ECE3614
Introduction to Communications Systems
Fall 2007

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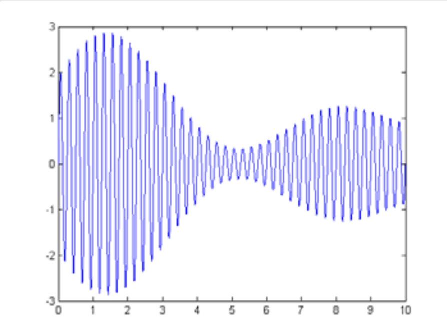
Lecture #20: Noise in FM/PM Receivers



Overview

- Today we will examine the performance (in terms of the output or post detection SNR) of FM and PM systems
- We will show that FM systems provide for better SNR gains when comparing post-detection SNR to pre-detection SNR than PM systems since the amount of phase deviation in PM systems is limited compared to frequency modulated systems
- We will also examine a concept known as *pre-emphasis* that further allows for SNR improvements for FM systems
- In general FM allows for a trade-off between SNR performance and bandwidth unlike AM systems
- Reading
  - 9.7-9.8

Angle Modulation

- Phase Modulation:

$$s_{PM}(t) = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$$

- Frequency Modulation:

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda\right)$$

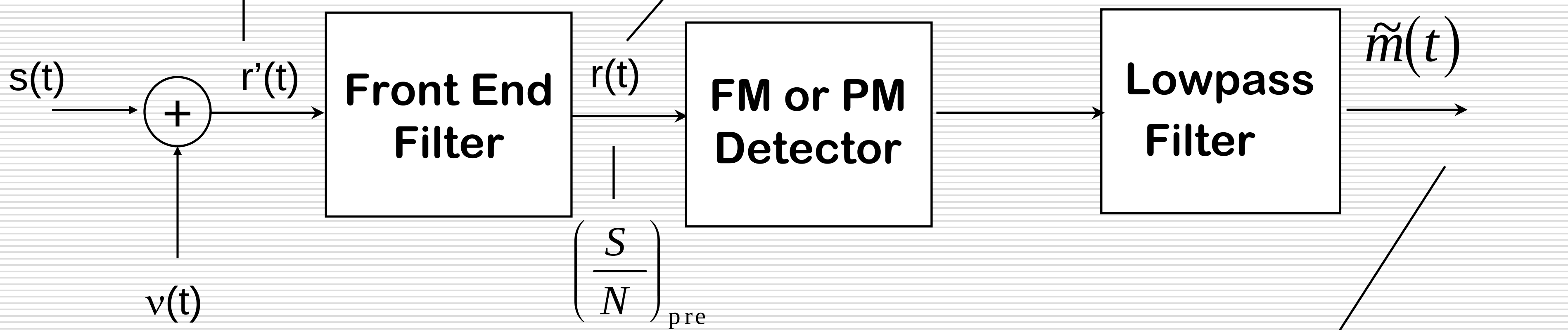
  - where

  - *m(t)* - message signal
  - A_c - signal amplitude
  - f_c - carrier frequency
  - k_p - phase sensitivity constant (radians/volt)
  - k_f - frequency deviation constant (radians/volt-second)

Receiver for FM/PM

$$r'(t) = \underbrace{s(t)}_{\text{FM or PM}} + \underbrace{\nu(t)}_{\text{AWGN}}$$

$$r(t) = \underbrace{s(t)}_{\text{FM or PM}} + \underbrace{n(t)}_{\text{bandpass AWGN}}$$



FM $\tilde{m}(t) = \frac{1}{2\pi k_f} \cdot \frac{d\angle r(t)}{dt} = \frac{1}{2\pi k_f} \cdot \frac{d}{dt}\phi_T(t) = m(t) + n_{FM}(t)$

PM $\tilde{m}(t) = \angle r(t) = \phi_T(t) = m(t) + n_{PM}(t)$

Pre-detection SNR

- For FM or PM the pre-detection SNR can be calculated as

$$S = \frac{A_c^2}{2}$$

$$N = \frac{N_o}{2} \cdot 2B_T = N_o B_T$$

$$\left(\frac{S}{N}\right)_{\text{pre}} = \frac{A_c^2}{2N_o B_T}$$

SNR for PM/FM

After front end filtering

$$r(t) = \underbrace{A_c \cos\left(2\pi f_c t + \phi(t)\right)}_{signal} + \underbrace{n(t)}_{noise}$$

Bandpass noise

$$n(t) = n_I(t)\cos\left(2\pi f_c t\right) - n_Q(t)\sin\left(2\pi f_c t\right)$$

$$= R_n(t)\cos\left(2\pi f_c t + \phi_n(t)\right)$$

$$\phi_T(t) = \phi(t) + \tan^{-1}\left\{\frac{R_n(t)\sin\left(\phi_n(t) - \phi(t)\right)}{A_c + R_n(t)\cos\left(\phi_n(t) - \phi(t)\right)}\right\}$$

For large input SNR, $A_c >> R_n(t)\cos\left(\theta_n(t)\right)$ $A_c >> R_n(t)\sin\left(\theta_n(t)\right)$

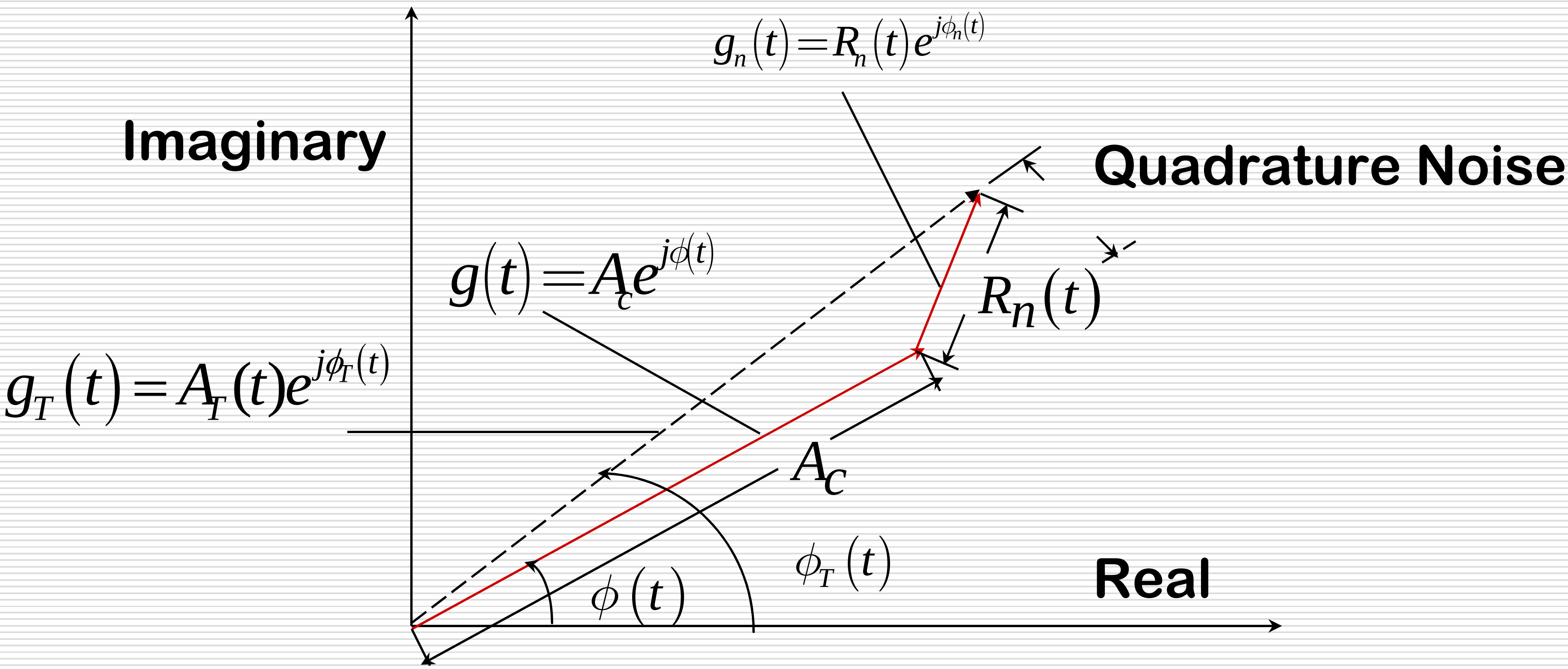
$$\phi_T(t) \approx \underbrace{\phi(t)}_{signal} + \underbrace{\frac{R_n(t)}{A_c}\sin\left(\phi_n(t) - \phi(t)\right)}_{noise} = \phi(t) + \frac{n_Q(t)}{A_c}$$

