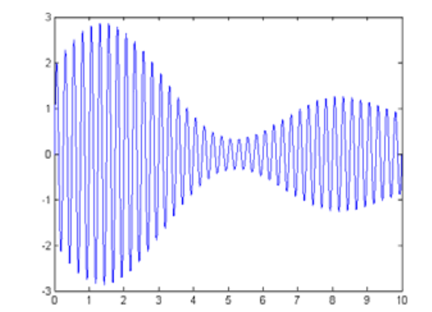
ECE3614 Introduction to Communications Systems Fall 2007

Instructor: Dr. R. Michael Buehrer

Lecture #14: Frequency Modulation



Overview

- Today we are going to examine two versions of Frequency Modulation
  - Narrowband FM
  - Wideband FM
- Traditionally, due to the non-linear nature of FM, it is not straightforward to analyze
- Thus, typically FM is taught using a simple sinusoidal message
  - We will follow that approach here
- Reading
  - 4.4 – 4.5

Angle Modulation

- Phase Modulation: $$s_{PM}(t) = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$$
- Frequency Modulation: $$s_{FM}(t) = A_c \cos\left(2\pi f_c t + k_f \int_{-\infty}^{t} m(\lambda)\,d\lambda\right)$$

where

- $m(t)$ - message signal
- A_c - signal amplitude
- f_c - carrier frequency
- k_p - phase sensitivity constant (radians/volt)
- k_f - frequency deviation constant (radians/volt-second)

FM - Sinusoidal Message Signal

- Consider a sinusoidal message signal

$$m(t) = A_m \cos(2\pi f_m t)$$

- The instantaneous frequency of the resulting FM wave can be written as

$$\begin{aligned} f(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned}$$

where $\Delta f = k_f A_m$ is the maximum difference between the instantaneous frequency of the carrier and the nominal carrier frequency f_c.

Sinusoidal Message Signal– cont.

- The instantaneous phase is simply 2π times the integral of the instantaneous frequency:

$$\theta(t) = 2\pi \int_0^t \left(f_c + \Delta f \cos(2\pi f_m \tau) \right) d\tau$$

$$= 2\pi f_c t + 2\pi \Delta f \frac{1}{2\pi f_m} \sin(2\pi f_m t)$$

$$= 2\pi f_c t + \beta \sin(2\pi f_m t)$$

$$\beta = \frac{\Delta f}{f_m}$$

where β is termed the *modulation index* and represents the maximum phase deviation from $2\pi f_c t$.

- Thus, the FM wave can be written as

$$s(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)$$

Narrowband FM

- Using trig identities we can write

$$s(t) = A_c \cos\left(2\pi f_c t + \beta \sin\left(2\pi f_m t\right)\right)$$

$$= A_c \cos\left(2\pi f_c t\right)\cos\left(\beta \sin\left(2\pi f_m t\right)\right) - A_c \sin\left(2\pi f_c t\right)\sin\left(\beta \sin\left(2\pi f_m t\right)\right)$$

- Now, if β << 1 (i.e., $k_f A_m << f_m$)

$$\cos\left(\beta \sin\left(2\pi f_m t\right)\right) \approx 1$$

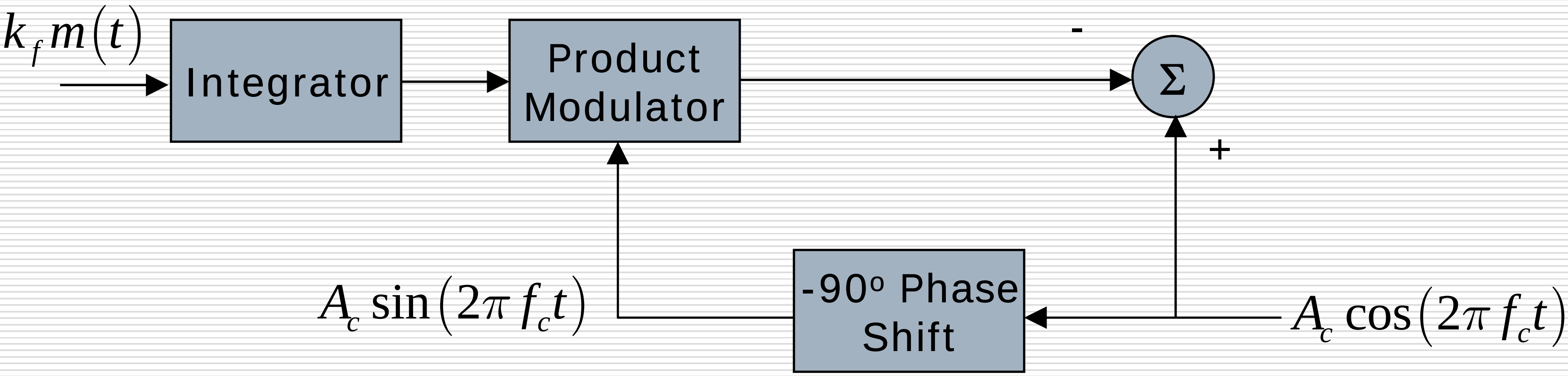
$$\sin\left(\beta \sin\left(2\pi f_m t\right)\right) \approx \beta \sin\left(2\pi f_m t\right)$$

- Which leads to

$$s(t) \approx A_c \cos\left(2\pi f_c t\right) - \beta A_c \sin\left(2\pi f_m t\right)\sin\left(2\pi f_c t\right)$$

Narrowband FM

- Thus, for small modulation indicies – (small β values which can also correspond to having a small frequency sensitivity factor k_f) we can create the following circuit for FM modulation:



$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_m t)\sin(2\pi f_c t)$$

Resulting Distortion

- The approximation given by the previous narrowband implementation results in both amplitude distortion and phase distortion.
  - It is instructive to examine the amount of distortion caused
- To do this we recall that any bandpass signal can be written using the following forms:

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

$$s(t) = R(t)\cos(2\pi f_c t + \theta(t))$$

where $s_I(t)$ and $s_Q(t)$ are the called the inphase and quadrature components while $R(t)$ and $\theta(t)$ are the magnitude (envelope) and phase components

- They are related as:

$$R(t) = \sqrt{s_I^2(t) + s_Q^2(t)}$$

$$\theta(t) = \tan^{-1}\left\{\frac{s_Q(t)}{s_I(t)}\right\}$$

$$s_I(t) = R(t)\cos(\theta(t))$$

$$s_Q(t) = R(t)\sin(\theta(t))$$

Narrowband FM Approximation

- A strict implementation of FM provides

$$s(t) = A_c \cos\left(2\pi f_c t + \beta \sin\left(2\pi f_m t\right)\right)$$

- Clearly the magnitude (i.e., envelope) is constant and equal to A_c while the phase is

$$\theta(t) = \beta \sin\left(2\pi f_m t\right)$$

- Now, from our approximation we have

$$s(t) \approx A_c \cos\left(2\pi f_c t\right) - \beta A_c \sin\left(2\pi f_m t\right) \sin\left(2\pi f_c t\right)$$

- The envelope is then

$$R(t) = \sqrt{A_c^2 + \beta^2 A_c^2 \sin^2\left(2\pi f_m t\right)}$$

$$= A_c \sqrt{1 + \beta^2 \sin^2\left(2\pi f_m t\right)}$$

Narrowband Approx. – cont.

