ECE3614
Introduction to
Communications Systems
Fall 2007

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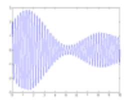
Lecture #22: Digital Communications – Quantization and Waveform Encoding

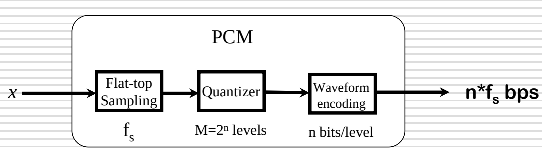


Overview

- Last class we introduced the concept of digital communications
- Digital communication of analog message signals requires analog-to-digital conversion which includes
  - Sampling
  - Quantization
  - Waveform encoding
- Today we discuss quantization and waveform encoding
- Reading
  - 5.4 – 5.5

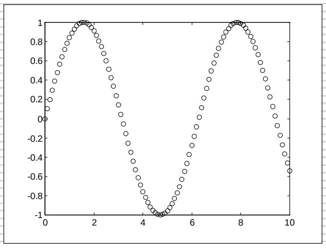
Pulse Code Modulation (PCM)

- Pulse Code Modulation refers to a system that creates a digital baseband signal from an analog signal using sampling, quantization and waveform encoding.



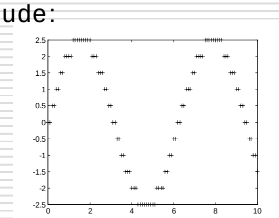
Digital Representation of Analog Signals

- Sampling analog signals makes them discrete in time:



Digital Representation of Analog Signals

- Quantization of sampled analog signals makes the samples discrete in amplitude:



Quantization

- Continuous time signals are sampled at discrete time intervals
- Sampling may be performed without distortion provided signal is sampled at Nyquist rate
- Continuous-valued samples of data require an infinite # of bits to represent with perfect precision.
- Quantization is the process of approximating continuous-valued samples with a finite # of bits.
- Quantization always introduces some distortion.

Notation Associated with Quantization

- Let X be a random variable representing a sample of data.
- Then $\tilde{X} = f_Q(X)$ is the quantized value of X.
- A quantizer has M quantization levels:

$$\tilde{X} \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_M\}$$

- The M levels correspond to M quantization regions.
- The endpoints of the quantization regions are specified by $M+1$ values: $\{x_0, x_1, \ldots, x_M\}$, where $x_0 = -\infty, x_M = \infty$
- Then: $x_{k-1} \le x < x_k \Rightarrow \tilde{x} = f_Q(x) = \tilde{x}_k$

Graphical Description of Quantization

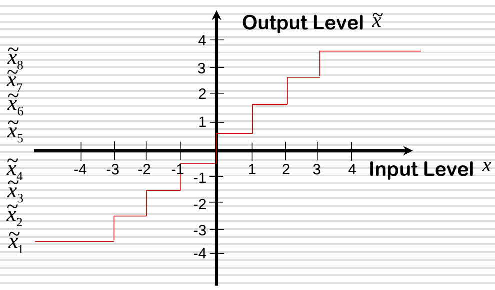


Table Representation of Quantizer

k	x_{k-1}	x_k	$\tilde{x}_k$	Output Bits
1	$-\infty$	-3	-3.5	000
2	-3	-2	-2.5	001
3	-2	-1	-1.5	010
4	-1	0	-0.5	011
5	0	1	0.5	100
6	1	2	1.5	101
7	2	3	2.5	110
8	3	∞	3.5	111

Concise Representation of Quantizer

- Usually, it is sufficient just to list the quantization levels.
- Example: {-3.5, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5}
- Why?
  - We assume that all points are quantized to the nearest quantization level
  - This determines where the borders of the quantization regions are
  - Any other borders would increase the error introduced by the quantizer

Practical Methods for Implementing Analog to Digital Converters

Lower Complexity ↑ / Faster ↓

- Counting or Ramp ADC
  - test value is incremented in equal steps until it is greater than input sample
- Serial or Successive Approximation ADC
  - uses binary search to narrow range of input sample until desired accuracy is reached
- Parallel or Flash ADC
  - input sample is compared with all possible quantization levels at once

Distortion

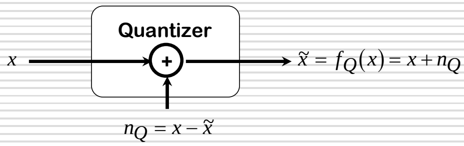
- Quantization introduces distortion into a signal.
- We want to minimize average distortion D, where

$$
\begin{aligned}
D &= E\left[\left(X-\tilde{X}\right)^2\right] \\
&= \int_{-\infty}^{\infty} \left(x-\tilde{x}\right)^2 f\left(x\right)dx \\
&= \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} \left(x-\tilde{x}_k\right)^2 f\left(x\right)dx
\end{aligned}
$$

