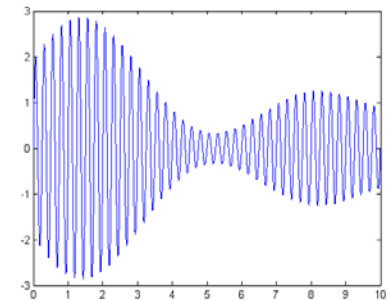


ECE3614

Introduction to Communications Systems

Fall 2007

Instructor: Dr. R. Michael Buehrer
Lecture #3: The Fourier Transform
Part I - definitions



Overview

- Last class we briefly reviewed the Fourier Series and introduced the Fourier Transform
- Today we will continue our discussion of the Fourier Transform by again examining its relationship to the Fourier Series
- The Fourier Transform introduces the *frequency domain* which is an equivalent but insightful means of representing signals
- Reading
 - 2.1, 2.3

Lecture Objectives

- In this lecture we will
 - define the Fourier Transform
 - related it to the Fourier Series
 - discuss its characteristics
 - examine several examples

Fourier Theory

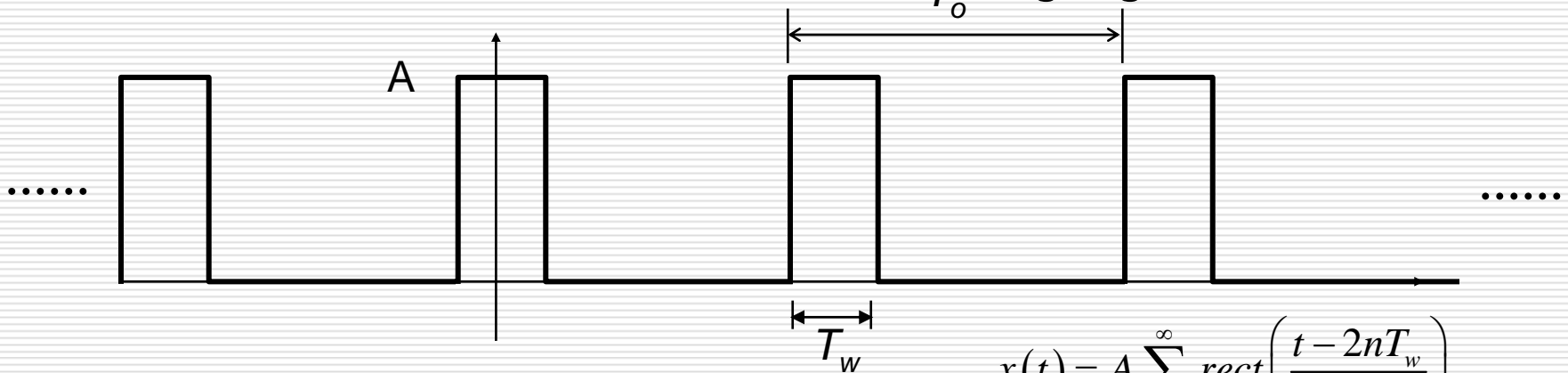
- Two basic types of signals
 - Periodic
 - Power signals
 - Aperiodic
 - Energy signals
- Fourier Domain Options
 - Fourier Series
 - Representation valid for all time if signal is periodic
 - Representation is valid only over a certain interval for aperiodic signals
 - Fourier Transform
 - Applies directly to aperiodic signals
 - Requires introduction of the impulse for application to periodic signals

Limitations of the Fourier Series

- The continuous time Fourier Series (CTFS) is a useful analytical tool but has limitations:
 - It can represent periodic signals for all time and can represent aperiodic signals for a finite time, but cannot represent an aperiodic signal for all time
 - It inherently depends on the fundamental frequency (i.e., the observation interval) chosen – signals with different fundamental periods must be converted to a common observation interval.
- The Fourier Transform will overcome these limitations by allowing us to represent periodic *and* aperiodic signals without depending on the observation interval

Illustrative Example

- To help us understand the relationship between the Continuous Time Fourier Series and the Continuous Time Fourier Transform consider the following signal



$$x(t) = A \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - 2nT_w}{T_w}\right)$$

- with $T_w = T_o/2$ and $t_o = 0$.