- The phase of the transmit signal is

$$\theta(t) = \tan^{-1}\left\{\beta \sin(2\pi f_m t)\right\}$$

- The power series allows for the approximation of the inverse tangent as

$$\tan^{-1}\{x\} \approx x - \frac{1}{3}x^3$$

$$\begin{aligned}\theta(t) &= \tan^{-1}\left\{\beta \sin(2\pi f_m t)\right\} \\ &\approx \beta \sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)\end{aligned}$$

Approximation Error

- The amplitude error is

$$e_R(t) = A_c - A_c\sqrt{1+\beta^2\sin^2(2\pi f_m t)}$$
$$= A_c\left(1-\sqrt{1+\beta^2\sin^2(2\pi f_m t)}\right)$$

- The maximum and minimum amplitudes are

$$\max\left(R(t)\right) = A_c\sqrt{1+\beta^2}$$
$$\min\left(R(t)\right) = A_c$$

- Thus, the modulation index determines the maximum deviation from constant amplitude.
- The phase error is

$$e_\theta(t) \approx \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

- which again depends on the value of β.

Example 14.1

- Assuming a sinusoidal message signal, let $f_c = 1000\text{Hz}$, $f_m = 10$, $A_m = 1$, $k_f = 1$
- Thus, $\beta = \Delta f/f_m = A_m k_f/f_m = 1/10 = 0.1$

$$s(t) = \cos\left(2000\pi t + 0.1\sin(20\pi t)\right)$$

$$s(t) \approx \cos(2000\pi t) - 0.1\sin(20\pi t)\sin(2000\pi t)$$

- Amplitude and Phase Error:

$$e_R(t) = \left(1 - \sqrt{1 + 0.01\sin^2(20\pi t)}\right)$$

$$\approx 1 - \left(1 + \frac{1}{2}0.01\sin^2(20\pi t)\right)$$

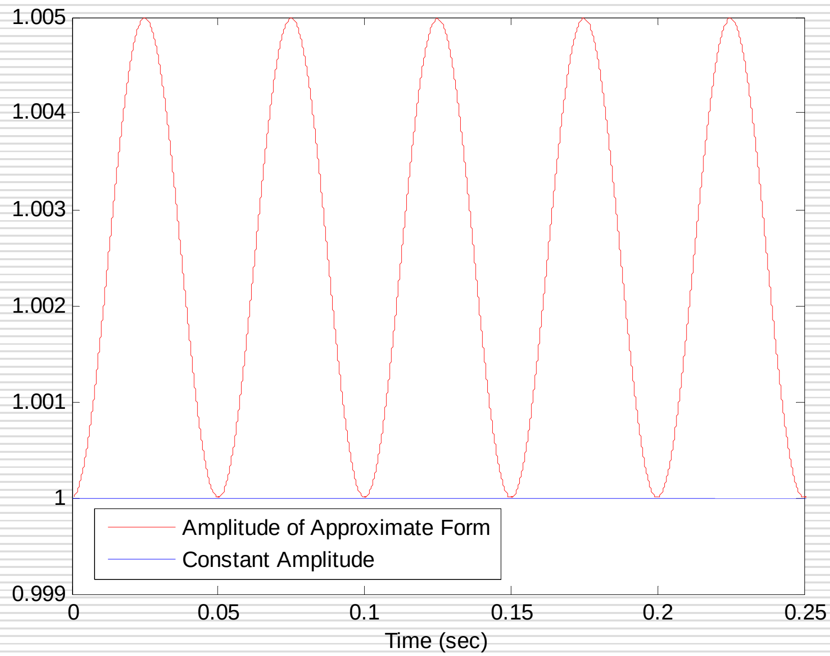
$$= -0.005\sin^2(20\pi t)$$

$$= -0.0025\left(1 - \cos(40\pi t)\right)$$

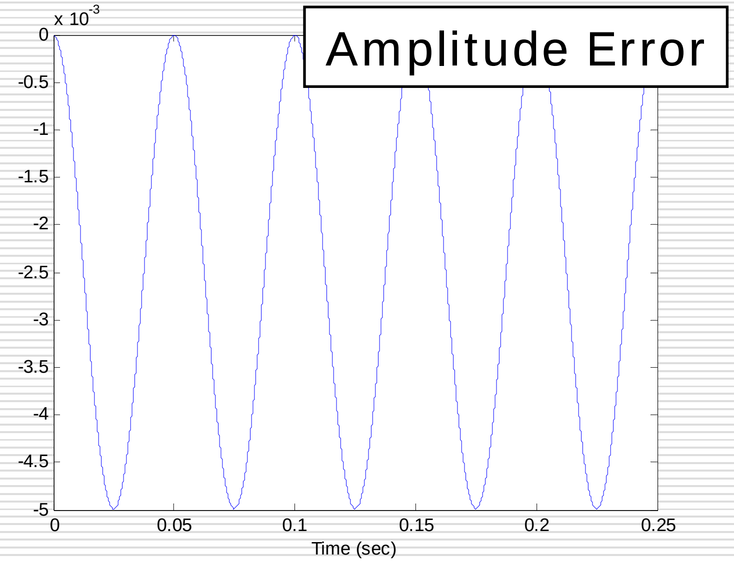
$$e_\theta(t) \approx \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

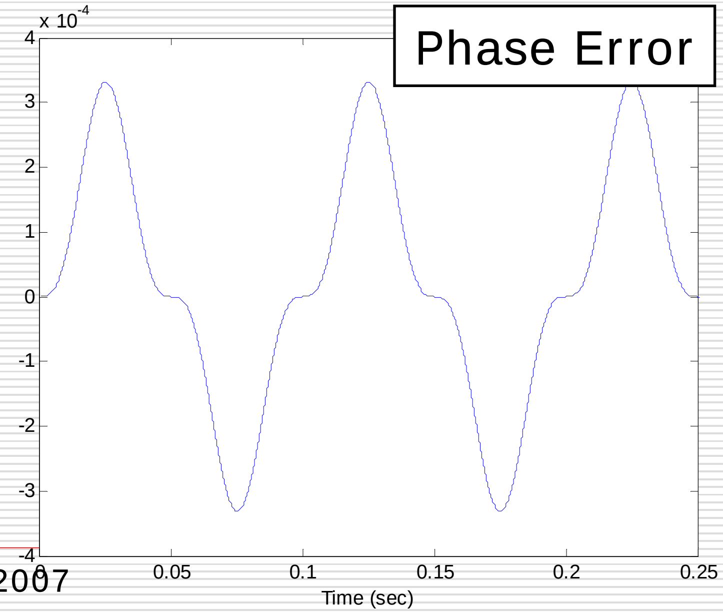
$$= 3.3e-4\sin^3(20\pi t)$$

Example 14.1 – cont.



