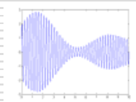
ECE3614
Introduction to
Communications Systems
Fall 2007

Instructor: Dr. R. Michael Buehrer

Lecture #15: Spectral Characteristics of Frequency Modulated Signals



Overview

- Today we continue our discussion of the spectral characteristics of FM signals
  - For a general measure of bandwidth, we can use Carson's Rule
  - For a more detailed description of the spectrum, we rely on approximations
  - Narrowband and wideband FM have very different behavior and thus we will examine them separately.
- Reading
  - Section 4.6

Angle Modulation

- Phase Modulation: $s_{PM}(t) = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$
- Frequency Modulation: $s_{FM}(t) = A_c \cos\left(2\pi f_c t + k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda\right)$

where

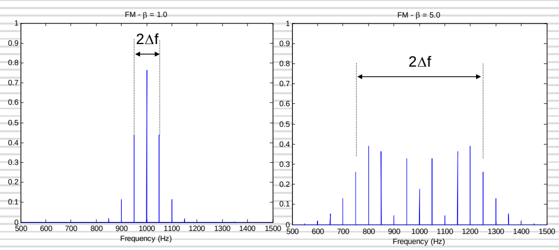
- $m(t)$ - message signal
- A_c - signal amplitude
- f_c - carrier frequency
- k_p - phase sensitivity constant (radians/volt)
- k_f - frequency deviation constant (radians/volt-second)

Exact Calculation of FM Bandwidth

- Because FM modulation is nonlinear, calculation of bandwidth is extremely difficult.
- Book shows example for $m(t) = \sin(2\pi f_m t)$
  - This is a standard textbook example
  - Result is somewhat complex and is in terms of the Bessel function
- The problem with this type of example
  - Doesn't tell us what the spectrum is for practical signals
  - Even adding another sinusoid is difficult because of nonlinear terms
- We would like a simple approximate formula for bandwidth

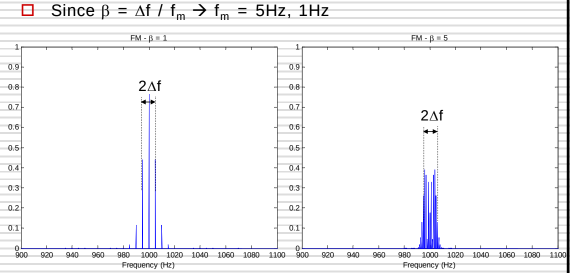
Example

- Recall our previous example for a sinusoidal message with f_c = 1kHz and f_m = 50Hz, β=1, β=5
- Since $\beta = \Delta f / f_m \rightarrow \Delta f$ = 50Hz, 250Hz



Example – cont.

- Further for a sinusoidal message with f_c = 1kHz if we hold Δf constant at Δf = 5Hz and let β=1, β=5
- Since $\beta = \Delta f / f_m \rightarrow f_m$ = 5Hz, 1Hz



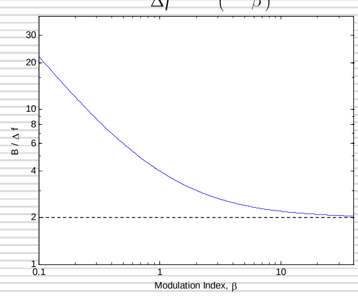
Bandwidth

- We can see that while the absolute bandwidth may be infinite, the number significant frequency components (i.e., the practical bandwidth) is finite and related to Δf.
- Increasing f_m increases bandwidth, but, $2\Delta f$ is a better measure for the significant bandwidth for higher values of β.
- This lead to Carson's rule for bandwidth of a sinusoidally modulated FM signal:

$$BW_{FM} \approx 2\Delta f + 2f_m = 2\Delta f\left(1+\frac{1}{\beta}\right)$$

Universal BW - Sinusoidal Message

$$\frac{BW_{FM}}{\Delta f} \approx 2\left(1+\frac{1}{\beta}\right)$$



99% Bandwidth

- Recall that the spectrum for a sinusoidal modulated FM wave is

$$S(f) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\left\{\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right\}$$

- The power in the signal is determined as

$$P = A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta)$$

- We can determine the number of side frequencies which contain 99% of the total power as

$$0.99P = A_c^2 \sum_{n=-k}^{k} J_n^2(\beta)$$

$$0.99 = J_0^2(\beta) + 2\sum_{n=1}^{k} J_n^2(\beta)$$

99% Bandwidth

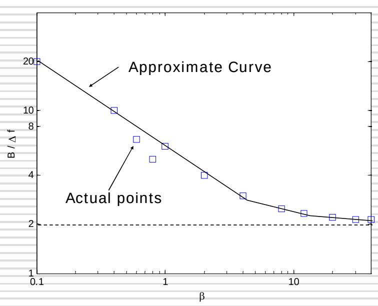
- The 99% bandwidth is then simply

$$B_{99} = 2kf_m$$

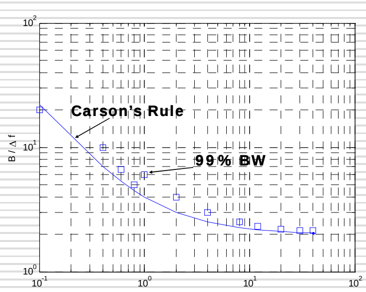
- The normalized 99% bandwidth is then

$$\bar{B}_{99} = \frac{2kf_m}{\Delta f} = \frac{2k}{\beta}$$

99% Bandwidth Curve



99% BW vs. Carson's Rule



- Carson's rule tends to slightly under-estimate bandwidth as compared to the 99% BW
- However, Carson's rule still provide a reasonable estimate