- We know that the Continuous Time Fourier Series is

$$c_n = \frac{A}{2} \text{sinc}\left(\frac{n}{2}\right)$$

Reducing the Duty Cycle

- Now let us reduce the duty cycle such that
 - T_w remains constant
 - $T_o = 10T_w$
 - The average power is kept constant (i.e., we increase the amplitude by T_o to maintain constant average power)

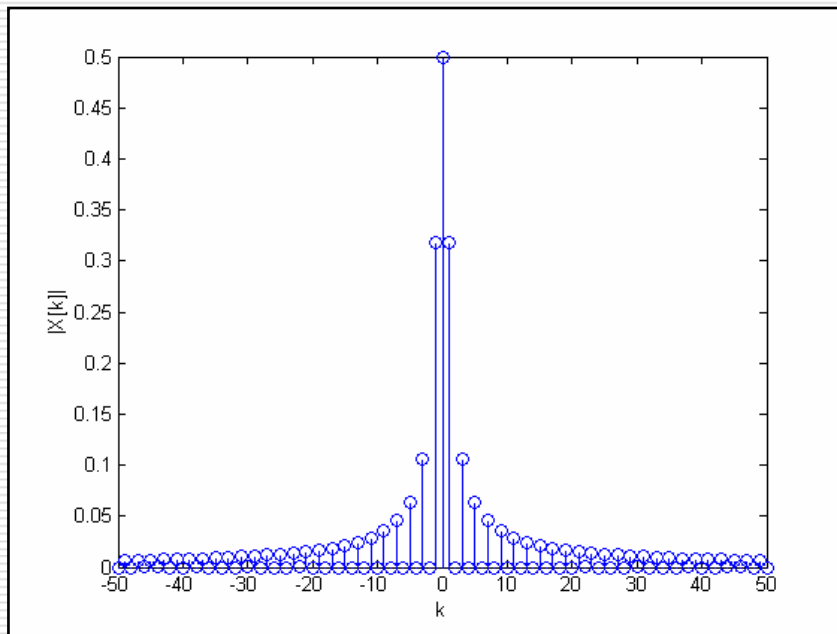
$$x(t) = 10A \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - 10nT_w}{T_w}\right) \quad c_n = \frac{A}{2} \text{sinc}\left(\frac{n}{10}\right)$$

- If we further reduce the duty cycle such that
 - T_w remains constant
 - $T_o = 1000T_w$
 - The average power is kept constant (i.e., we increase the amplitude by T_o)

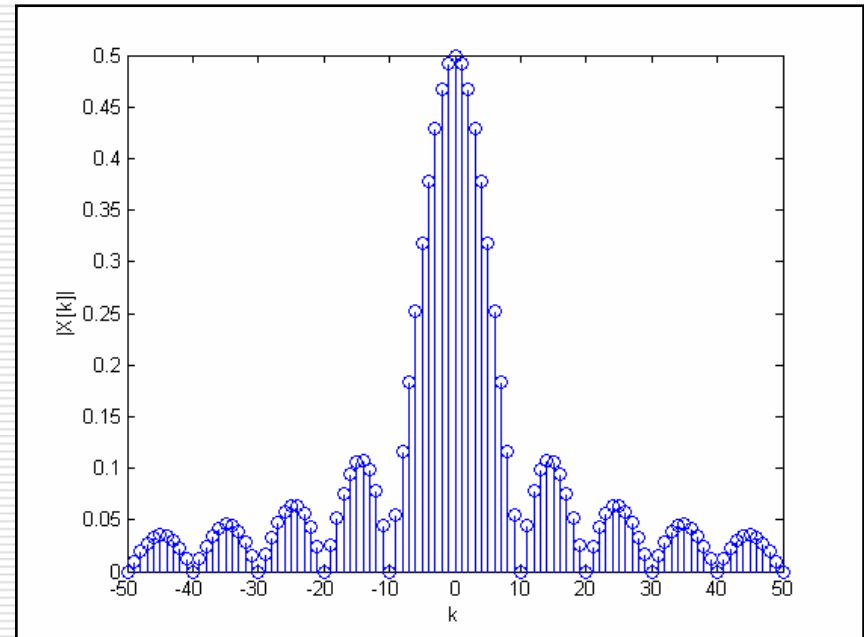
$$x(t) = 1000A \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t - 1000nT_w}{T_w}\right) \quad c_n = \frac{A}{2} \text{sinc}\left(\frac{n}{1000}\right)$$