Amplitudes





The plots represent the true error, which match our calculations. Further, we see that the errors are rather small for $\beta = 0.1$

Example 14.2

- Assuming a sinusoidal message signal, now let f_c = 1000Hz, f_m = 10, A_m = 1, k_f = 5
- Thus, $\beta = \Delta f/f_m = A_m k_f/f_m = 5/10 = 0.5$

$$s(t) = \cos\left(2000\pi t + 0.5\sin(20\pi t)\right)$$

$$s(t) \approx \cos(2000\pi t) - 0.5\sin(20\pi t)\sin(2000\pi t)$$

- Amplitude and Phase Error:

$$e_R(t) = \left(1 - \sqrt{1 + 0.25\sin^2(20\pi t)}\right)$$

$$\approx 1 - \left(1 + \frac{1}{2}0.25\sin^2(20\pi t)\right)$$

$$= -0.125\sin^2(20\pi t)$$

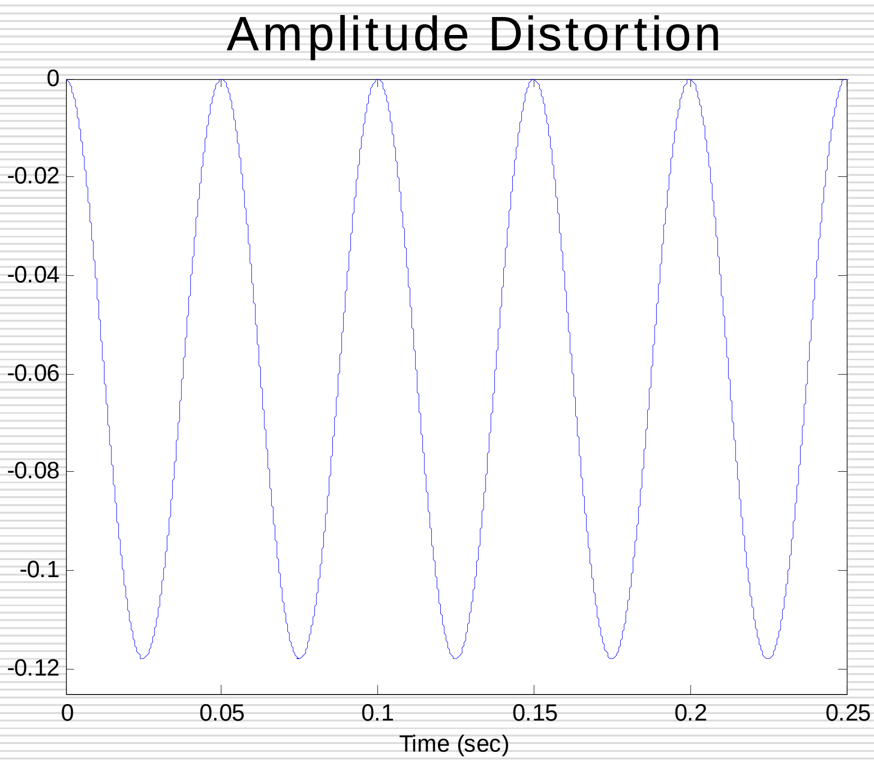
$$= -0.125\left(1 - \cos(40\pi t)\right)$$

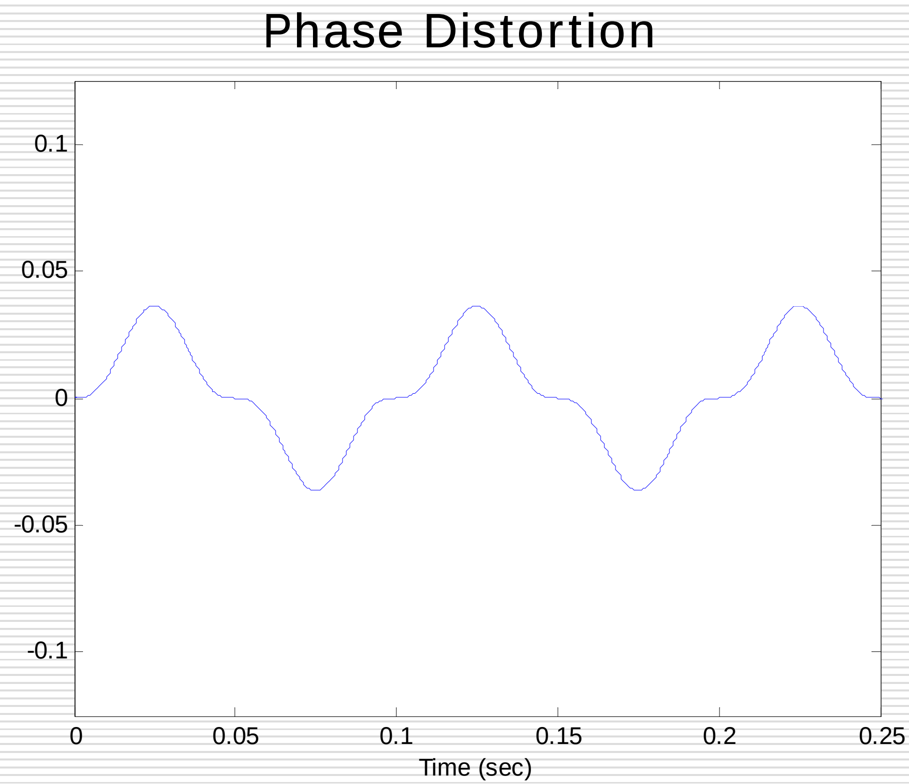
$$e_\theta(t) \approx \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

$$= 0.005\sin^3(20\pi t)$$

Example 14.2 – cont.

- Error has increased significantly, but approximation is still not too bad





Complex Baseband Representation

- Earlier we stated that any bandpass signal can be represented using either of two forms:

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_I(t)\sin(2\pi f_c t)$$

$$s(t) = R(t)\cos(2\pi f_c t + \theta(t))$$

- There is a third form that can also be used:

$$s(t) = \mathrm{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\}$$

- This is termed the complex envelope or complex baseband representation and $\tilde{s}(t)$ is called the complex envelope.
- For FM signals,

$$\tilde{s}(t) = A_c e^{j\theta(t)} = A_c e^{j\left(2\pi k_f \int_0^t m(\tau)d\tau\right)}$$

Sinusoidal Modulation

- For sinusoidal modulation the complex baseband becomes:

$$\tilde{s}(t) = A_c e^{j\beta \sin(2\pi f_m t)}$$

- Now, the spectrum of a bandpass signal is related to the spectrum of the complex baseband signal as

$$S(f) = \tilde{S}(f - f_c) + \tilde{S}^*(-f - f_c)$$

- Thus, we can examine the spectral properties of the complex baseband signal in order to determine the spectral properties of the bandpass signal

Spectrum of Sinusoidal Modualation

- T signal of interest $\tilde{s}(t)$ is periodic and thus doesn't have a Fourier Transform in the strict sense. However, we can determine the Fourier Transform using the Fourier Series:

$$\tilde{S}(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_o)$$

- In this case, the fundamental frequency is the message frequency: $f_o = f_m$
- The Fourier Series coefficients are determined as

$$c_n = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} \tilde{s}(t) e^{-j2\pi n f_o t} dt$$

$$= f_m \int_{-1/2f_m}^{1/2f_m} A_c e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi n f_o t} dt$$

$$= f_m \int_{-1/2f_m}^{1/2f_m} A_c e^{j[\beta \sin(2\pi f_m t) - 2\pi n f_o t]} dt$$

Spectrum - cont.