Signal Plus Noise for Angle Modulation

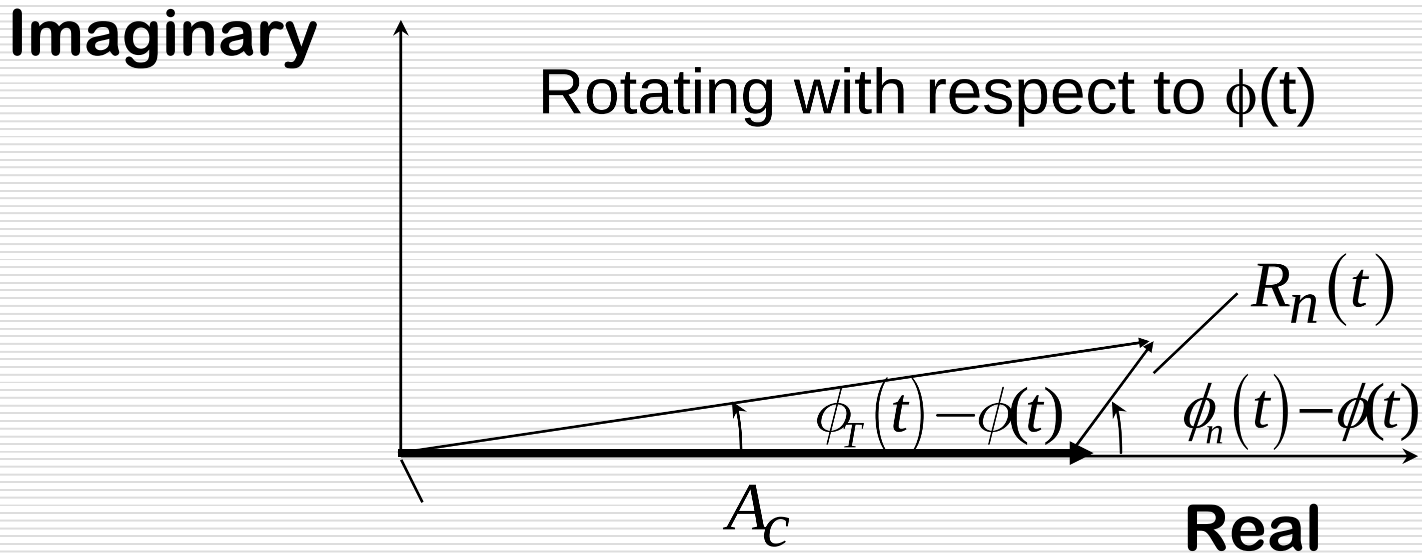
Complex baseband representation

$$r(t) = \mathrm{Re}\left\{g_T(t)e^{j2\pi f_c t}\right\} \quad n(t) = \mathrm{Re}\left\{g_n(t)e^{j2\pi f_c t}\right\} \quad s(t) = \mathrm{Re}\left\{g(t)e^{j2\pi f_c t}\right\}$$

$$g_T(t) = A_T(t)e^{j\phi_T(t)} \quad g_n(t) = R_n(t)e^{j\phi_n(t)} \quad g(t) = A_c e^{j\phi(t)}$$



Signal Plus Noise for Angle Modulation



$$\phi_T(t)-\phi(t) = \tan^{-1}\left\{\frac{y_T(t)}{x_T(t)}\right\}$$

$$= \tan^{-1}\left\{\frac{R_n(t)\sin\left(\phi_n(t)-\phi(t)\right)}{A_c + R_n(t)\cos\left(\phi_n(t)-\phi(t)\right)}\right\}$$

SNR for PM

For PM: $$\phi(t) = k_p m(t)$$

Desired signal power *S* is:

$$S = k_p^2 \overline{m^2(t)}$$

$$= D_p^2 \overline{\left(\frac{m(t)}{V_p} \right)^2}$$

$$\boxed{D_p = k_p V_p}$$

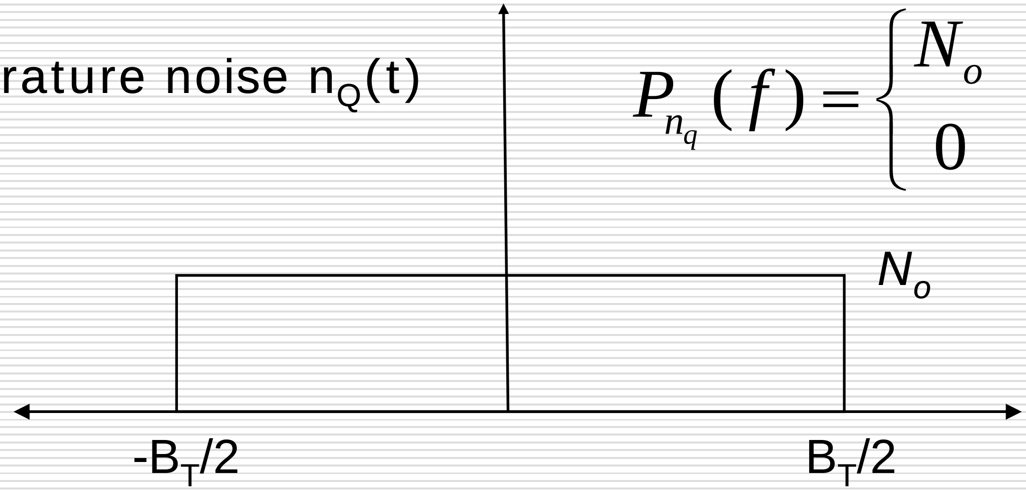
The noise power spectral density can be determined from the PSD of $n_Q(t)$:

$$n_{PM}(t) = \frac{n_Q(t)}{A_c}$$

Noise

PSD of Quadrature noise $n_Q(t)$
$N_Q = N_o B_T$

$$P_{n_q}(f) = \begin{cases} N_o & |f| \le B_T/2 \\ 0 & else \end{cases}$$



Now, since the noise is uniformly distributed in angle, the offset angle $\phi(t)$ does not change the statistics of $R_n(t)\sin(\phi_n(t)-\phi(t))$ from those of $R_n(t)\sin(\phi_n(t))$. Thus,

$$P_n(f) = \begin{cases} \dfrac{1}{A_c^2} N_o & |f| \le B_T/2 \\ 0 & else \end{cases}$$

SNR for PM

Noise power *N* after low pass filter is:

$$N = \int_{-W}^{W} P_n(f)\,df$$

$$= \frac{2WN_o}{A_c^2}$$

Note that increasing the signal amplitude decreases the noise power.