The Magnitude Response

□ Plotting for $T_o = 2T_w$ and $T_o = 10T_w$



$$c_n = \frac{A}{2} \operatorname{sinc}\left(\frac{n}{2}\right)$$

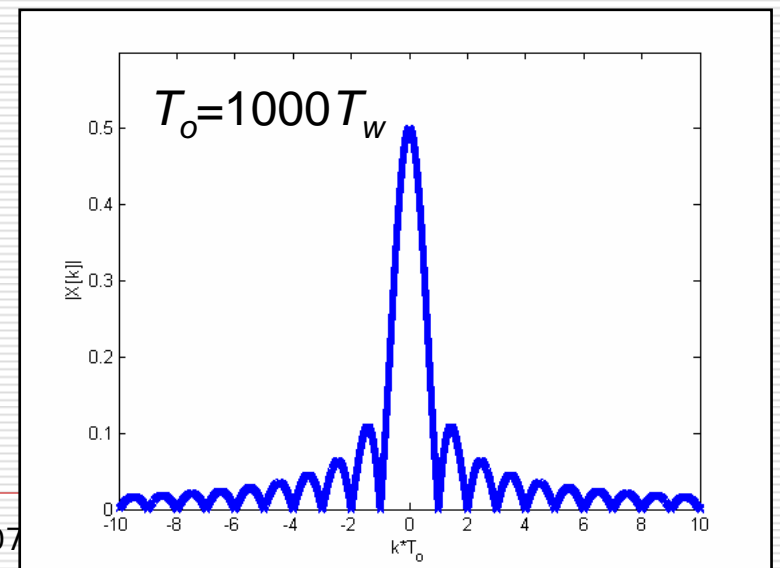
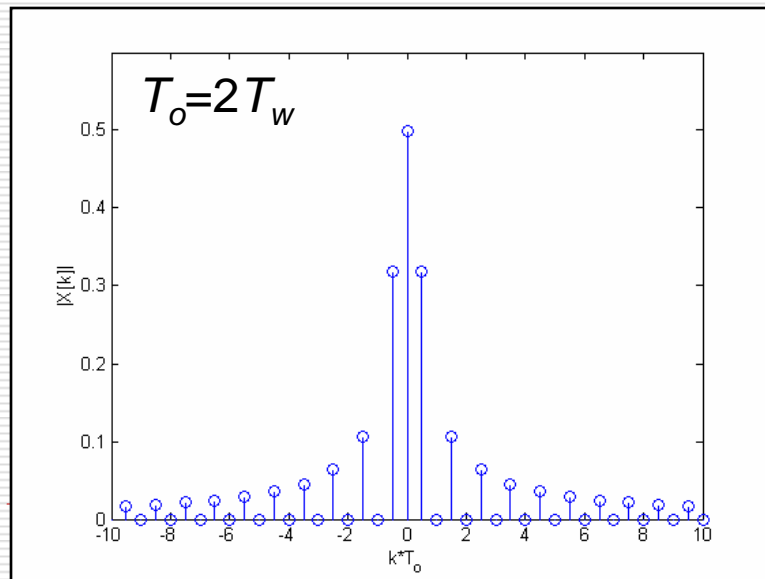
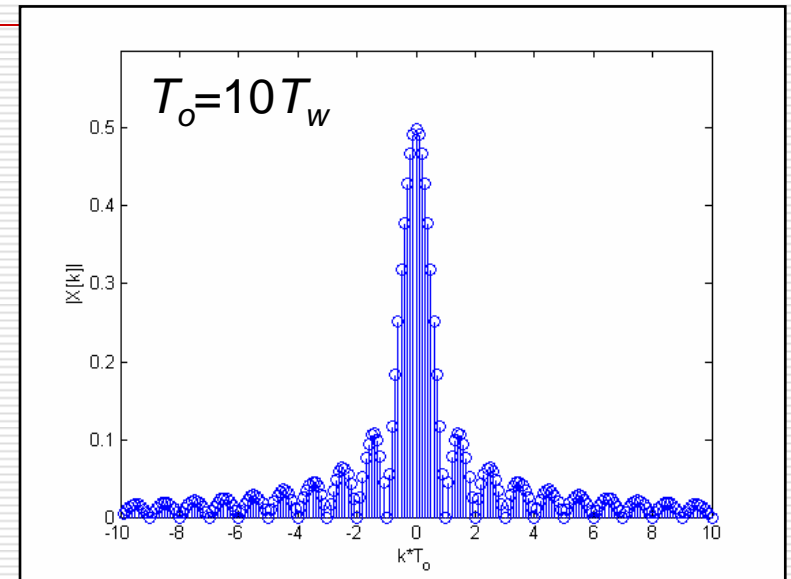