- Continuing with the coefficient derivation:

$$c_n = f_m \int_{-1/2 f_m}^{1/2 f_m} A_c e^{j[\beta \sin(2\pi f_m t) - 2\pi n f_o t]} dt$$

- We can make substitution of variables $x = 2\pi f_m t$

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j[\beta \sin(x) - nx]} dx$$

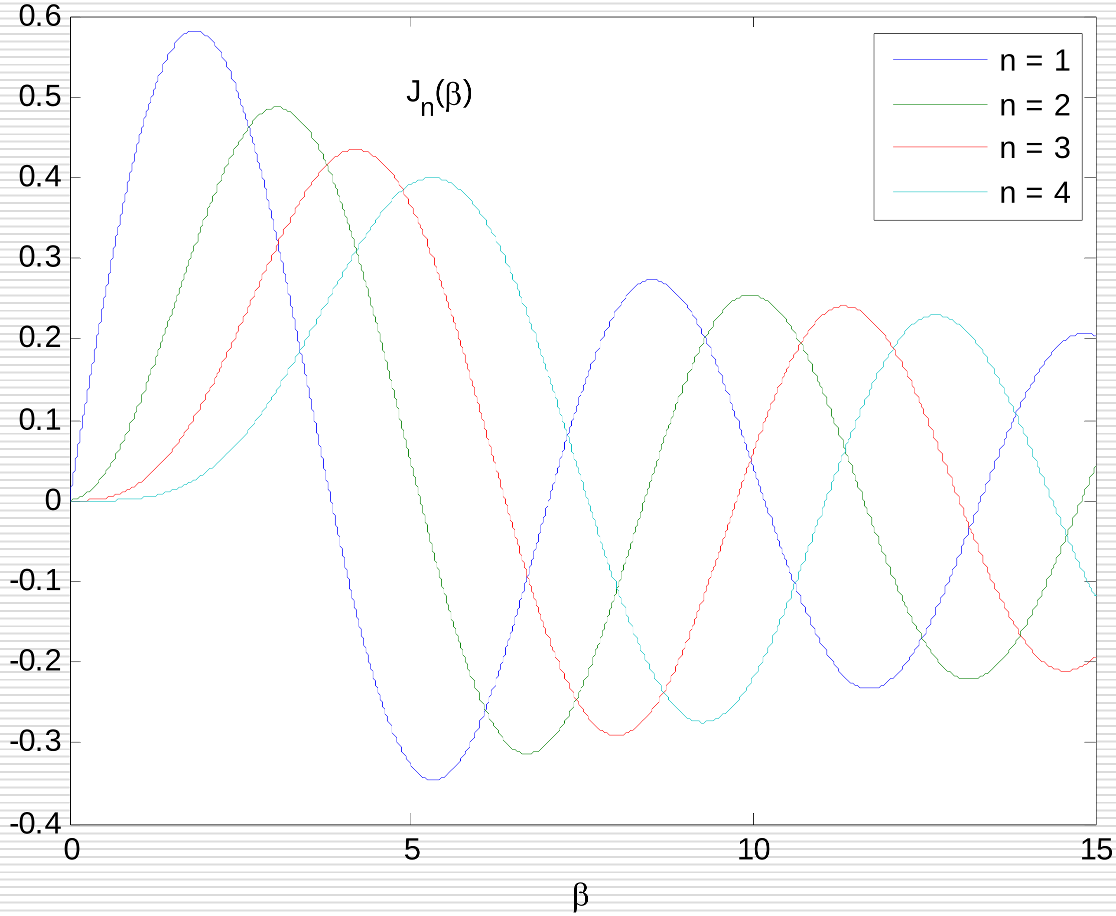
- However, this is simply a known integral known as a Bessel Function $J_n(\beta)$ scaled by A_c:

$$c_n = A_c J_n(\beta)$$

- Thus, the spectrum of the complex baseband signal may be written as

$$\tilde{S}(f) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\delta(f - n f_m)$$

Sidebar – Bessel Function



- The Bessel function is tabulated in Appendix 3 in your text
- This function is also available in Matlab using besselj(n,beta)

Spectrum – cont.

- Thus, the spectrum of the bandpass signal can be written as

$$S(f) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\left\{\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right\}$$

- Notice the difference between AM and FM
  - With AM, sinusoidal modulation results in a single pair of side frequencies
  - With FM, there are an infinite number of side frequencies, although only a finite number are significant.
  - The number of significant side frequencies will depend on the modulation index β.
  - If β is small, only $J_0(\beta)$ and $J_1(\beta)$ will have significant values and the FM wave will be similar to an AM in the frequency domain
  - Also note that for any value of β

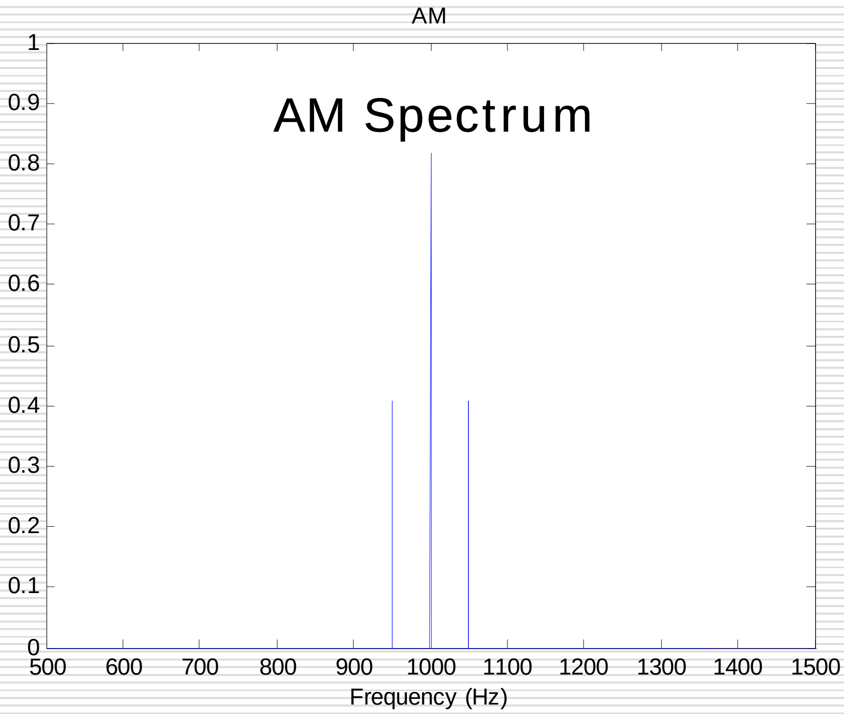
$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

