_c t)$$

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Resulting Distortion

- The approximation given by the previous narrowband implementation results in both amplitude distortion and phase distortion.
 - It is instructive to examine the amount of distortion caused
- To do this we recall that any bandpass signal can be written using the following forms:

$$s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t)$$

$$s(t) = R(t) \cos(2\pi f_c t + \theta(t))$$

where $s_I(t)$ and $s_Q(t)$ are called the inphase and quadrature components while $R(t)$ and $\theta(t)$ are the magnitude (envelope) and phase components

- They are related as:

$$R(t) = \sqrt{s_I^2(t) + s_Q^2(t)} \quad s_I(t) = R(t) \cos(\theta(t))$$

$$\theta(t) = \tan^{-1} \left\{ \frac{s_Q(t)}{s_I(t)} \right\} \quad s_Q(t) = R(t) \sin(\theta(t))$$

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Narrowband FM Approximation

- A strict implementation of FM provides

$$s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$$
- Clearly the magnitude (i.e., envelope) is constant and equal to A_c while the phase is

$$\theta(t) = \beta \sin(2\pi f_m t)$$

- Now, from our approximation we have

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_m t) \sin(2\pi f_c t)$$

- The envelope is then

$$R(t) = \sqrt{A_c^2 + \beta^2 A_c^2 \sin^2(2\pi f_m t)}$$

$$= A_c \sqrt{1 + \beta^2 \sin^2(2\pi f_m t)}$$

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Narrowband Approx. – cont.

- The phase of the transmit signal is

$$\theta(t) = \tan^{-1}\{\beta \sin(2\pi f_m t)\}$$

- The power series allows for the approximation of the inverse tangent as

$$\tan^{-1}\{x\} \approx x - \frac{1}{3}x^3$$

$$\begin{aligned} \theta(t) &= \tan^{-1}\{\beta \sin(2\pi f_m t)\} \\ &\approx \beta \sin(2\pi f_m t) - \frac{\beta^3}{3} \sin^3(2\pi f_m t) \end{aligned}$$

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Approximation Error

- The amplitude error is

$$\begin{aligned} e_R(t) &= A_c - A_c \sqrt{1 + \beta^2 \sin^2(2\pi f_m t)} \\ &= A_c \left(1 - \sqrt{1 + \beta^2 \sin^2(2\pi f_m t)}\right) \end{aligned}$$

- The maximum and minimum amplitudes are

$$\max(R(t)) = A_c \sqrt{1 + \beta^2}$$

$$\min(R(t)) = A_c$$

- Thus, the modulation index determines the maximum deviation from constant amplitude.
- The phase error is

$$e_\theta(t) \approx \frac{\beta^3}{3} \sin^3(2\pi f_m t)$$

- which again depends on the value of β .

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Example 14.1

- Assuming a sinusoidal message signal, let $f_c = 1000\text{Hz}$, $f_m = 10$, $A_m = 1$, $k_f = 1$

- Thus, $\beta = \Delta f/f_m = A_m k_f/f_m = 1/10 = 0.1$

$$s(t) = \cos(2000\pi t + 0.1 \sin(20\pi t))$$

$$s(t) \approx \cos(2000\pi t) - 0.1 \sin(20\pi t) \sin(2000\pi t)$$

- Amplitude and Phase Error:

$$e_R(t) = \left(1 - \sqrt{1 + 0.01 \sin^2(20\pi t)}\right) \quad e_\theta(t) \approx \frac{\beta^3}{3} \sin^3(2\pi f_m t)$$

$$\approx 1 - \left(1 + \frac{1}{2} 0.01 \sin^2(20\pi t)\right) = 3.3e - 4 \sin^3(20\pi t)$$

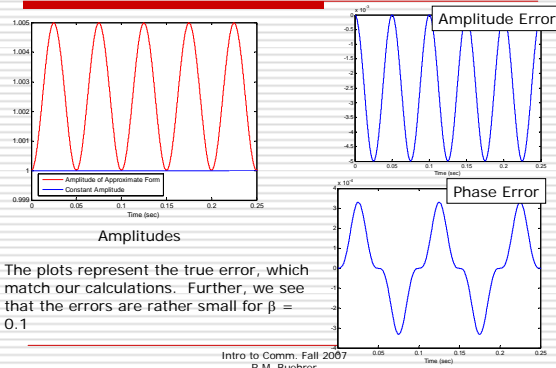
$$= -0.005 \sin^2(20\pi t)$$

$$= -0.0025(1 - \cos(40\pi t))$$

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Example 14.1 – cont.