- This measure of distortion is sometimes also called mean square error (MSE)
- MSE penalizes large errors more than small errors

Another Way of Viewing Quantization

- Quantization adds a random “noise” to the true value of the sample point
- Then $\text{MSE} = E\left[n_Q^2\right]$ may be thought of as noise power
- We can define a signal-to-noise ratio (SNR) to measure performance



Signal to Noise Ratio Calculations for Quantizers

- Average SNR

$$\left(\frac{S}{N}\right)_{avg} = \frac{E\left[X^2\right]}{E\left[n_Q^2\right]} = \frac{E\left[X^2\right]}{D} = \frac{\int_{-\infty}^{\infty} x^2 f_X(x)dx}{\int_{-\infty}^{\infty} (x-\tilde{x})^2 f_X(x)dx}$$

Example of SNR Calculation

- Let: $\{\tilde{x}_1 = -3.5, \tilde{x}_2 = -2.5, \tilde{x}_3 = -1.5, \tilde{x}_4 = -0.5, \tilde{x}_5 = 0.5, \tilde{x}_6 = 1.5, \tilde{x}_7 = 2.5, \tilde{x}_8 = 3.5\}$
- Let: $f(x) = \begin{cases} 1/8, & -4 \le x \le 4 \\ 0, & \text{else} \end{cases}$

$$E\left[X^2\right] = \int_{-4}^{4} x^2 \cdot \frac{1}{8} dx = \left.\frac{x^3}{24}\right|_{-4}^{4} = \frac{128}{24} = \frac{16}{3}$$

Example of SNR Calculation (cont.)

- Distortion: $D = MSE = \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} \left(x - \tilde{x}_k\right)^2 f_X(x)\, dx$

$$= \sum_{k=1}^{8} \int_{-5+k}^{-4+k} \left(x - (-4.5+k)\right)^2 \cdot \frac{1}{8} dx$$

- Examine the $k = 5$ term:

$$\int_0^1 (x-0.5)^2 \cdot \frac{1}{8} dx = \frac{1}{8} \cdot \int_0^1 x^2 - x + 0.25 dx = \frac{1}{8} \cdot \left[\frac{x^3}{3} - \frac{x^2}{2} + 0.25x \right]_0^1$$

$$= \frac{1}{8} \cdot \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right] = \frac{1}{8} \cdot \frac{1}{12}$$

Example of SNR Calculation (continued)

- All 8 terms are identical. $\therefore \text{MSE} = 1/12$

$$\left(\frac{S}{N}\right)_{\text{avg}} = 10\log_{10} \frac{E\left[X^2\right]}{D} = 10\log_{10}\left(\frac{16/3}{1/12}\right) = 18.1\text{dB}$$

- A standard rule of thumb is that each additional bit adds 6 dB to the SNR of uniform quantizers operating on a uniform pdf.

SNR for Uniform Quantization

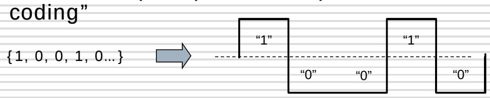
- General result: $\left(\frac{S}{N}\right)_{\text{avg}} = M^2$
  - Assumes uniform quantizer with *M* levels
  - Assumes that input samples have uniform distribution with identical range as quantizer
  - Though some texts do not make it clear, this result applies only when these special set of conditions hold. Otherwise, we have to use integral formula
- However, a useful Rule of Thumb:
  - Each additional bit (doubling *M*) increases SNR by 6 dB

$$\left(\frac{S}{N}\right)_{avg} = 6.02n + \alpha$$

  - Where α depends on the distribution of the signal

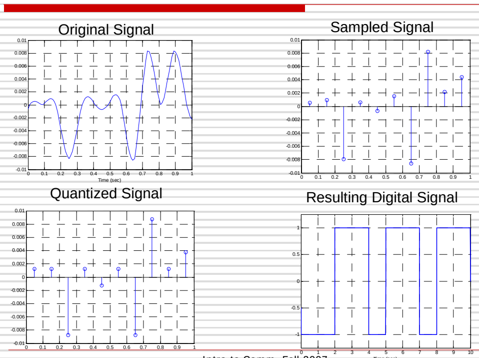
Waveform Encoding

- Once the information is converted to bits, it must be mapped onto waveforms
- If each bit is mapped to one of two different waveforms, we term this *binary encoding*
- If m bits are mapped to $M = 2^m$ waveforms, we term this M-ary encoding
- The mapping of bits to pulse-based waveforms (i.e., pulse trains) is termed “line coding”



Example: Non-return to zero polar signaling

PCM – Example



Bandwidth of PCM Signals

- Sample rate: f_s samples/second.
- Bit rate out of the quantizer:

$$f_s \log_2 M = f_s \cdot n \text{ bits / second}$$

- Bandwidth of the resulting digital signal depends on the waveform encoding used
- Minimum theoretical bandwidth with optimal waveform (baseband waveforms) : $f_s \cdot n/2$ Hz
- First null bandwidth (with rectangular pulse waveforms) for baseband waveforms: $f_s \cdot n$ Hz

Note that the resulting bandwidth depends on the digital waveform that is used.