The SNR is then:

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{D_p^2 \overline{\left(\frac{m(t)}{V_p}\right)^2}}{\frac{2WN_o}{A_c^2}}$$

$$= \frac{A_c^2 D_p^2 \overline{\left(\frac{m(t)}{V_p}\right)^2}}{2WN_o}$$

SNR for FM

Now for FM we have $$\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\lambda) d\lambda$$

At the output of the FM detector we have (for large SNR_{in})

$$\frac{1}{2\pi}\frac{d}{dt}\phi_T(t) = k_f m(t) + \frac{1}{2\pi}\frac{d}{dt}\left\{\frac{n_Q(t)}{A_c}\right\}$$

Thus, the desired signal power is:

$$S = k_f^2 \overline{m^2(t)}$$

$$= D_f^2 W^2 \overline{\left(\frac{m(t)}{V_p}\right)^2}$$

$$D_f = \frac{k_f V_p}{W}$$

SNR for FM

Using the PSD from the PM case, we know that the derivative is equivalent to multiplying by $2\pi f$ in frequency:

$$P_n(f) = \begin{cases} \frac{(2\pi f)^2}{4\pi^2} \frac{1}{A_c^2} N_o & |f| \le B_T/2 \\ 0 & else \end{cases}$$

The noise power after low pass filtering is then:

$$\begin{aligned} N &= \int_{-W}^{W} P_n(f)\,df \\ &= \int_{-W}^{W} \frac{f^2}{A_c^2} N_o\,df \\ &= \left. \frac{f^3}{3A_c^2} N_o \right|_{-W}^{W} \\ &= \frac{2W^3}{3A_c^2} N_o \end{aligned}$$

Note that increasing the signal amplitude decreases the noise power.

Output SNR for FM

The output SNR is then:

$$\left(\frac{S}{N}\right)_{out} = \frac{S}{N} = \frac{\left(D_f^2 W^2 \frac{\overline{m^2(t)}}{V_p^2}\right)}{\left(\frac{2N_o W^3}{3A_c^2}\right)}$$

$$= \frac{3A_c^2 D_f^{\ 2} \overline{\left(m/V_p\right)^2}}{2N_0 W}$$

Comparison between PM and FM

- Output SNR for FM:

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{3A_c^2 D_f^2 \overline{\left(m/V_p \right)^2}}{2N_o W}$$

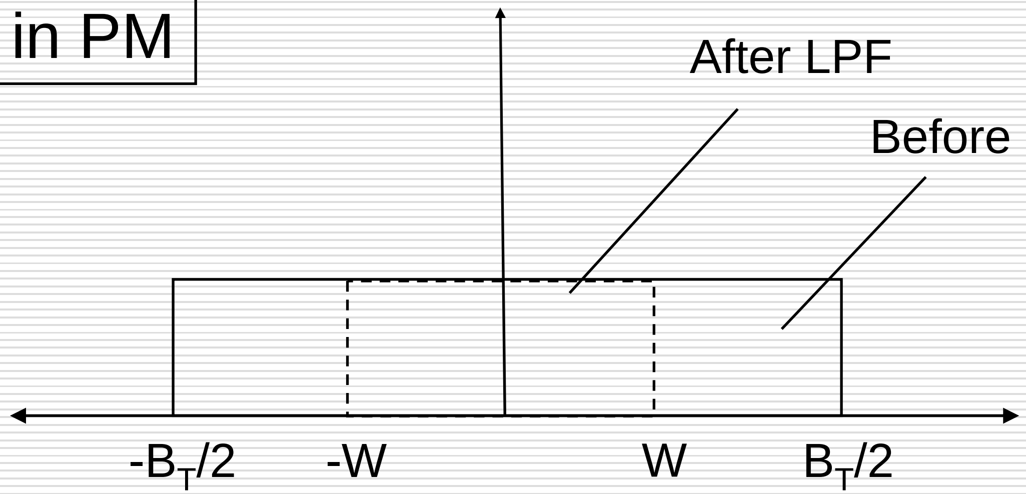
- Output SNR for PM:

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{A_c^2 D_p^2 \overline{\left(m/V_p \right)^2}}{2N_o W}$$

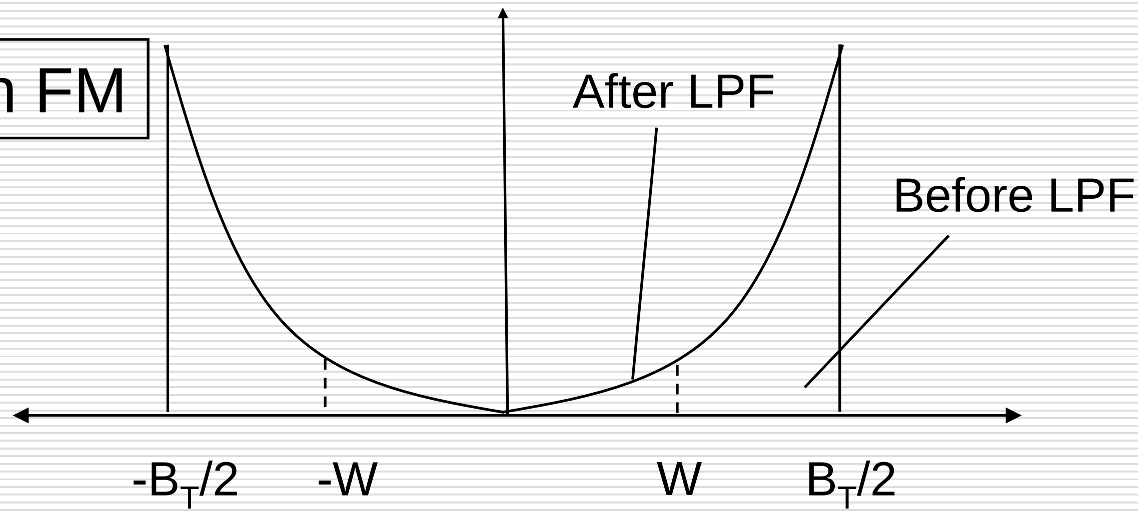
  - Factor of 3 arises because of integration of noise pdf
  - Further, D_p is more limited than D_f.
  - This is another reason for preferring FM over PM

Noise Spectrum for PM and FM

PSD for noise in PM



PSD for noise in FM



Input SNR (both PM and FM)

$$r(t) = \underbrace{A_c \cos\left(2\pi f_c t + \theta(t)\right)}_{signal} + \underbrace{n(t)}_{noise}$$

$$S = \frac{A_c^2}{2}$$

$$N = N_o B_T$$
$$= N_o 2(D+1)W$$

$$\left(\frac{S}{N}\right)_{\text{pre}} = \frac{\left(\frac{A_c^2}{2}\right)}{2N_o(D+1)W}$$
$$= \frac{A_c^2}{4N_o(D+1)W}$$