$$c_n = \frac{A}{2} \operatorname{sinc}\left(\frac{n}{10}\right)$$

Decreasing duty cycle slows the CTFS in frequency and shows more detail of the underlying sinc function

Magnitude Response (cont.)

- If we normalize the x-axis by the fundamental period, we see that increasing the period increases the sampling rate of the underlying function
- If we let the period go to infinity (i.e., we let the square pulse train approach a single pulse), the Fourier Series representation approaches a continuous function – this is the Fourier Transform

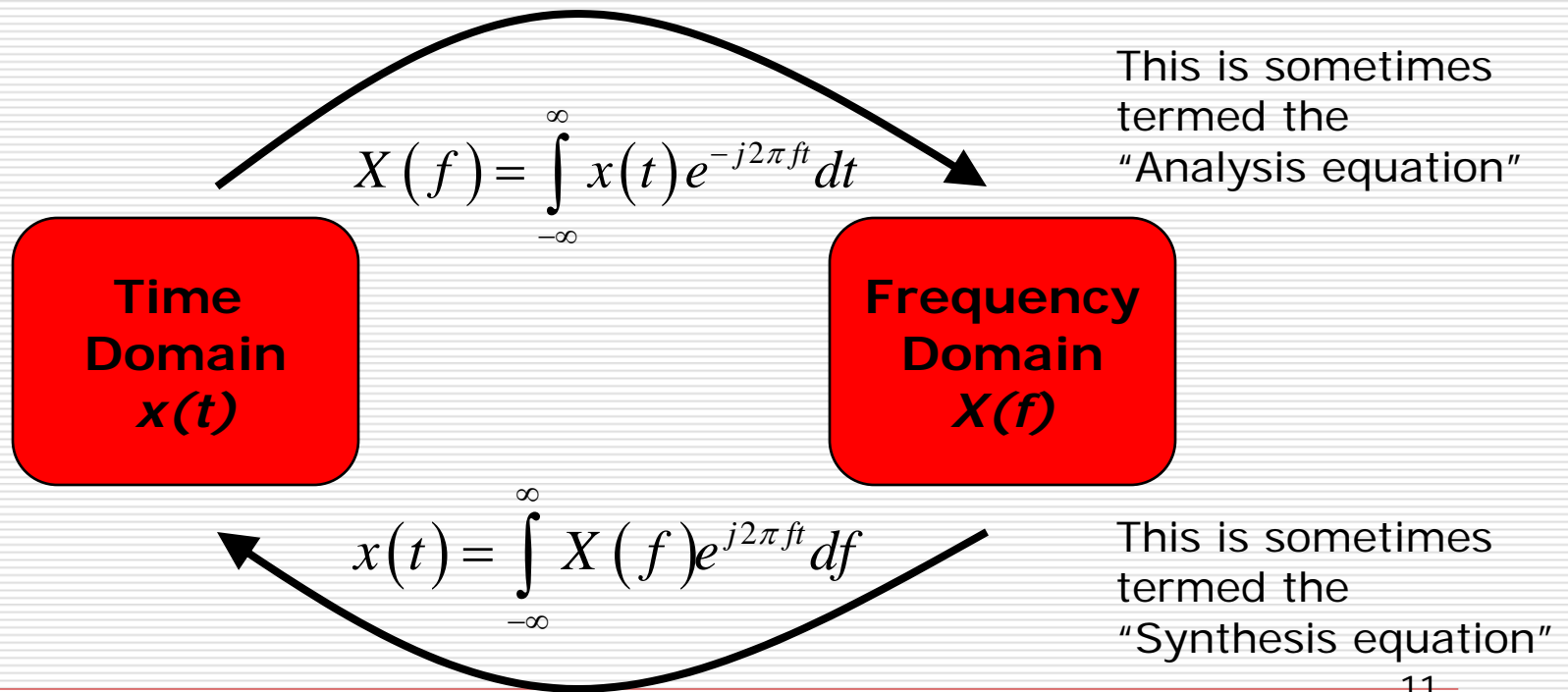


The Frequency Domain

- The original signal $x(t)$ is said to be in the *time domain* since its argument represents time
- The Fourier Transform $X(f)$ representation is said to be in the *frequency domain* since its argument f represents frequency
- Notes:
 - Frequency is the reciprocal of time
 - The Fourier Transform is referred to as an *analysis* of the signal $x(t)$ since it extracts the frequency components of $x(t)$ at each value of f
