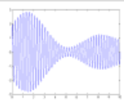
ECE3614
Introduction to
Communications Systems
Fall 2007

Instructor: Dr. R. Michael Buehrer
Lecture #3: The Fourier Transform
Part I - definitions



Overview

- Last class we briefly reviewed the Fourier Series and introduced the Fourier Transform
- Today we will continue our discussion of the Fourier Transform by again examining its relationship to the Fourier Series
- The Fourier Transform introduces the *frequency domain* which is an equivalent but insightful means of representing signals
- Reading
  - 2.1, 2.3

Lecture Objectives

- In this lecture we will
  - define the Fourier Transform
  - related it to the Fourier Series
  - discuss its characteristics
  - examine several examples

Fourier Theory

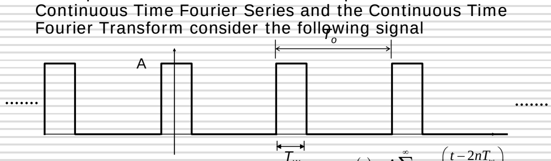
- Two basic types of signals
  - Periodic
    - Power signals
  - Aperiodic
    - Energy signals
- Fourier Domain Options
  - Fourier Series
    - Representation valid for all time if signal is periodic
    - Representation is valid only over a certain interval for aperiodic signals
  - Fourier Transform
    - Applies directly to aperiodic signals
    - Requires introduction of the impulse for application to periodic signals

Limitations of the Fourier Series

- The continuous time Fourier Series (CTFS) is a useful analytical tool but has limitations:
  - It can represent periodic signals for all time and can represent aperiodic signals for a finite time, but cannot represent an aperiodic signal for all time
  - It inherently depends on the fundamental frequency (i.e., the observation interval) chosen – signals with different fundamental periods must be converted to a common observation interval.
- The Fourier Transform will overcome these limitations by allowing us to represent periodic *and* aperiodic signals without depending on the observation interval

Illustrative Example

- To help us understand the relationship between the Continuous Time Fourier Series and the Continuous Time Fourier Transform consider the following signal



$$x(t) = A\sum_{n=-\infty}^{\infty} rect\left(\frac{t-2nT_w}{T_w}\right)$$

- with $T_w = T_o/2$ and $t_o = 0$.
- We know that the Continuous Time Fourier Series is

$$c_n = \frac{A}{2}\text{sinc}\left(\frac{n}{2}\right)$$

Reducing the Duty Cycle

- Now let us reduce the duty cycle such that
  - T_w *remains constant*
  - $T_o = 10T_w$
  - The average power is kept constant (i.e., we increase the amplitude by T_o *to maintain constant average power)*

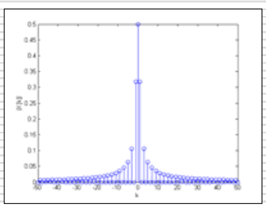
$$x(t) = 10A\sum_{n=-\infty}^{\infty} rect\left(\frac{t-10nT_w}{T_w}\right) \qquad c_n = \frac{A}{2}\text{sinc}\left(\frac{n}{10}\right)$$

- If we further reduce the duty cycle such that
  - T_w *remains constant*
  - $T_o = 1000T_w$
  - The average power is kept constant (i.e., we increase the amplitude by T_o

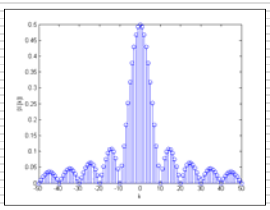
$$x(t) = 1000A\sum_{n=-\infty}^{\infty} rect\left(\frac{t-1000nT_w}{T_w}\right) \qquad c_n = \frac{A}{2}\text{sinc}\left(\frac{n}{1000}\right)$$

The Magnitude Response

- Plotting for $T_o = 2T_w$ and $T_o = 10T_w$



$$c_n = \frac{A}{2}\text{sinc}\left(\frac{n}{2}\right)$$

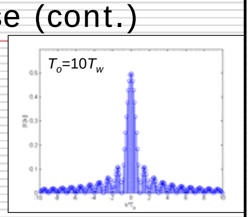


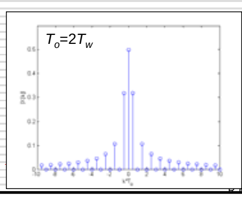
$$c_n = \frac{A}{2}\text{sinc}\left(\frac{n}{10}\right)$$

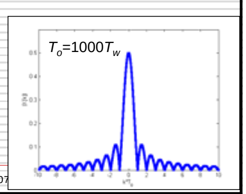
Decreasing duty cycle slows the CTFS in frequency and shows more detail of the underlying sinc function

Magnitude Response (cont.)