SNR Gain (both PM and FM)

FM

$$\frac{(S/N)_{\text{post}}}{(S/N)_{\text{pre}}} = \left(\frac{\dfrac{3A_c^2 D_f^{\,2} \overline{(m/V_p)^2}}{2N_0 W}}{\dfrac{A_c^2}{4N_o(D_f+1)W}} \right)$$

$$= 6D_f^{\,2}(D_f+1)\overline{(m/V_p)^2}$$

PM

$$\frac{(S/N)_{\text{post}}}{(S/N)_{\text{pre}}} = \left(\frac{\dfrac{A_c^2 D_p^2 \overline{(m/V_p)^2}}{2N_o W}}{\dfrac{A_c^2}{4N_o(D_p+1)W}} \right)$$

$$= 2D_p^{\,2}(D_p+1)\overline{(m/V_p)^2}$$

FM provides an additional factor of 3 in SNR provided that *input SNR is sufficiently high.*

SNR Gain (both PM and FM)

FM

$$\frac{(S/N)_{\text{post}}}{(S/N)_{\text{ref}}} = \left(\frac{\dfrac{3A_c^2 D_f^{\,2} \overline{(m/V_p)^2}}{2N_0W}}{\dfrac{A_c^2}{2N_oW}} \right)$$

$$= 3D_f^{\,2}\overline{(m/V_p)^2}$$

PM

$$\frac{(S/N)_{\text{post}}}{(S/N)_{\text{ref}}} = \left(\frac{\dfrac{A_c^2 D_p^2 \overline{(m/V_p)^2}}{2N_oW}}{\dfrac{A_c^2}{2N_oW}} \right)$$

$$= D_p^{\,2}\overline{(m/V_p)^2}$$

FM provides an additional factor of 3 in SNR provided that *input SNR is sufficiently high.*

$$\left(\frac{S}{N}\right)_{\text{ref}} = \frac{A_c^2}{2N_oW}$$

Role of Modulation Index in the design of FM

- SNR increases with modulation index D_f:

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{3A_c^2 D_f^{\,2}\,\overline{\left(m/V_p\right)^2}}{2N_0 W}$$

- Bandwidth increases with modulation index D_f:

$$BW_{FM} = 2\left(D_f + 1\right)W$$

- Selection of D_f allows tradeoff of bandwidth efficiency for power efficiency

SNR for FM - example

- Let's assume that FM is used
- let $m(t) = A_c \sin(\omega_c t)$

$$\overline{\left(\frac{m(t)}{V_p}\right)^2} = \frac{1}{2} \qquad \left(\frac{S}{N}\right)_{post} = \frac{3A_c^2 D_f^{\,2} \frac{1}{2}}{2N_0 W}$$

$$\left(\frac{S}{N}\right)_{post} = \frac{3}{2} D_f^{\,2} \left(\frac{S}{N}\right)_{ref}$$

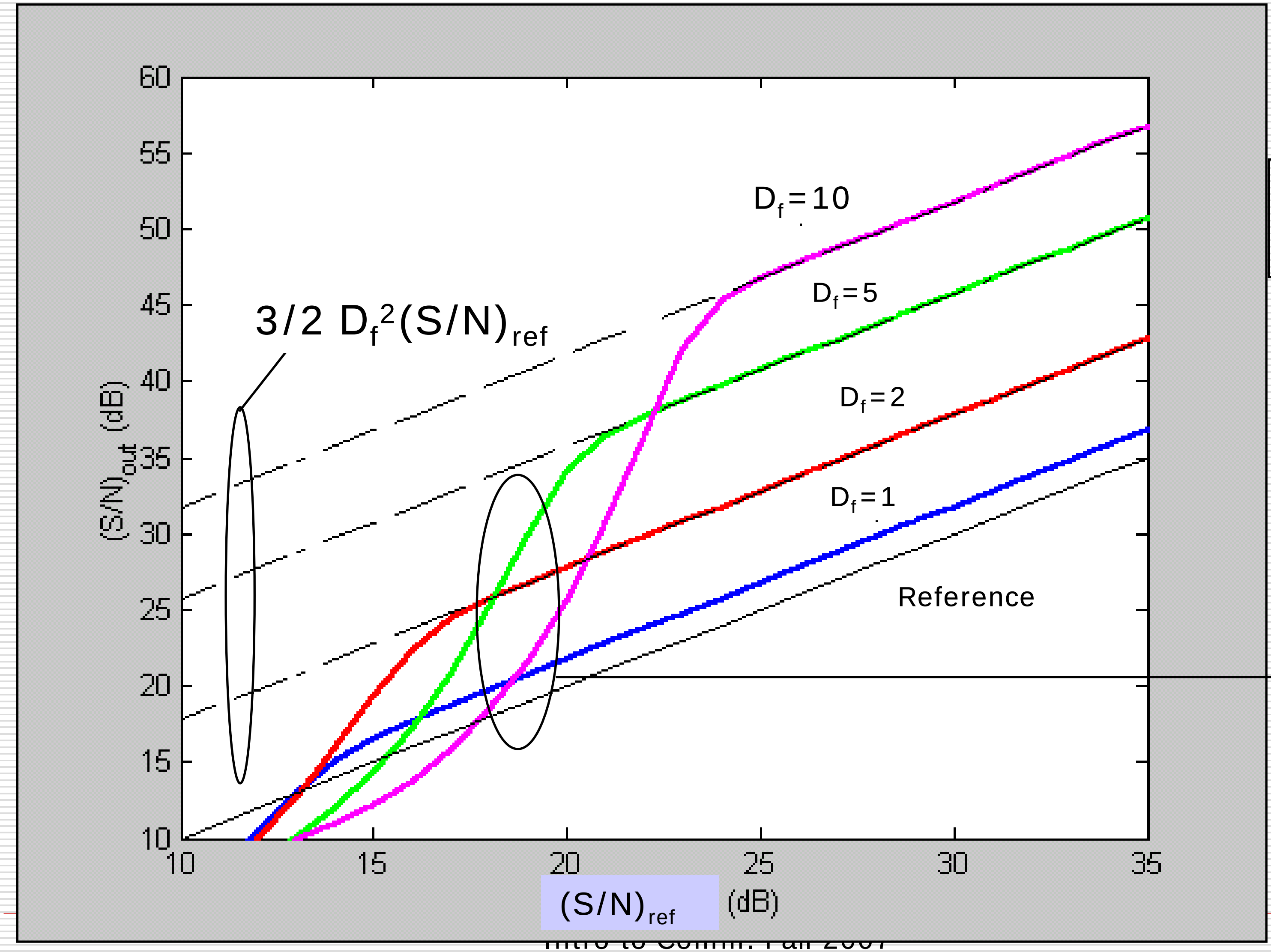
However, this assumes that SNR_{ore} is sufficiently high to allow our approximations to hold.