Example 14.2

□ Assuming a sinusoidal message signal, now let $f_c = 1000\text{Hz}$, $f_m = 10$, $A_m = 1$, $k_f = 5$

□ Thus, $\beta = \Delta f/f_m = A_m k_f/f_m = 5/10 = 0.5$

$$s(t) = \cos(2000\pi t + 0.5 \sin(20\pi t))$$

$$s(t) \approx \cos(2000\pi t) - 0.5 \sin(20\pi t) \sin(2000\pi t)$$

□ Amplitude and Phase Error:

$$e_R(t) = \left(1 - \sqrt{1 + 0.25 \sin^2(20\pi t)}\right) \quad e_\phi(t) \approx \frac{\beta^3}{3} \sin^3(2\pi f_m t)$$

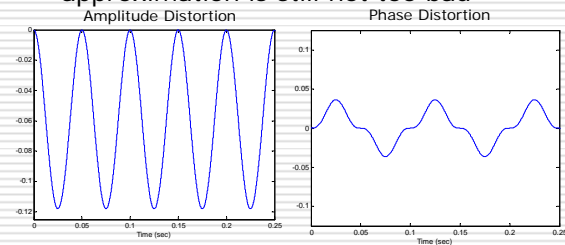
$$\approx 1 - \left(1 + \frac{1}{2} 0.25 \sin^2(20\pi t)\right) \quad = 0.005 \sin^3(20\pi t)$$

$$= -0.125 \sin^2(20\pi t)$$

$$= -0.125(1 - \cos(40\pi t))$$

Example 14.2 – cont.

□ Error has increased significantly, but approximation is still not too bad



Complex Baseband Representation

- Earlier we stated that any bandpass signal can be represented using either of two forms:

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

$$s(t) = R(t)\cos(2\pi f_c t + \theta(t))$$

- There is a third form that can also be used:

$$s(t) = \text{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\}$$

- This is termed the complex envelope or complex baseband representation and $\tilde{s}(t)$ is called the complex envelope.

- For FM signals,

$$\tilde{s}(t) = A_c e^{j\theta(t)} = A_c e^{j\left[2\pi k_f \int_0^t m(\tau) d\tau\right]}$$

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Sinusoidal Modulation

- For sinusoidal modulation the complex baseband becomes:

$$\tilde{s}(t) = A_c e^{j\beta \sin(2\pi f_m t)}$$

- Now, the spectrum of a bandpass signal is related to the spectrum of the complex baseband signal as

$$S(f) = \tilde{S}(f - f_c) + \tilde{S}^*(-f - f_c)$$

- Thus, we can examine the spectral properties of the complex baseband signal in order to determine the spectral properties of the bandpass signal

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Spectrum of Sinusoidal Modulation

- The signal of interest $s(t)$ is periodic and thus doesn't have a Fourier Transform in the strict sense. However, we can determine the Fourier Transform using the Fourier Series:

$$\tilde{S}(f) = \sum_{-\infty}^{\infty} c_n \delta(f - nf_o)$$

- In this case, the fundamental frequency is the message frequency: $f_o = f_m$
- The Fourier Series coefficients are determined as

$$\begin{aligned} c_n &= \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} s(t) e^{-j2\pi n f_o t} dt \\ &= f_m \int_{-1/2f_m}^{1/2f_m} A_c e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt \\ &= f_m \int_{-1/2f_m}^{1/2f_m} A_c e^{j\beta \sin(2\pi f_m t) - 2\pi n f_m t} dt \end{aligned}$$

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Spectrum - cont.

- Continuing with the coefficient derivation:

$$c_n = f_m \int_{-1/2f_m}^{1/2f_m} A_c e^{j[\beta \sin(2\pi f_m t) - 2\pi n f_m t]} dt$$

- We can make substitution of variables $x = 2\pi f_m t$

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j[\beta \sin(x) - nx]} dx$$

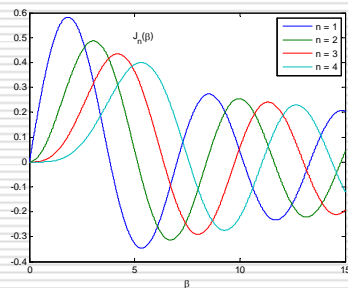
- However, this is simply a known integral known as a Bessel Function $J_n(\beta)$ scaled by A_c :

$$c_n = A_c J_n(\beta)$$

- Thus, the spectrum of the complex baseband signal may be written as

$$\tilde{S}(f) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \delta(f - n f_m)$$

Sidebar – Bessel Function



- The Bessel function is tabulated in Appendix 3 in your text
- This function is also available in Matlab using `besselj(n,beta)`

Spectrum – cont.

- Thus, the spectrum of the bandpass signal can be written as

$$S(f) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \{ \delta(f - f_c - n f_m) + \delta(f + f_c + n f_m) \}$$

- Notice the difference between AM and FM
 - With AM, sinusoidal modulation results in a single pair of side frequencies
 - With FM, there are an infinite number of side frequencies, although only a finite number are significant.
 - The number of significant side frequencies will depend on the modulation index β .
 - If β is small, only $J_0(\beta)$ and $J_1(\beta)$ will have significant values and the FM wave will be similar to an AM in the frequency domain
 - Also note that for any value of β

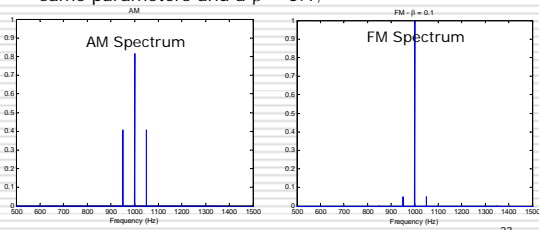
$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

Narrowband vs. Wideband FM

- We have seen previously that if β is small ($\ll 1$) the signal can be approximated as a linear modulation scheme similar to AM
- If β is bigger than 1 we call the resulting modulation *wideband FM* and the modulation is nonlinear as expected.
- The spectral properties of wideband FM are substantially different from AM.
- Let's examine the spectrum for various values of β when the message is sinusoidal.