Carson's Rule – General Signals

- Carson's rule as described applies to sinusoidal signals
- However, we can define a ratio similar to the modulation index for general message signals
- Deviation Ratio *D:*

$$D = \frac{\Delta f}{W}$$

where Δf is the highest instantaneous frequency deviation and *W* is the bandwidth of the message signal.

- The generalized Carson Rule is then written as

$$BW_{FM} = 2\Delta f + 2W$$

Carson's Rule – General Signals

- Approximately 98% of the power in an FM signal lies within the bandwidth:

$$BW_{FM} = 2\Delta f + 2W = 2(D+1)W$$

- This formula provides a very good approximation to bandwidth
  - No information concerning the shape of spectrum
- Interpretation:

$$BW_{FM} = 2(D+1)W = 2\left(\frac{\Delta f}{W}+1\right)W = 2\Delta f + 2W$$

  - Bandwidth corresponds to frequency deviation around carrier frequency plus bandwidth of message signal
  - Deviation ratio tells us the extra bandwidth required by the FM signal (compared to 2*W* required by DSBSC)

Wideband and Narrowband FM

- Depending on value of *D* FM performs quite differently
  - Identical to changing β for sinusoidal message signals
- Wideband FM
  - *D* > 1
  - We will usually consider this case
- Narrowband FM
  - D < 0.2
  - Behaves similarly to AM
- There is an in-between region: 0.2 < *D* < 1
  - For simplicity, we will consider *D* < 1 to be **narrowband**

Spectrum - Narrowband FM

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda\right)$$

- For $\theta(t) << 1 \rightarrow$ $\cos(\omega_c t + \theta(t)) \approx \cos(\omega_c t) - \theta(t)\sin(\omega_c t)$
- Small *D* implies small k_f. Therefore for *D* << *1*

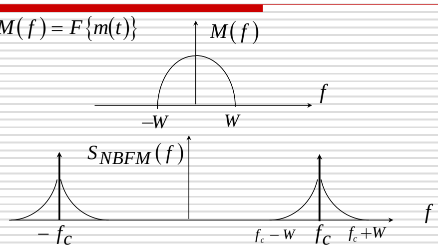
$$s_{NBFM}(t) = A_c \cos(2\pi f_c t) - \left(A_c k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda\right)\sin(2\pi f_c t)$$

  - This is resembles large carrier AM except that the message is in quadrature
- Spectrum: $S_{NBFM}(f) = \frac{A_c}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right]$

$$+\frac{A_c k_f}{2}\left[\frac{1}{(f-f_c)}M(f-f_c) + \frac{1}{(f+f_c)}M(f+f_c)\right]$$

Example of NBFM Spectrum

- $M(f) = F\{m(t)\}$



- BW of NBFM: 2*W* (close to Carson's Rule)

Carson's Rule: $BW_{FM} = 2(D+1)W \approx 2W$
since $D << 1$

Spectrum of Wideband FM

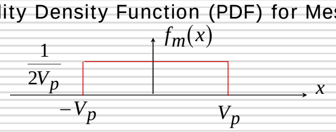
- The instantaneous frequency of the signal is directly proportional to the value of *m*(*t*)
- If we know the **probability distribution function** (pdf) of *m*(*t*) we can determine the average spectral properties of *s*(*t*) – (i.e., power spectral density)

$$S_{WBFM}(f) = \frac{A_c^2}{4k_f}\left[p_m\left(\frac{1}{k_f}(f-f_c)\right) + p_m\left(\frac{1}{k_f}(-f-f_c)\right)\right]$$

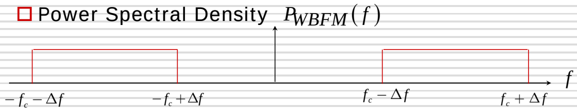
  - where $p_m(x)$ is the pdf of the message signal
- Careful!
  - PDFs and PSDs are two very different things
  - *For this case only, the PSD is proportional to the PDF*

Example of PSD for WBFM

- Probability Density Function (PDF) for Message Signal



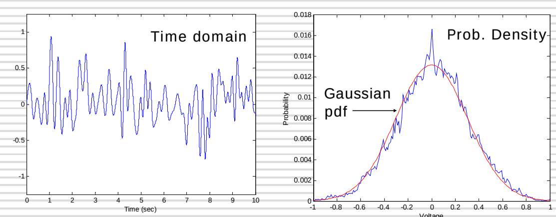
- Power Spectral Density $P_{WBFM}(f)$



Carson's Rule: $BW_{FM} = 2\Delta f + 2W \approx 2\Delta f$
since $W \ll \Delta f \rightarrow D \gg 1$