Example PCM Calculation

- **Problem:**
  - Suppose that an analog music signal is found to have a bandwidth of 15 kHz and that samples of the signal may be modeled as having a uniform distribution.
  - Find the minimum first-null bandwidth (assuming the use of square pulses as the waveform) at which it would be possible to transmit a PCM signal while maintaining an average SNR of at least 58 dB.

Example PCM Calculation (cont.)

- **Solution:**

$$f_S \geq 2B = 30{,}000 \text{ samples / sec}$$

$$10\log_{10} M^2 \geq 58\text{dB} \Rightarrow M \geq 794 \Rightarrow M \geq 1024$$

$$\Rightarrow n \geq 10 \text{ bits/sampl e}$$

  - We will assume *n* must be an integer, thus *n* = 10 $R = f_s n \geq 300 kbps$
  - Minimum data rate:
  - First null BW: $B = f_s n = 300 kHz$
  - Assumes rectangular pulses are used as the waveforms

Note: Bandwidth comes into play twice here. Once for the original analog signal and once for the resulting digital signal.

Bandpass Digital Modulation

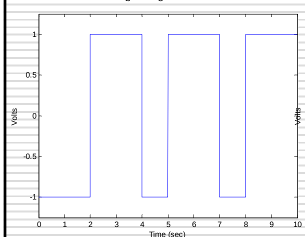
- Baseband digital maps bits to pulses
- Bandpass digital maps bits to sinusoids with different
  - phases
  - frequencies
  - amplitudes
- Bandpass digital techniques can be viewed as AM or FM with digital baseband waveforms as the message signal

Binary Frequency Shift Keying

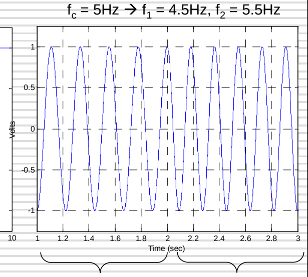
- While FM is an analog communications technique, the basic idea can also be used with digital modulation
- First we first map bits to square pulses with amplitudes +1/-1 representing data bits 1/0.
- Second, we use the square pulse baseband signal as our message and use $k_f = \Delta f = R_b/2$ where R_b is the bit rate.
- The result is a digital modulation scheme known as Binary Frequency Shift Keying (BFSK)

Example

$R_b = R_s = 1$bps



$f_c = 5$Hz → $f_1 = 4.5$Hz, $f_2 = 5.5$Hz

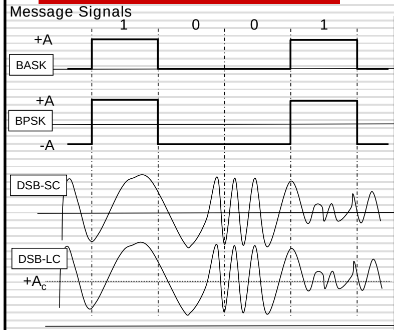


$f_1 = 4.5$Hz , $f_2 = 5.5$Hz

Relationship between AM and Digital Modulation

- DSB-LC AM → BASK
  - Message is entirely in the amplitude
- DSB-SC AM → BPSK, BASK
  - Message is in the phase and amplitude
  - In BPSK, amplitude is always 1 since the message is square wave.
  - In BASK the message signal doesn't go negative so there are no phase changes.
- SSB AM → QPSK
  - Since message is all real, we can eliminate ½ of spectrum
  - QPSK is based on fact that since a real message only needs ½ of the spectrum, 2 messages can be sent in quadrature in original BW

Relationship Between Analog and Digital Modulation Schemes



Message	What is modulated?
Unipolar NRZ	Amplitude Only
Polar NRZ	Phase Only
Arbitrary Analog signal	Amplitude and Phase
Arbitrary Analog signal with offset	Amplitude Only

Summary

- Transmitting an analog signal over a Digital Communication system involves
  - Sampling
  - Quantizing
  - Waveform encoding (i.e., mapping the bits to transmit waveforms)
- Today we examined the last two items
  - Sampling/Quantization convert analog signals to a stream of 1's and 0's
  - Waveform encoding maps bits by modulating
    - A pulse train in baseband digital communications
    - A sinusoid in bandpass digital communicaitons