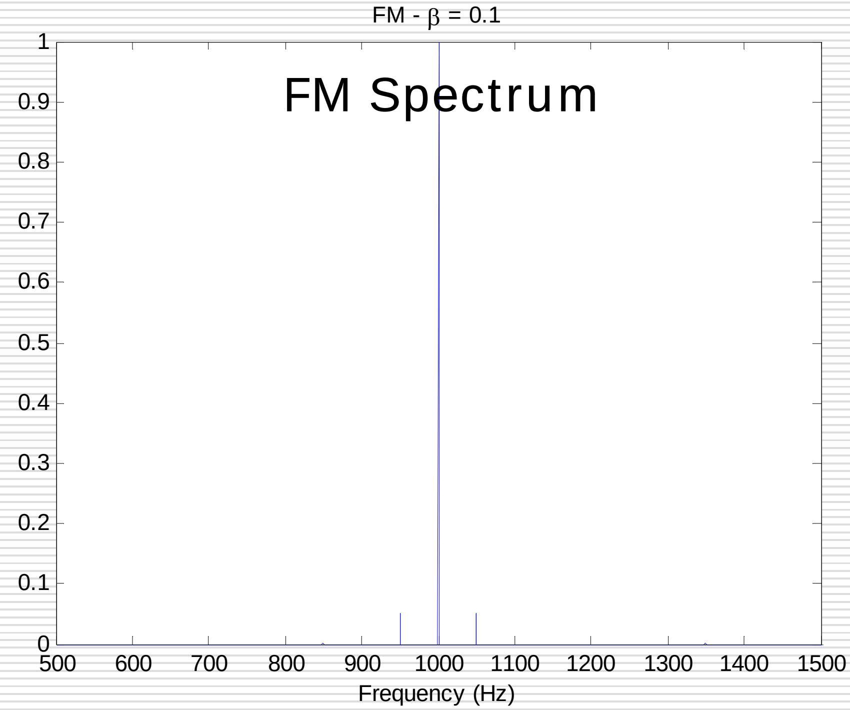
Narrowband vs. Wideband FM

- We have seen previously that if β is small ($<< 1$) the signal can be approximated as a linear modulation scheme similar to AM
- If β is bigger than 1 we call the resulting modulation *wideband FM* and the modulation is nonlinear as expected.
- The spectral properties of wideband FM are substantially different from AM.
- Let's examine the spectrum for various values of β when the message is sinusoidal.

Example 14.3

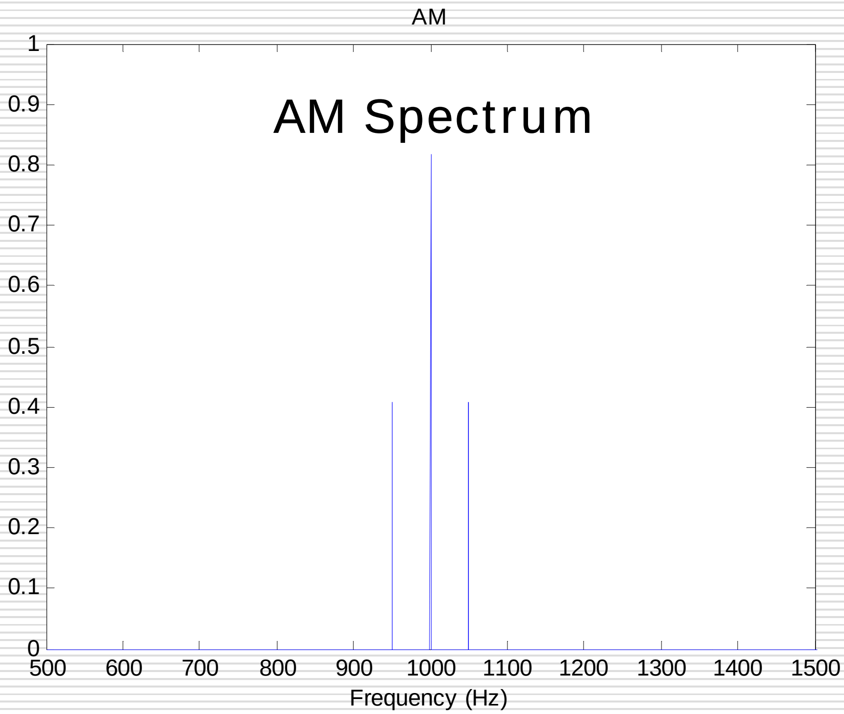
- Consider a LC-AM wave modulated by a sinusoidal message assuming f_c = 1kHz and f_m = 50Hz
- Compare the spectrum with an FM wave with the same parameters and a $\beta = 0.1$;

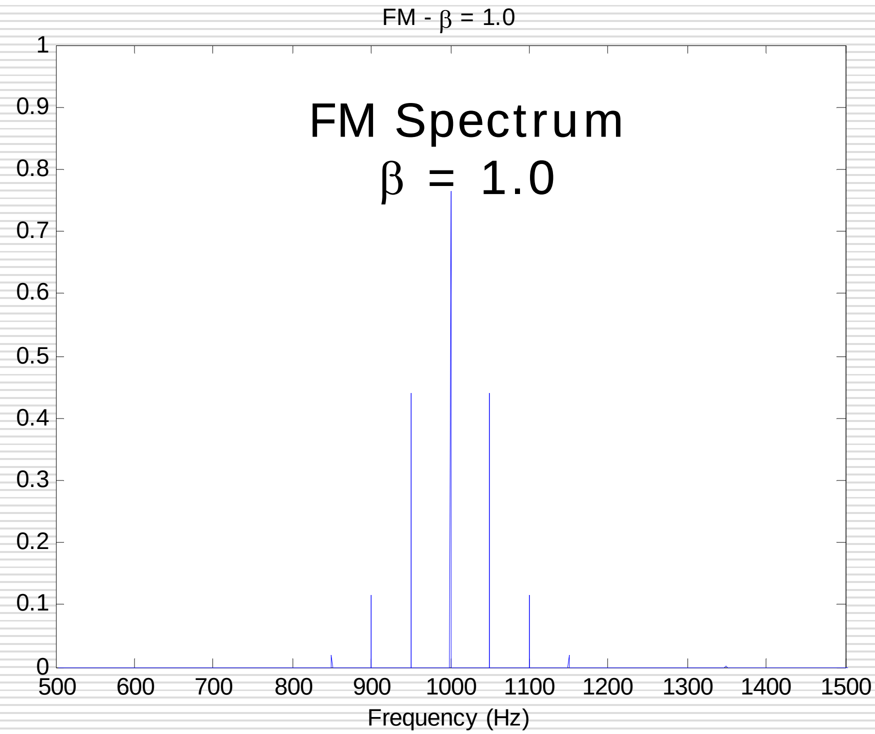




Example 14.3 – cont.

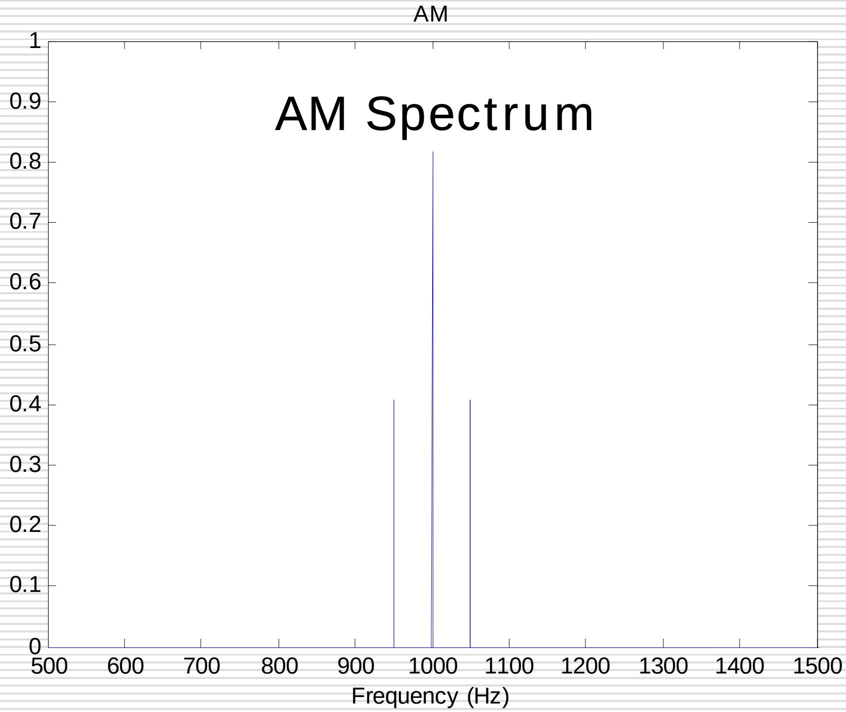
- Now compare the AM spectrum with FM when β = 1.0, 2.0 and 5.0
- Keep f_m constant (i.e., Δf is increased since $\beta = \Delta f / f_m$)