SNR for FM - example

- When SNR_{pre} is not sufficiently high, the output SNR drops substantially. This is called the *threshold effect.*
- It has been shown for sinusoidal modulation that the relationship between pre-detection and reference SNR when the pre-detection SNR is *not* sufficiently high is:

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{\frac{3}{2} D_f^{\,2} \left(S/N\right)_{\text{ref}}}{1 + \frac{12}{\pi}\left(S/N\right)_{\text{ref}} \, e^{\left\{-\left[\frac{1}{2(D_f+1)}\left(S/N\right)_{\text{ref}}\right]\right\}}}$$

SNR of FM Systems



Sinusoidal modulation

Threshold Effect

Threshold Effect

- We can see that SNR_{post} is a linear function of SNR_{ref} (or SNR_{pre}) only for high values of SNR_{pre}.
- Further, the required value of SNR_{pre} increases with D_f
- Thus, while SNR gain increases with D_f ,the SNR requirements also increase with D_f
- However, for constant transmit power input (pre-detection) SNR *decreases* with increasing D_f
- The bottom line is that we cannot increase D_f indefinitely to increase performance, even if we had sufficient bandwidth.
- Threshold is defined as the minimum input SNR (or baseband SNR) yielding an FM improvement that is not significantly deteriorated from the value predicted.

FM Design Example 20.1

- A given message signal has:
  - Bandwidth W=15 kHz
  - $\overline{\left(m/V_p\right)^2} = 0.1$
- The signal is transmitted using FM modulation over a channel with $N_0 = 10^{-7}$ W/Hz with a power $\frac{A_c^2}{2} = 1$ W
- If we require an output SNR of at least 30 dB for sufficient fidelity, find the smallest possible bandwidth for the resulting signal.

FM Design Example 20.1 (continued)

- Express SNR requirement as a linear value:

$$\left(\frac{S}{N}\right)_{\text{post}} \geq 30dB \Rightarrow \left(\frac{S}{N}\right)_{\text{post}} \geq 10^{30/10} = 1000$$

- Evaluate SNR expression for D_f :

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{3A_c^2 D_f{}^2 \overline{\left(m/V_p\right)^2}}{2N_0W} = \frac{3\cdot 1\cdot D_f{}^2\cdot 0.1}{10^{-7}\cdot 15000} \geq 1000$$

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{3\cdot 1\cdot D_f{}^2\cdot 0.1}{10^{-7}\cdot 15000} \geq 1000 \Rightarrow D_f{}^2 \geq 50 \Rightarrow D_f \geq 7.1$$

FM Design Example 20.1 (continued)

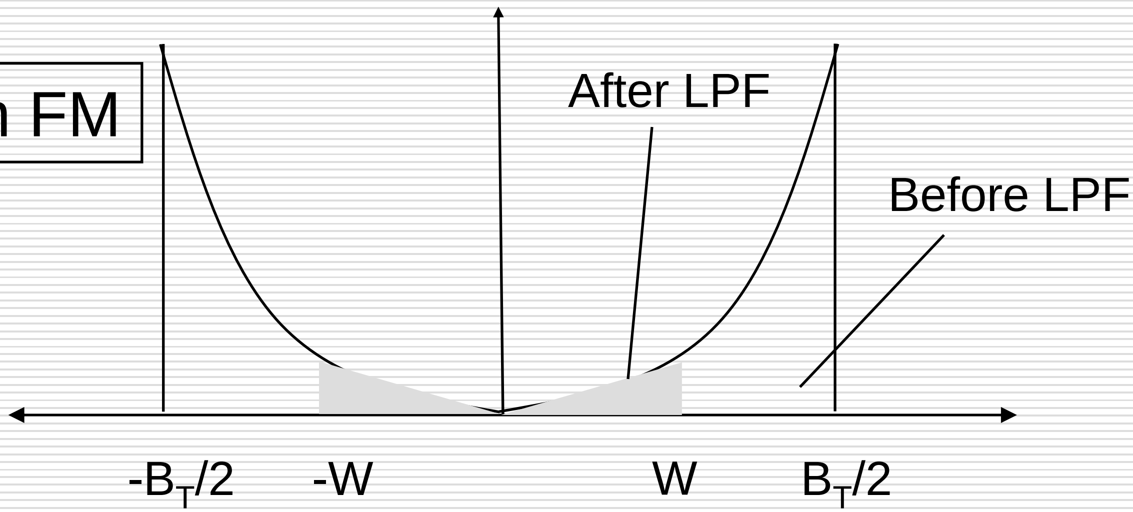
- Apply Carson's Rule to compute the final bandwidth:

$$\begin{aligned} B_{FM} &= 2\left(D_f + 1\right)B \\ &= 2\cdot\left(7.1+1\right)\cdot 15000 \\ &= 243\ \text{kHz} \end{aligned}$$

Noise Spectrum for FM

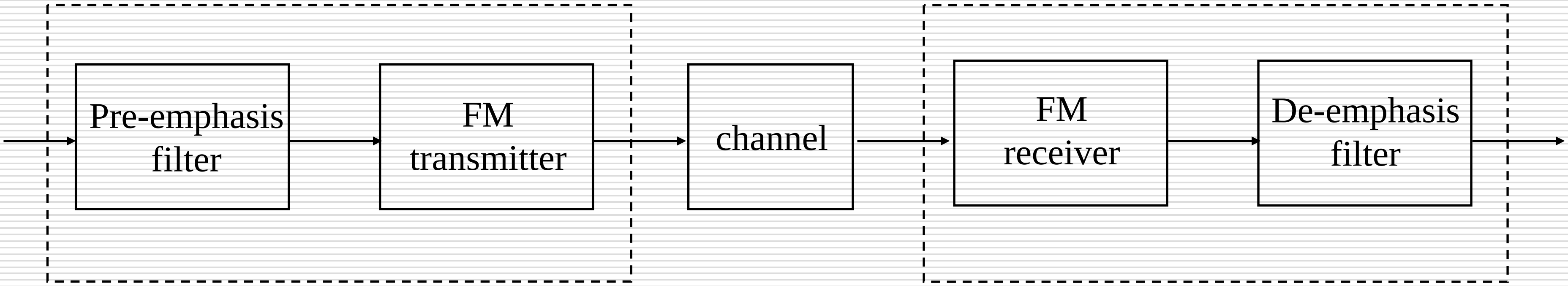
Recall that the noise PSD has a parabolic shape at the output of the FM detector:

PSD for noise in FM



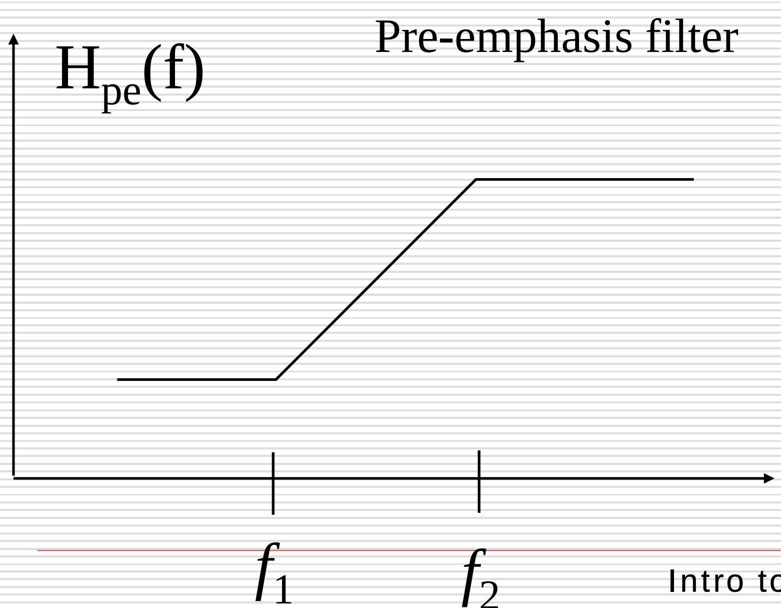
Pre-emphasis

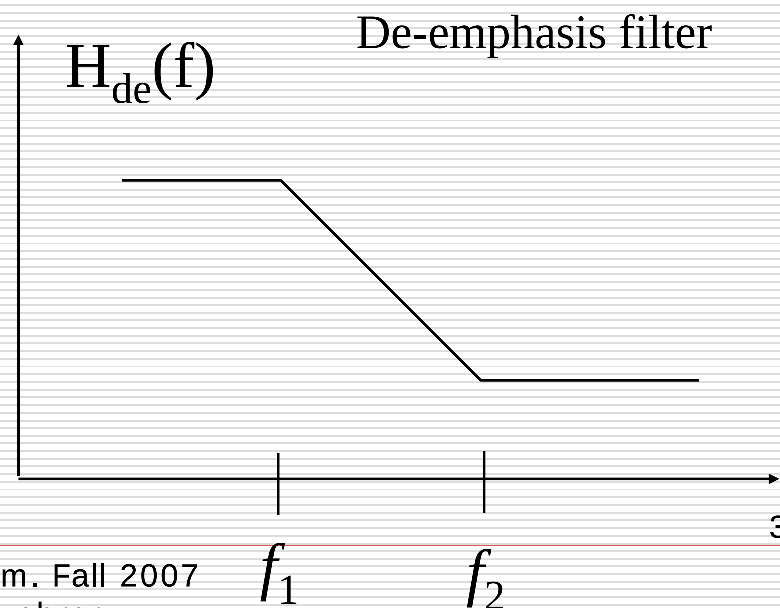
- In frequency-modulated systems, the signal-to-noise ratio can be improved if the level of modulation is boosted at the top end of the message spectrum.
- This is called *pre-emphasis* and is used in broadcast FM



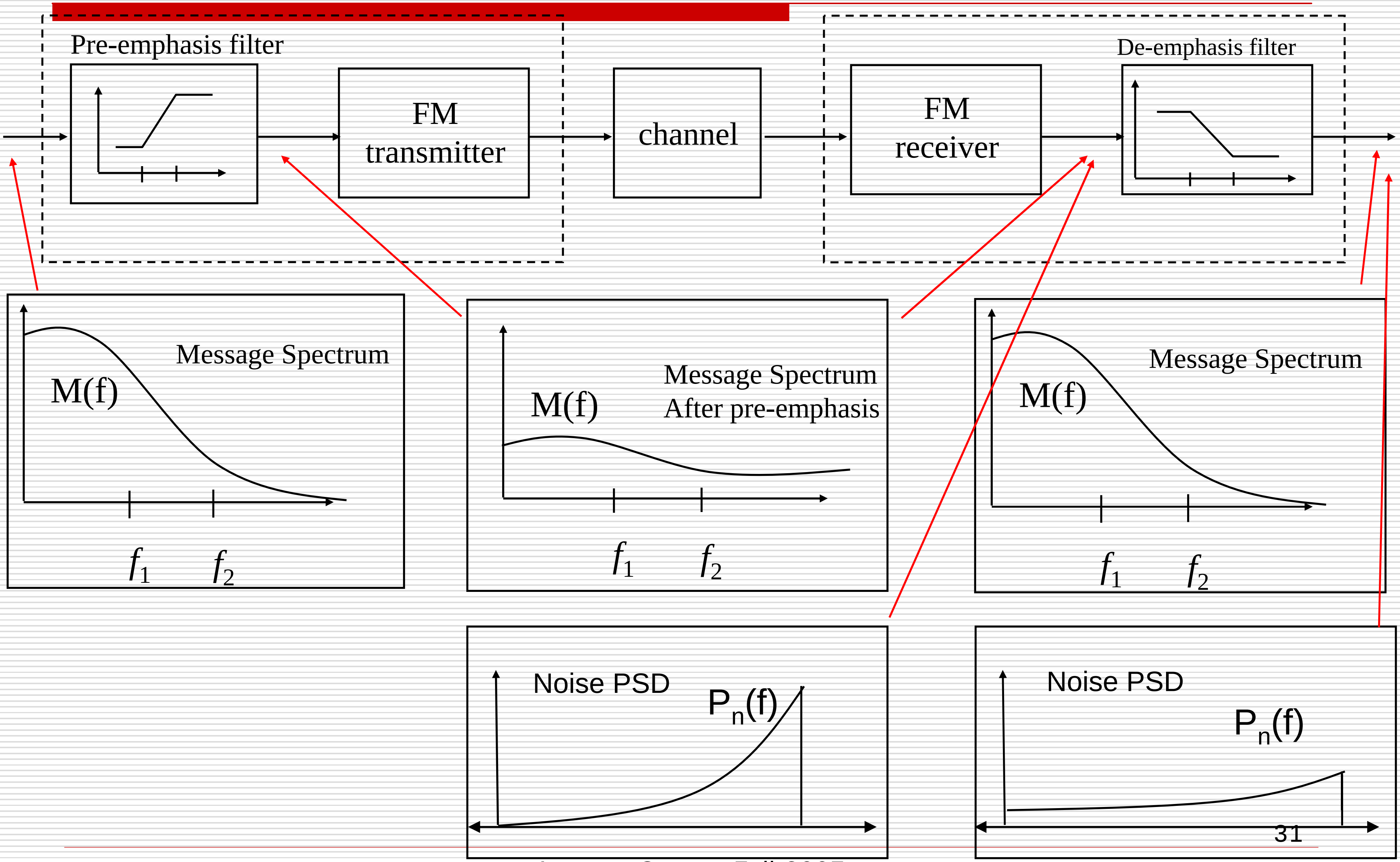
Pre-emphasis

- Pre-emphasis filter acts as a differentiator over the band f_1 to f_2. This creates *phase modulation* over that band
- We will see later that this improves the SNR substantially.
- This helps SNR because the noise PSD at the output of the FM detector is parabolic. Thus, we accentuate the signal where the noise is highest.