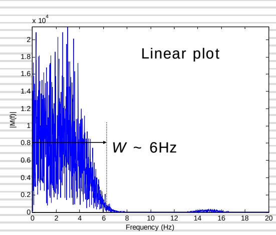
Example A

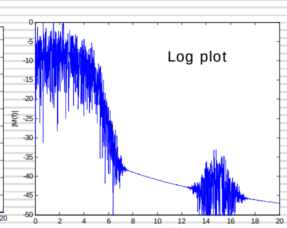
- Consider the following message signal



Example A – cont.

- Message Signal – Frequency domain



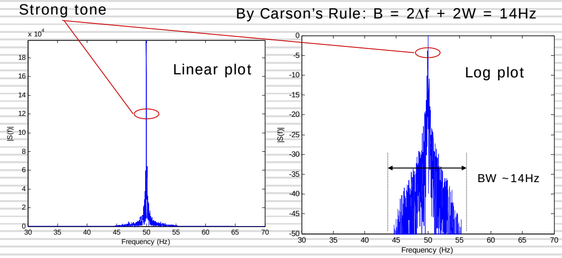


Example A – Narrowband FM Signal

- For D = 0.16 and Δf = 1

Strong tone

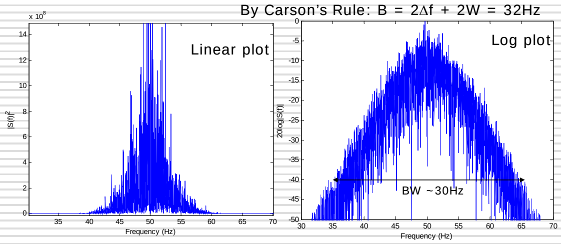
By Carson's Rule: $B = 2\Delta f + 2W$ = 14Hz



Example A – Wideband FM Signal

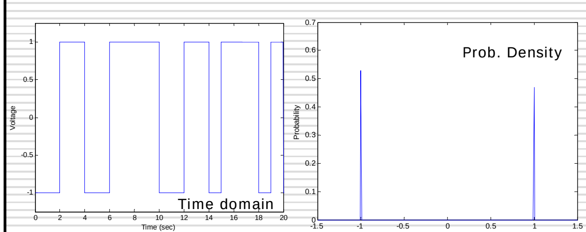
- For D= 1.6 and Δf = 10

By Carson's Rule: $B = 2\Delta f + 2W$ = 32Hz



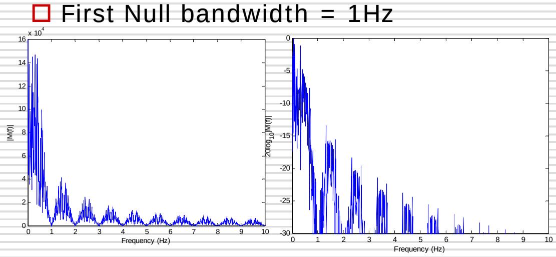
Example B – Digital Message

- Message Signal



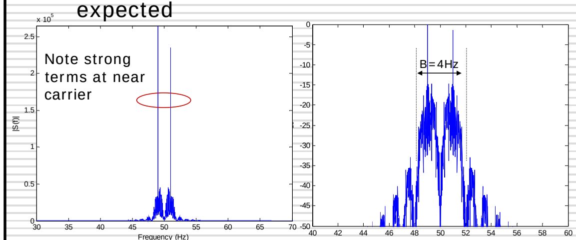
Example B – Digital Message

- Message Spectrum
- First Null bandwidth = 1Hz



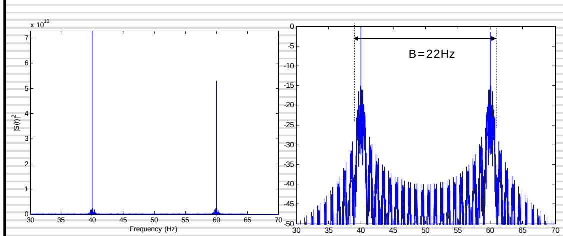
Example B – Narrowband FM

- Δf = 1Hz, W = 1Hz
- Carson's Rule: B = 2Δf + 2W = 4Hz
- Spectrum resembles message spectrum as expected



Spectrum – Wideband FM

- Δf = 10Hz, W = 1Hz
- Carson's Rule: B = 2Δf + 2W = 22Hz
- Spectrum more closely resembles the pdf



Summary

- If all you need is decent measure of BW, use Carson's Rule
- If you need to determine a better approximation of the spectrum, determine from *D* whether you have WBFM or NBFM
  - If the signal is narrowband, use spectrum of the message signal to approximate the spectrum of the FM signal, similar to AM
  - If the signal is wideband, use the probability density function to approximate the spectrum (PSD)