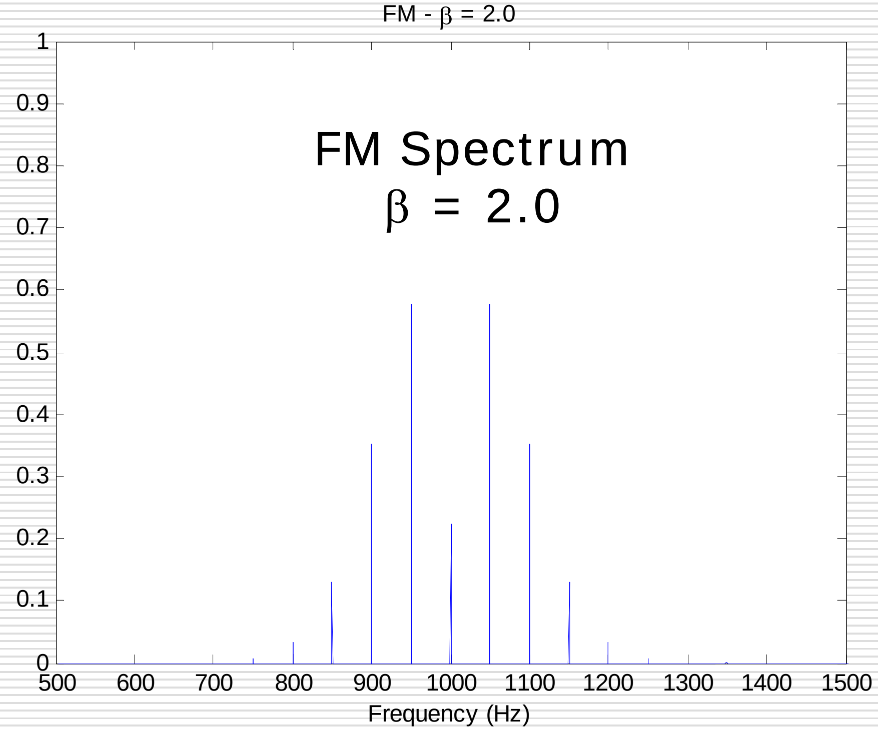




Example 14.3 – cont.

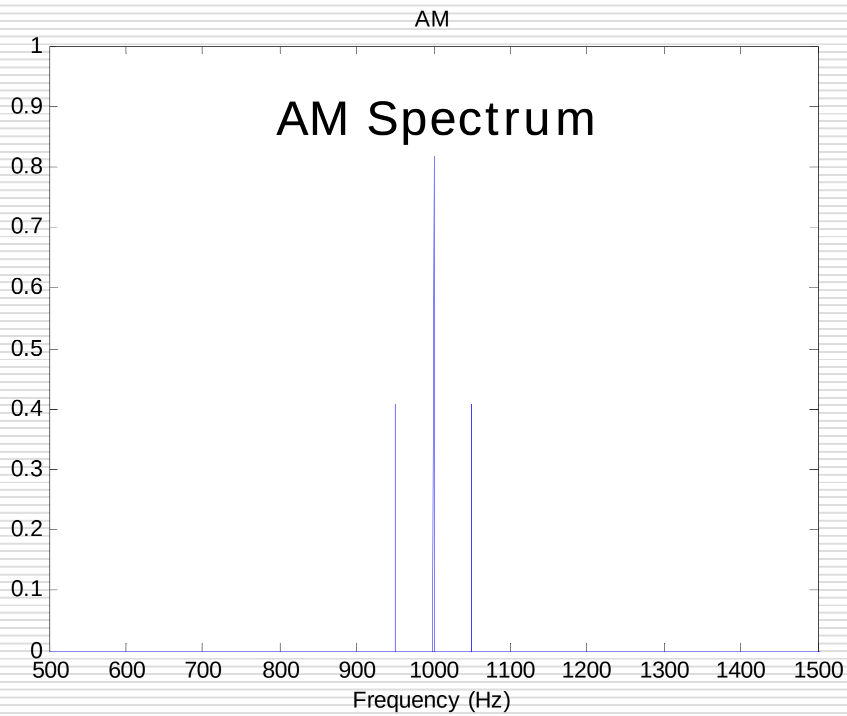
- For $\beta = 2.0$

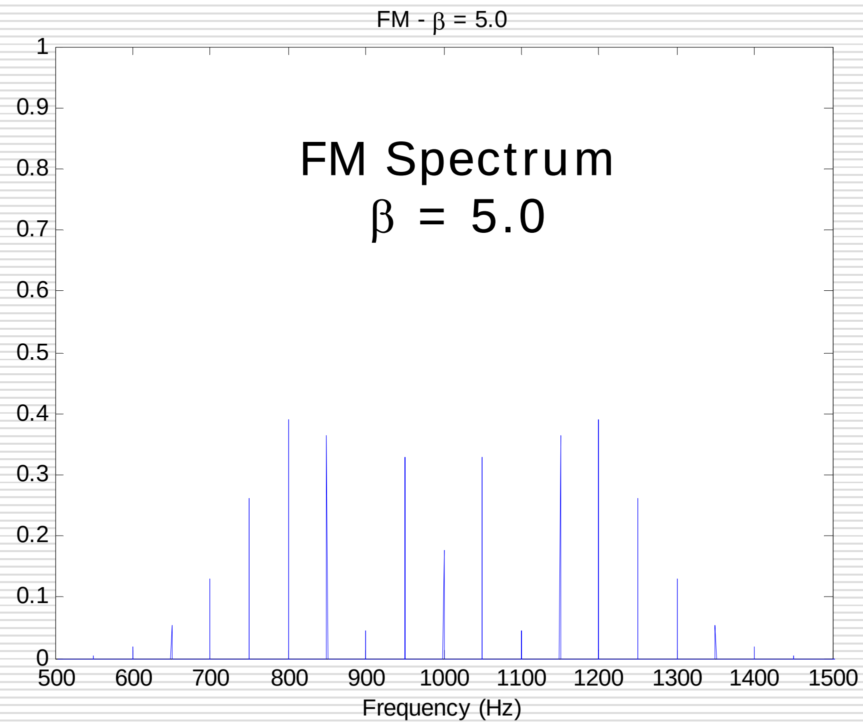




Example 14.3 – cont.