 - The Inverse Fourier Transform is referred to as *synthesis* since it recombines the components $X(f)$ to obtain the original signal $x(t)$
 - The physical meaning of $X(f)$ depends on the meaning of $x(t)$. If $x(t)$ has units of volts, $X(f)$ has units volts/Hz.
 - Thus it represents how much of the over all voltage signal is present at each frequency.

The Frequency Domain

- We can think of the Fourier Transform and the Inverse Fourier Transform as means for moving between the time and frequency domains
- Note that no information is lost in the transformation and both are equivalent representations of a signal



Further Notes

- The function $X(f)$ is also sometimes referred to as the *amplitude spectral density* or the *spectrum* of $x(t)$
- We often represent Fourier Transform pairs using the notation

$$x(t) \iff X(f)$$

and we refer to $x(t)$ and $X(f)$ as a Fourier Transform pair

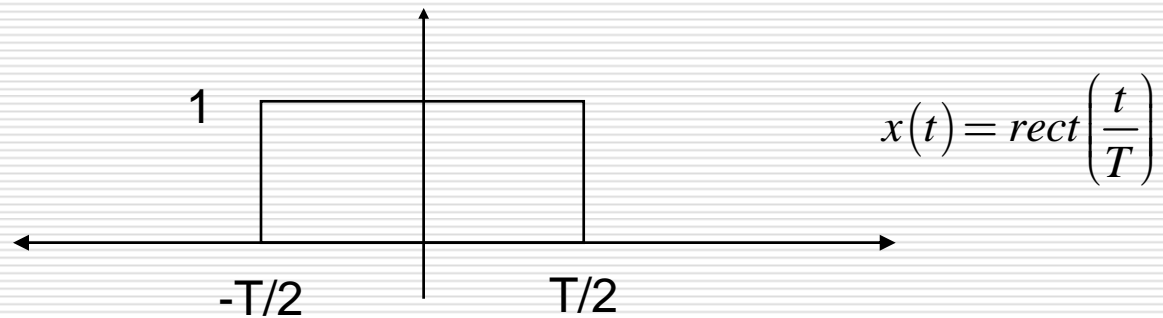
- Sometimes the Fourier Transform is defined in terms of radian frequency:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Example 3.1