Pre-emphasis



SNR for FM with Pre-emphasis

For FM we have $$\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\lambda)d\lambda$$

The pre-emphasis/de-emphasis process does not impact the desired signal since the combined response is flat. Thus, the desired part of the output of the FM detector is

$$\frac{1}{2\pi}\frac{d}{dt}\phi(t) = k_f m(t)$$

Thus, the desired signal power is:

$$S = k_f^2 \overline{m^2(t)}$$

$$= D_f^2 W^2 \overline{\left(\frac{m(t)}{V_p}\right)^2}$$

$$D_f = \frac{k_f V_p}{W}$$

SNR for FM

:Let's use a de-emphasis filter that is a simple RC circuit:

$$H_{de}(f) = \frac{1}{H_{pre}(f)} = \frac{1}{1 + j(f / f_{3dB})}$$

The noise power after filtering with de-emphasis filter is:

$$N = \int_{-W}^{W} P_n(f) \left| H_{de}(f) \right|^2 df$$

$$= \int_{-W}^{W} \frac{f^2}{A_c^2} N_o \left| \frac{1}{1 + j(f / f_{3dB})} \right|^2 df$$

$$= \int_{-W}^{W} \frac{f^2}{A_c^2} N_o \frac{1}{1 + (f / f_{3dB})^2} df$$

$$= \frac{2 N_o f_{3dB}^3}{A_c^2} \left[\frac{W}{f_{3dB}} - \tan^{-1}\left(\frac{W}{f_{3dB}} \right) \right]$$

$$\approx \frac{2 N_o f_{3dB}^2}{A_c^2} W$$

Output SNR for FM

The output SNR is then:

$$\left(\frac{S}{N}\right)_{\text{post}} = \frac{S}{N} = \frac{\left(D_f^2 W^2 \frac{\overline{m^2(t)}}{V_p^2}\right)}{\left(\frac{2N_o f_{3dB}^2}{A_c^2} W\right)}$$

$$= \frac{A_c^2 D_f^{\,2} \left(W \big/ f_{3dB} \right)^2 \overline{\left(m/V_p\right)^2}}{2N_0 W}$$

Input SNR

$$r(t) = \underbrace{A_c \cos\left(2\pi f_c t + \theta(t)\right)}_{signal} + \underbrace{n(t)}_{noise}$$

$$S = \frac{A_c^2}{2}$$

$$N = N_o B_T$$
$$= N_o 2(D_f + 1)B$$

$$\left(\frac{S}{N}\right)_{pre} = \frac{\left(\frac{A_c^2}{2}\right)}{2N_o(D_f + 1)W} = \frac{A_c^2}{4N_o(D_f + 1)W}$$

SNR Gain

$$\frac{(S/N)_{post}}{(S/N)_{pre}} = \left(\frac{\dfrac{A_c^2 D_f^{\ 2} \left(W / f_{3dB} \right)^2 \overline{\left(m/V_p \right)^2}}{2N_0 W}}{\dfrac{A_c^2}{4N_o (D_f + 1) W}} \right)$$

$$= 2D_f^{\ 2} (D_f + 1) \left(W / f_{3dB} \right)^2 \overline{\left(m/V_p \right)^2}$$

Without De-emphasis: $\dfrac{(S/N)_{post}}{(S/N)_{pre}} = 6D_f^{\ 2} (D_f + 1) \overline{\left(m/V_p \right)^2}$

SNR Gain

$$\frac{(S/N)_{post}}{(S/N)_{ref}} = \left(\frac{\dfrac{A_c^2 D_f^{\,2} \left(W/f_{3dB} \right)^2 \overline{\left(m/V_p \right)^2}}{2N_0 W}}{\dfrac{A_c^2}{2N_o W}} \right)$$

$$= D_f^{\,2} \left(W/f_{3dB} \right)^2 \overline{\left(m/V_p \right)^2}$$

De-emphasis provides significant benefit provided the input SNR is sufficiently large and $\frac{W}{f_{3dB}} > \sqrt{3}$

$$\left(\frac{S}{N} \right)_{ref} = \frac{A_c^2}{2N_o W}$$

Example 20.2

- Standard FM broadcasting uses D_f = 5, W = 15kHz and f_{3dB} = 2.1kHz

$$\frac{(S/N)_{post}}{(S/N)_{ref}} = D_f^{\,2}\left(W/f_{3dB}\right)^2 \overline{\left(m/V_p\right)^2}$$

$$= 25 * \left(15000/2100\right)^2 \overline{\left(m/V_p\right)^2}$$

$$= 1275\overline{\left(m/V_p\right)^2}$$

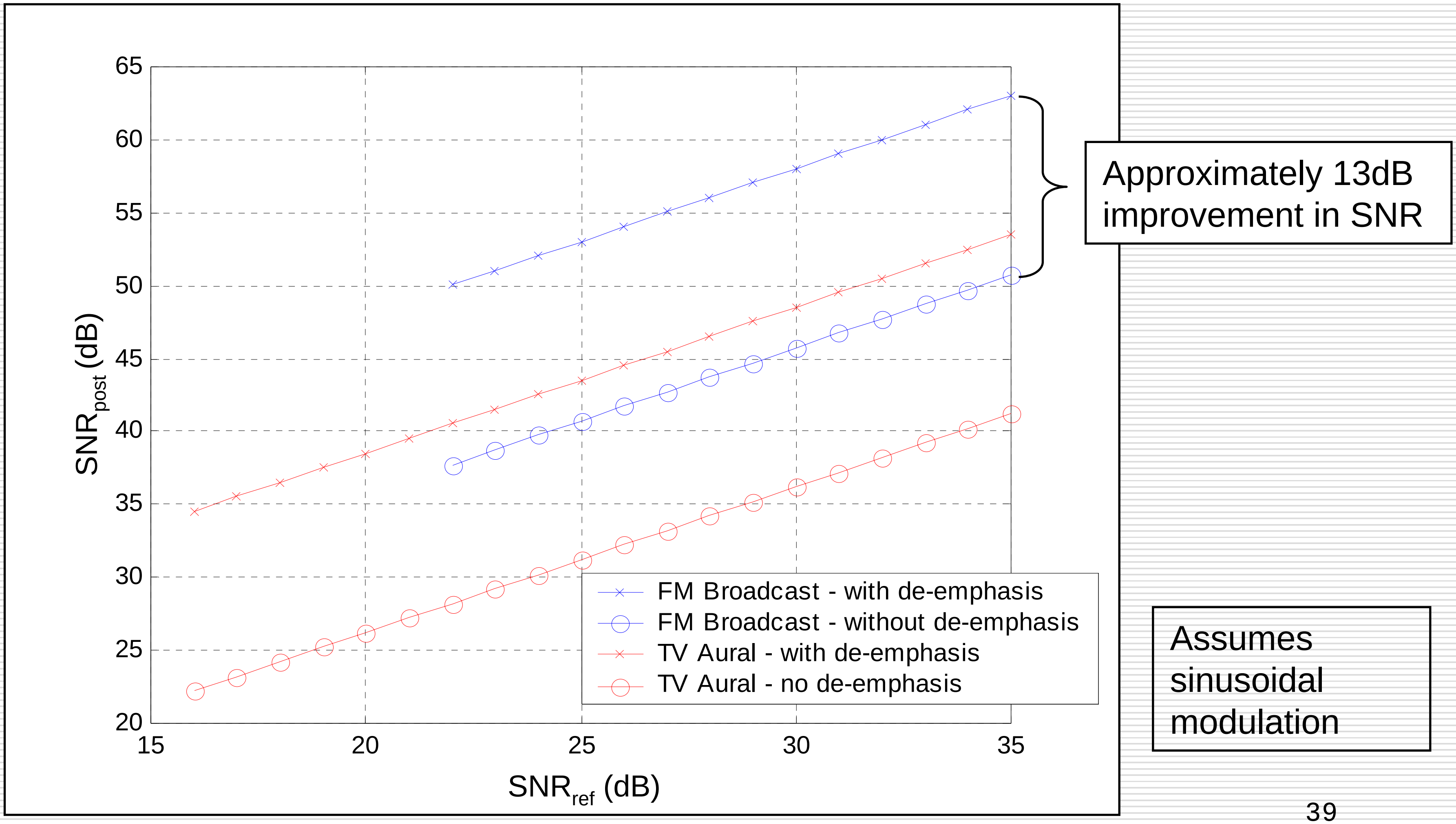
- TV aural broadcasting uses D_f = 1.67, W = 15kHz and f_{3dB} = 2.1kHz