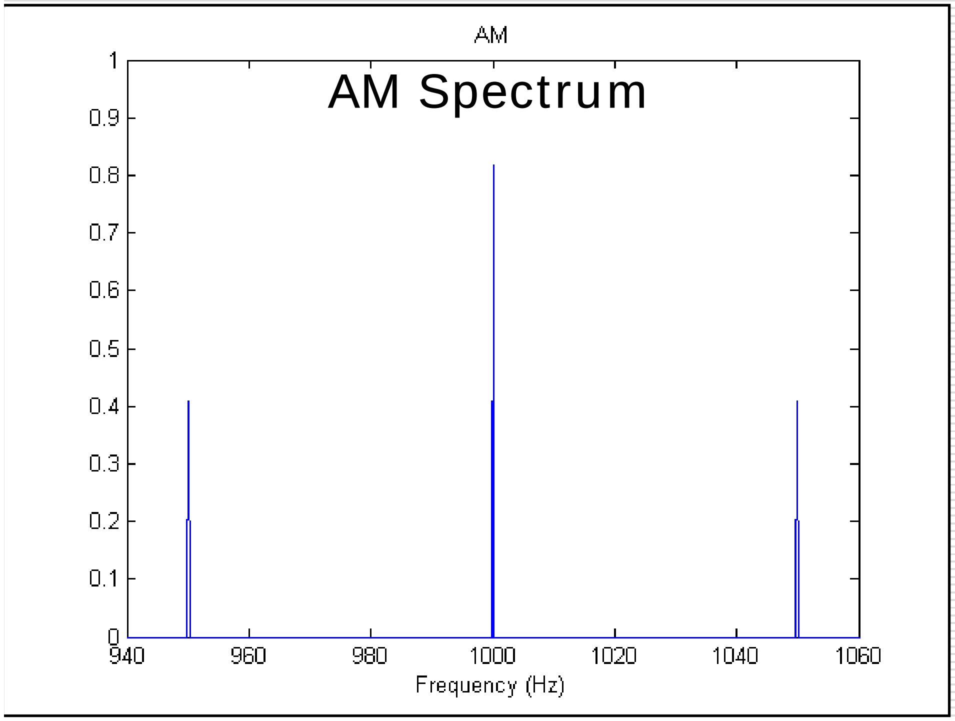
- For β = 5.0

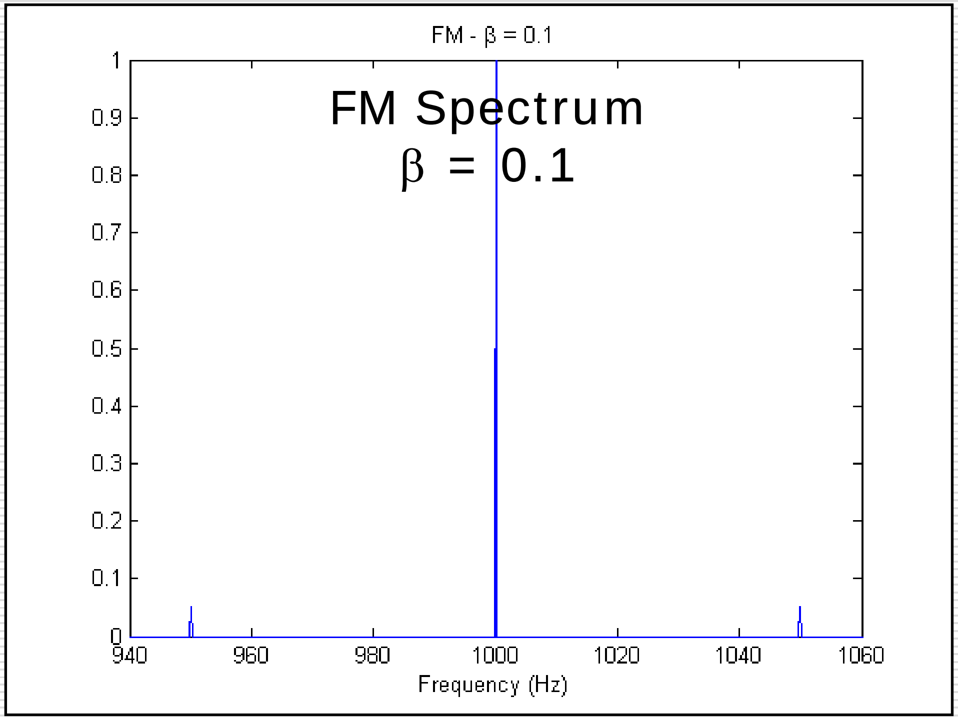




Example 14.4

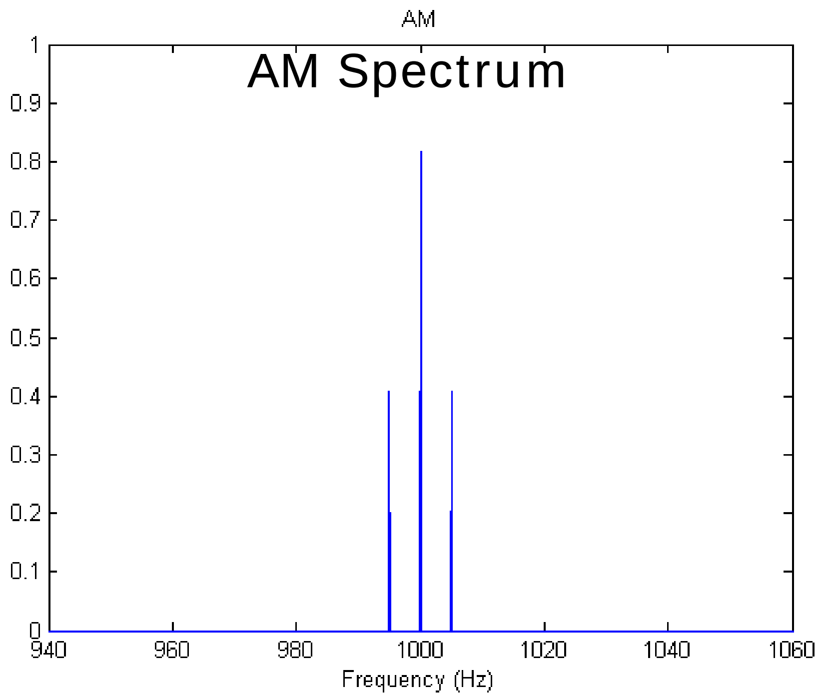
- Now let us vary β by decreasing f_m but keeping Δf constant

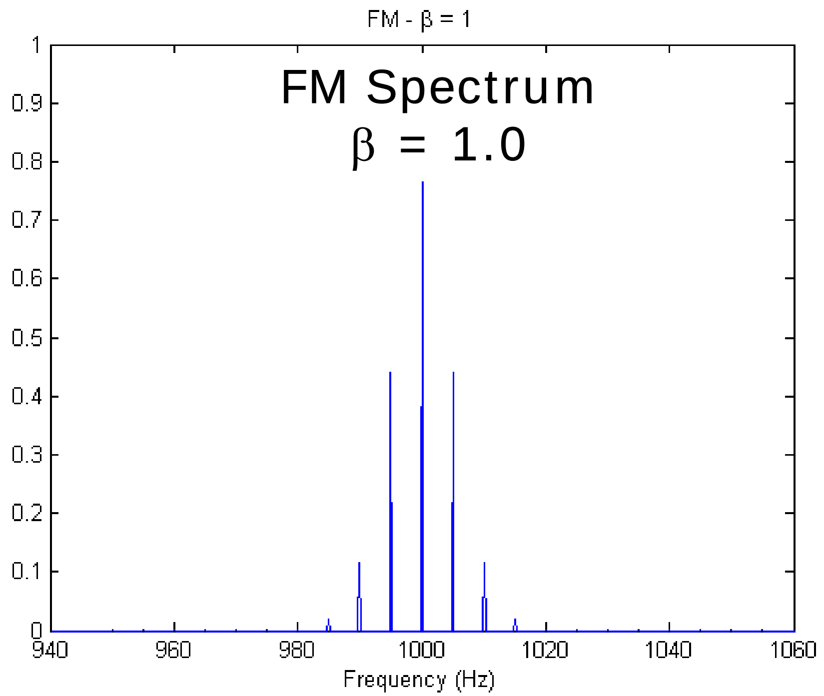




Example 14.4 – cont.