□ Consider the rectangular pulse



□ Find the Continuous Time Fourier Transform

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ &= \int_{-T/2}^{T/2} e^{-j2\pi ft} dt \end{aligned}$$

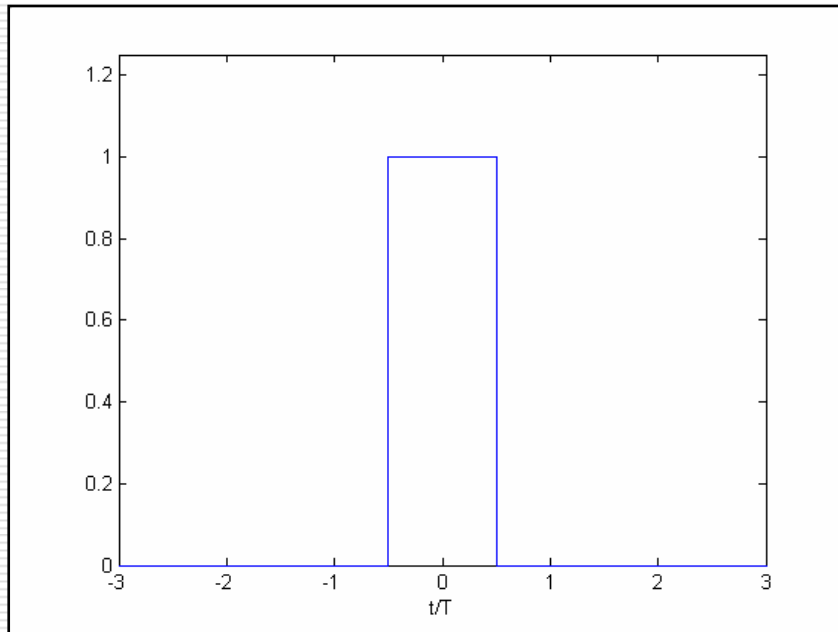
Example 3.1 (cont.)

$$\begin{aligned} X(f) &= \int_{-T/2}^{T/2} (\cos(2\pi ft) - j \sin(2\pi ft)) dt \\ &= \left[\frac{1}{2\pi f} \sin(2\pi ft) + j \frac{1}{2\pi f} \cos(2\pi ft) \right]_{-T/2}^{T/2} \\ &= \frac{1}{2\pi f} [\sin(2\pi fT/2) - \sin(-2\pi fT/2)] + \dots \\ &\quad j \frac{1}{2\pi f} [\cos(2\pi fT/2) - \cos(-2\pi fT/2)] \\ &= \frac{1}{\pi f} \sin(\pi fT) \\ &= T \operatorname{sinc}(fT) \end{aligned}$$