- If we normalize the *x*-axis by the fundamental period, we see that increasing the period increases the sampling rate of the underlying function
- If we let the period go to infinity (i.e., we let the square pulse train approach a single pulse), the Fourier Series representation approaches a continuous function – this is the Fourier Transform





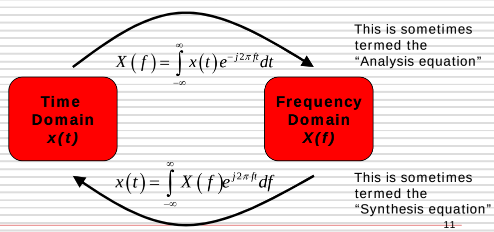


The Frequency Domain

- The original signal *x(t)* is said to be in the *time domain* since its argument represents time
- The Fourier Transform *X(f)* representation is said to be in the *frequency domain* since its argument *f* represents frequency
- Notes:
  - Frequency is the reciprocal of time
  - The Fourier Transform is referred to as an *analysis* of the signal *x(t)* since it extracts the frequency components of *x(t)* at each value of *f*
  - The Inverse Fourier Transform is referred to as *synthesis* since it recombines the components *X(f)* to obtain the original signal *x(t)*
  - The physical meaning of X(f) depends on the meaning of x(t). If x(t) has units of volts, X(f) has units volts/Hz.
    - Thus it represents how much of the over all voltage signal is present at each frequency.

The Frequency Domain

- We can think of the Fourier Transform and the Inverse Fourier Transform as means for moving between the time and frequency domains
- Note that no information is lost in the transformation and both are equivalent representations of a signal



Further Notes

- The function *X(f)* is also sometimes referred to as the *amplitude spectral density* or the *spectrum* of *x(t)*
- We often represent Fourier Transform pairs using the notation

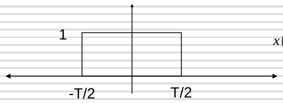
$$x(t) \Longleftrightarrow X(f)$$

  and we refer to *x(t)* and *X(f)* as a Fourier Transform pair
- Sometimes the Fourier Transform is defined in terms of radian frequency:

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad x(t)=\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}df$$

Example 3.1

- Consider the rectangular pulse



$$x(t) = rect\left(\frac{t}{T}\right)$$

- Find the Continuous Time Fourier Transform

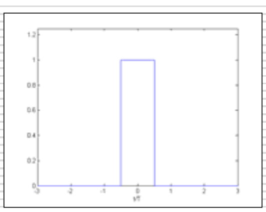
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$
$$= \int_{-T/2}^{T/2} e^{-j2\pi ft} dt$$

Example 3.1(cont.)

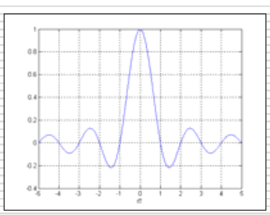
$$X(f) = \int_{-T/2}^{T/2} \left(\cos(2\pi ft) - j\sin(2\pi ft)\right) dt$$
$$= \left[\frac{1}{2\pi f}\sin(2\pi ft) + j\frac{1}{2\pi f}\cos(2\pi ft)\right]\Bigg|_{-T/2}^{T/2}$$
$$= \frac{1}{2\pi f}\left[\sin(2\pi fT/2) - \sin(-2\pi fT/2)\right] + \ldots$$
$$j\frac{1}{2\pi f}\left[\cos(2\pi fT/2) - \cos(-2\pi fT/2)\right]$$
$$= \frac{1}{\pi f}\sin(\pi fT)$$
$$= T\text{sinc}(fT)$$

$$\boxed{rect\left(\frac{t}{T}\right) \Longleftrightarrow T\text{sinc}(fT)}$$ Fourier Transform Pair