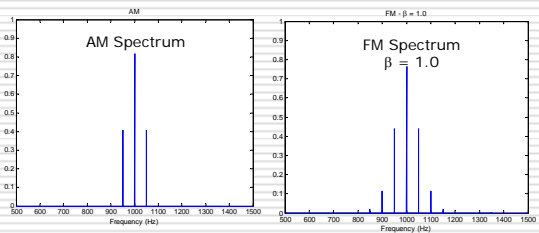
Example 14.3

- Consider a LC-AM wave modulated by a sinusoidal message assuming $f_c = 1\text{kHz}$ and $f_m = 50\text{Hz}$
- Compare the spectrum with an FM wave with the same parameters and a $\beta = 0.1$;



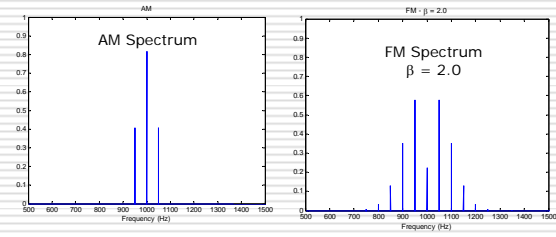
Example 14.3 – cont.

- Now compare the AM spectrum with FM when $\beta = 1.0$, 2.0 and 5.0
- Keep f_m constant (i.e., Δf is increased since $\beta = \Delta f / f_m$)



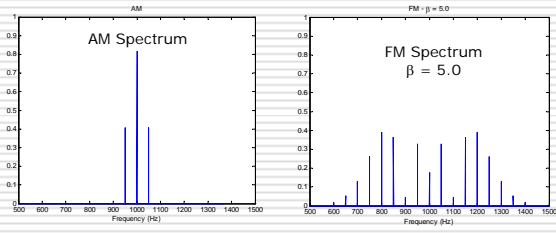
Example 14.3 – cont.

□ For $\beta = 2.0$



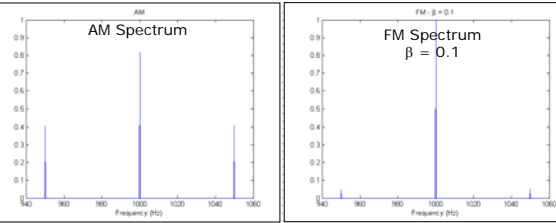
Example 14.3 – cont.

□ For $\beta = 5.0$



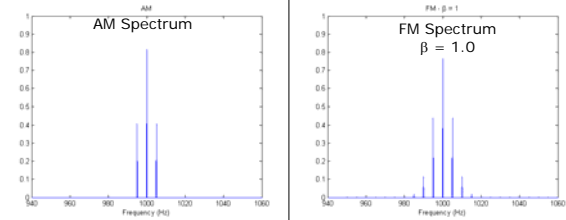
Example 14.4

□ Now let us vary β by decreasing f_m but keeping Δf constant



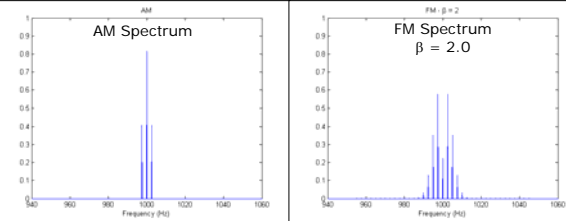
Example 14.4 – cont.

□ $\beta = 1.0$ ($f_m = 5\text{Hz}$)



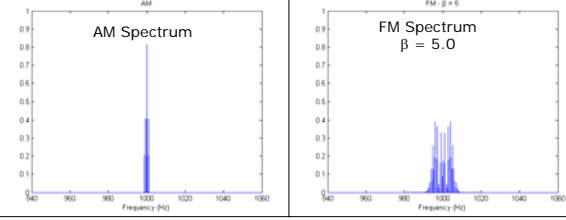
Example 14.4 – cont.

□ $\beta = 2.0$ ($f_m = 2.5\text{Hz}$)



Example 14.4 – cont.

□ $\beta = 5.0$ ($f_m = 1\text{Hz}$)



Summary

- Today we have investigated two forms of frequency modulation, namely narrowband FM and wideband FM
 - Narrowband FM approaches linear modulation and is similar to AM
 - Wideband FM is decidedly non-linear and occupies a band much larger than the bandwidth of the message.
- We restricted ourselves to a sinusoidal message signal since the non-linear nature makes analysis extremely difficult.