$$\frac{(S/N)_{post}}{(S/N)_{ref}} = D_f^{\,2}\left(W/f_{3dB}\right)^2 \overline{\left(m/V_p\right)^2}$$

$$= 2.78 * \left(15000/2100\right)^2 \overline{\left(m/V_p\right)^2}$$

$$= 142\overline{\left(m/V_p\right)^2}$$

Broadcast Examples



Assumes sinusoidal modulation

Comparing Analog Modulation

Modulation Type	Tx Bandwidth	$\frac{SNR_{post}}{SNR_{ref}}$	Comments
AM	$2W$	$\frac{k_a^2 \overline{m^2}}{1+k_a^2 \overline{m^2}}$	Valid for any SNR_{in} and coherent detection. Must be above threshold for envelope detection
DSB-SC	$2W$	1	Coherent detection required
SSB	W	1	Coherent detection required
PM	$2(D_p+1)W$	$D_p^2 \overline{\left(\frac{m}{V_p}\right)^2}$	Coherent detection required; valid for SNR_{pre} above threshold
FM	$2(D_f+1)W$	$3D_f^2 \overline{\left(\frac{m}{V_p}\right)^2}$	Valid for SNR_{pre} above threshold
FM with demphasis	$2(D_f+1)W$	$D_f^2 \left(\frac{B}{f_1}\right)^2 \overline{\left(\frac{m}{V_p}\right)^2}$	Valid for SNR_{pre} above threshold
Baseband	W	1	No Modulation

Ideal Performance

- FM shows us that we can trade bandwidth for performance.
- De-emphasis shows that this trade-off can be improved over standard FM
- What is the most improvement that we can obtain?
- We can answer this using Shannon's Channel Capacity Theorem

$$C = W \log_2\left(1 + \frac{S}{N}\right)$$

Ideal Performance (cont.)

- An ideal system is one in which there is no capacity lost in the detection process (*i.e.*, $C_{pre} = C_{post}$)

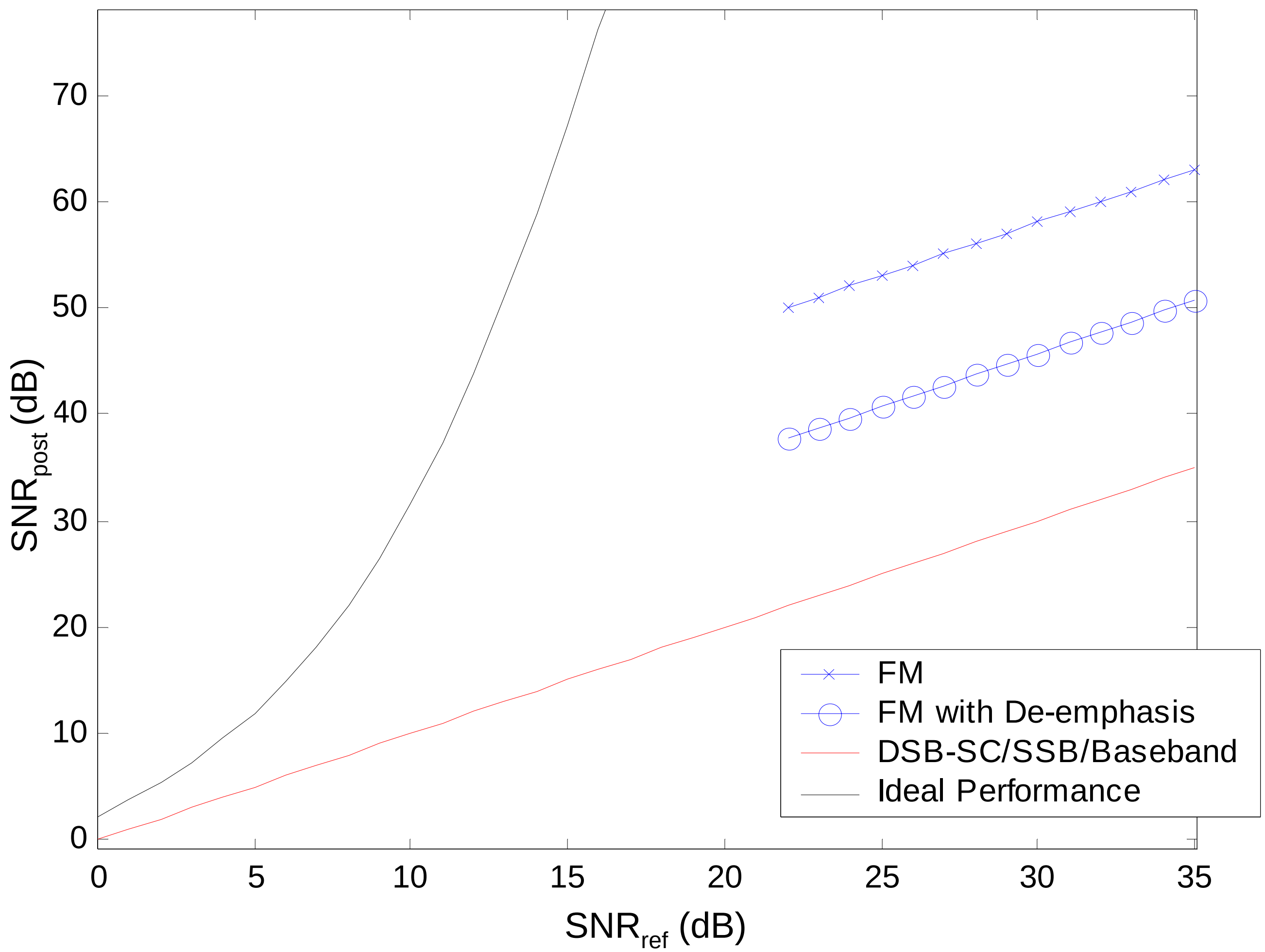
$$C_{pre} = C_{post}$$

$$B_T \log_2\left(1+\left(\frac{S}{N}\right)_{pre}\right) = W \log_2\left(1+\left(\frac{S}{N}\right)_{post}\right)$$

- Solving for C_{out}:

$$\left(\frac{S}{N}\right)_{post} = \left(1+\left(\frac{S}{N}\right)_{pre}\right)^{B_T/W} - 1$$

Example



$B_T/B = 12$

$D_f = 5$

$W/f_{3dB} = 7.1$

$$\left(\frac{S}{N}\right)_{pre} = \frac{P_s}{N_o B_T} = \frac{W}{B_T}\frac{P_s}{N_o W} = \frac{W}{B_T}\left(\frac{S}{N}\right)_{ref}$$

Summary

- Today we have examined the performance (in terms of the output or post detection SNR) of FM and PM systems
- FM systems allow for a trade-off between SNR performance and bandwidth unlike AM systems for which both are relatively fixed
  - However, we cannot increase bandwidth arbitrarily in hope of improving SNR performance
- We have also examined the concept known as *pre-emphasis* which further allows for SNR improvements for FM systems