Example 3.1 - Plots

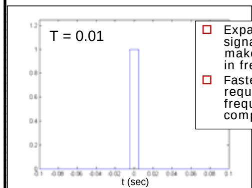


$rect\left(\frac{t}{T}\right)$

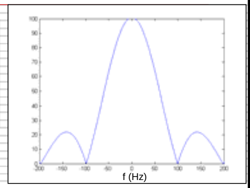


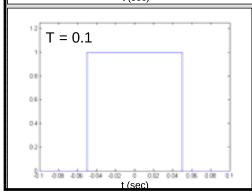
$T\text{sinc}(fT)$

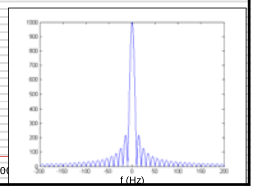
Time vs. Frequency



- Expanding a signal in time makes it narrow in frequency
- Faster changes require higher frequency components







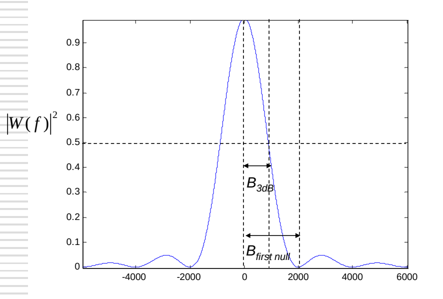
Time vs. Frequency

- As mentioned earlier, time and frequency are reciprocal
- If a function speeds up in time, it slows down in frequency
  - If a signal changes rapidly it requires more high-frequency components
  - Signals which change rapidly in time are said to have a *large bandwidth* (a measure of the frequency content)
- If a function slows down time, it speeds up in frequency
  - If a signal changes slowly in time it requires less high-frequency components and more low-frequency components
  - Signals which change slowly in time are said to have a *small bandwidth*

Definitions of Bandwidth for Baseband Signals

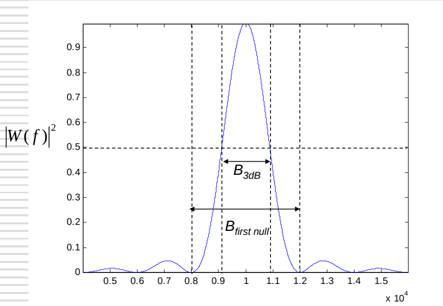
- Bandwidth is a term used to describe a *positive* frequency range over which the signal has significant content. There are various definitions for bandwidth including:
  - Absolute Bandwidth (B_{abs})
    - Defined as ***B*** where $W(f)=0 \quad f>B$
  - *3*-dB Bandwidth (half-power bandwidth - (B_{3dB}))
    - Defined as ***B*** where $|W(f)|^2 < \frac{|W(f)|^2_{\max}}{2} \quad f>B$
  - *X-dB Bandwidth*
    - *Defined as **B** where* $20\log_{10}(|W(f)|) < \left[20\log_{10}(|W(f)|_{\max}) - X\right] \quad f>B$
  - First Null Bandwidth ($B_{first\ null}$)
    - For baseband systems this is equal to the frequency of the first null in the spectrum

Bandwidth - Baseband



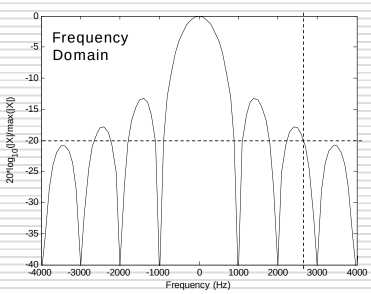
B_{3dB} = 900Hz
$B_{first\ null}$ = 2kHz
B_{abs} = infinity

Bandwidth - Bandpass



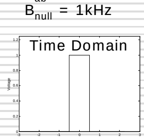
B_{3dB} = 1.8kHz
$B_{first\ null}$ = 4kHz
B_{abs} = infinity