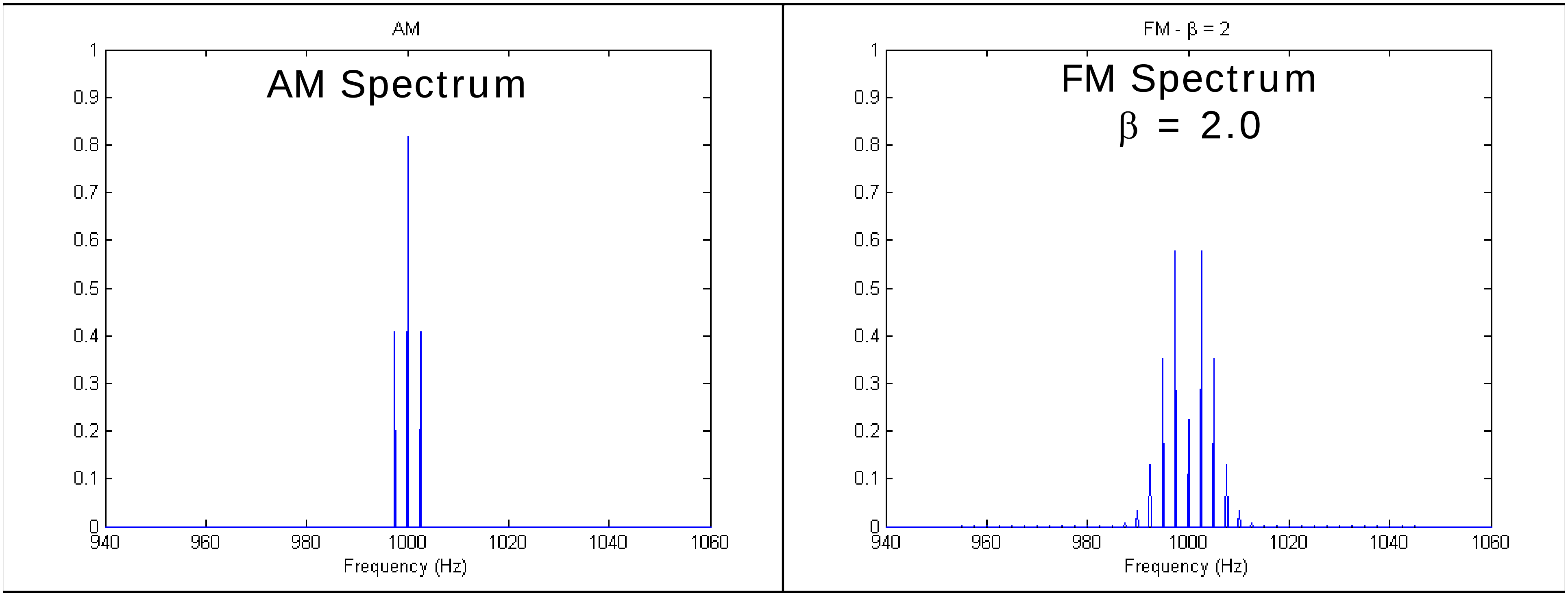
- $\beta = 1.0$ (f_m = 5Hz)





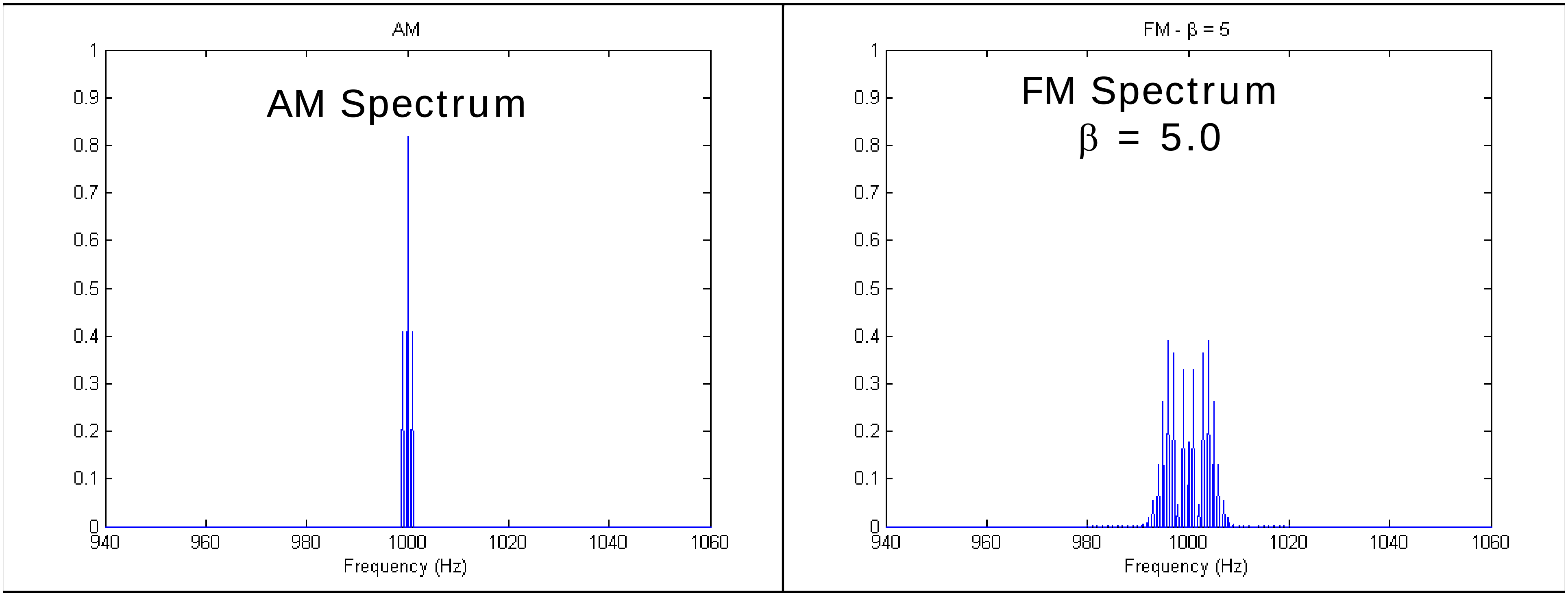
Example 14.4 – cont.

- $\beta = 2.0$ ($f_m = 2.5$Hz)



Example 14.4 – cont.

- $\beta = 5.0$ ($f_m = 1$Hz)



Summary

- Today we have investigated two forms of frequency modulation, namely narrowband FM and wideband FM
  - Narrowband FM approaches linear modulation and is similar to AM
  - Wideband FM is decidedly non-linear and occupies a band much larger than the bandwidth of the message.
- We restricted ourselves to a sinusoidal message signal since the non-linear nature makes analysis extremely difficult.