Example 3.2

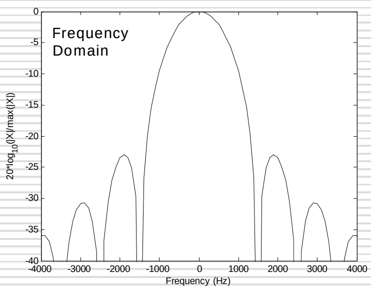


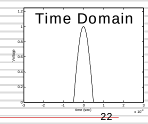
- Find the 20dB bandwidth, the absolute bandwidth and the first-null bandwidth

B_{20dB} = 2.75kHz
$B_{ab}s$ = infinite
B_{null} = 1kHz



In-class drill





Example 3.3: FT of Exponential

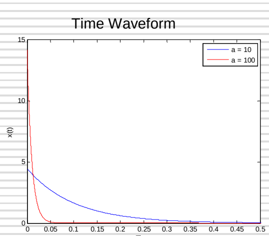
$$w(t) = u(t)e^{-at} = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

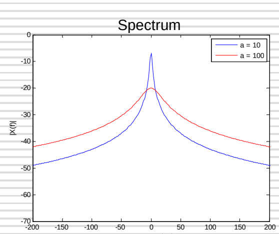
$$W(f) = \int_{-\infty}^{\infty} u(t)e^{-at}e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(a+j2f\pi)t}dt$$

$$= \left[\frac{e^{-(a+j2f\pi)t}}{a+j2f\pi}\right]_{\infty}^{0} = \frac{1}{a+j2f\pi}$$

$$|W(f)| = \frac{1}{\sqrt{a^2+(2f\pi)^2}} \qquad \arg(W(f)) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

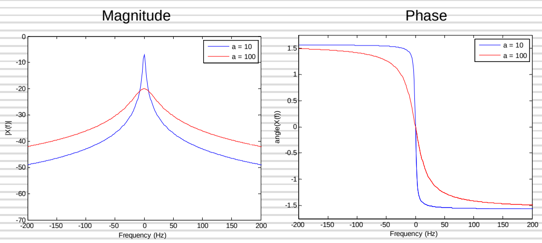
Example 3.3 (cont.)





As time waveform decreases more slowly, the *more low frequency content in the wave.*

Example 3.3 (cont.)



$$|W(f)| = \frac{1}{\sqrt{a^2 + (2f\pi)^2}} \qquad \arg(W(f)) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

Approach to Finding Fourier Transform Pairs

- We could continue to find transform pairs according to the definition, but this is inefficient
- In general, we compile a table of known transform pairs
- We also compile a table of simple rules for modifying transform pairs (i.e., properties).
- We will study properties next class.
- Using the known pairs and transform properties we can find most transforms needed.

Fourier Transform Pairs

Rectangular Pulse	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T[\mathrm{sinc}(fT)]$
Triangular Pulse	$\mathrm{tri}\left(\frac{t}{T}\right)$	$T[\mathrm{sinc}(fT)]^2$
Unit Step	$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
Signum	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at t_o	$\delta(t - t_o)$	$e^{-j2\pi f t_o}$
Sinc	$\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
Phasor	$e^{j\omega_o t + \varphi}$	$e^{j\varphi}\delta(f - f_o)$
Sinusoid	$\cos(2\pi f t + \varphi)$	$\frac{1}{2}e^{j\varphi}\delta(f - f_o) + \frac{1}{2}e^{-j\varphi}\delta(f + f_o)$
Gaussian	$e^{-\pi(t/t_o)^2}$	$t_o e^{-\pi(f t_o)^2}$

Note: Think of a constant as a sinusoid with an infinite period (f = 0). Does the transform make sense?

In-class drill

- Determine the first-null bandwidth, absolute bandwidth and 30dB bandwidth of the signal

Summary

- In this lecture we have discussed a vital tool in communication system analysis termed the *Fourier Transform.*
- The Fourier Transform is useful for providing a *frequency domain* representation of periodic and aperiodic signals that is valid for *all time.*
  - We will examine periodic signals next week.
- The Fourier Transform is an incredibly useful tool in many fields of engineering.
- Understanding the relationship between time and frequency is perhaps one of the most important concepts in this course.