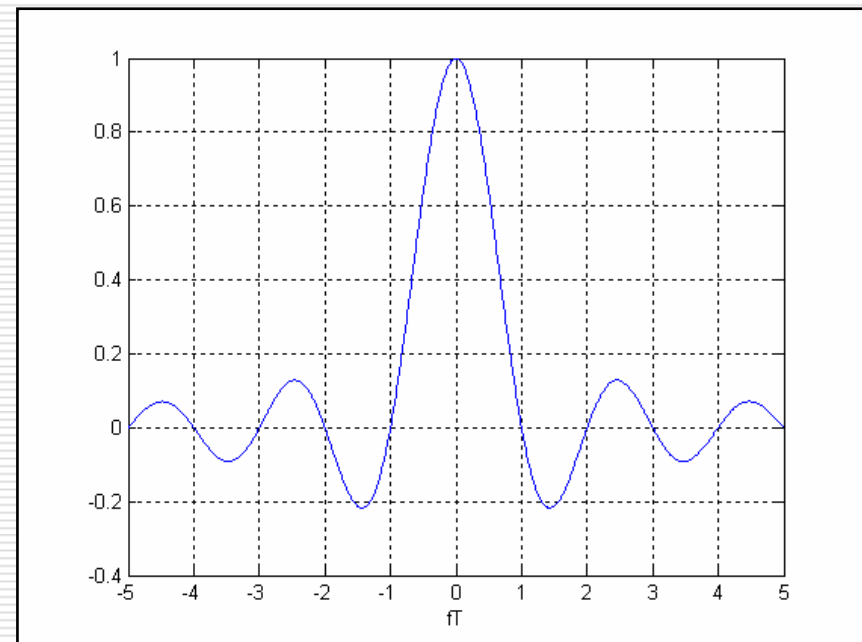
$$\boxed{\operatorname{rect}\left(\frac{t}{T}\right) \iff T \operatorname{sinc}(fT)}$$

Fourier Transform Pair

Example 3.1 - Plots

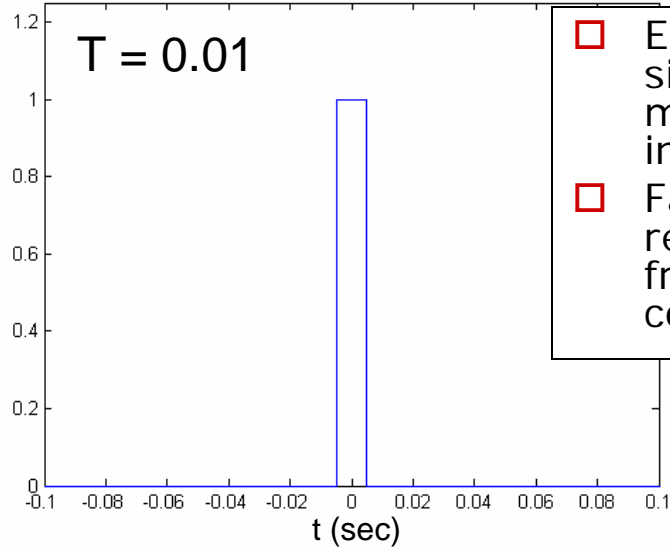


$$\text{rect}\left(\frac{t}{T}\right)$$

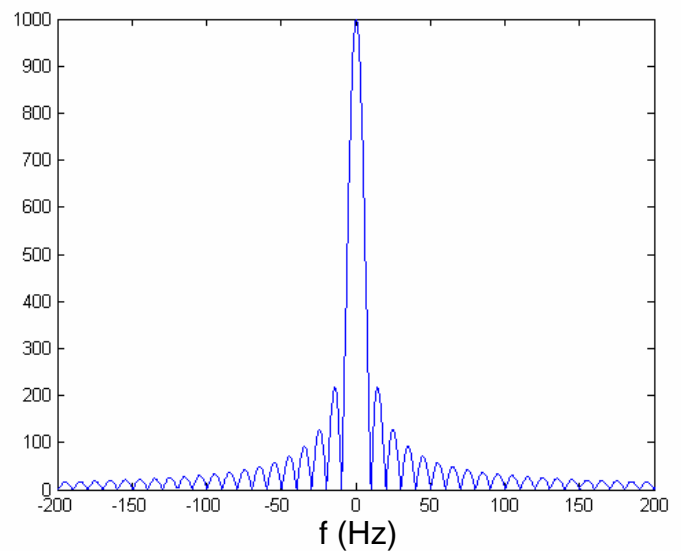
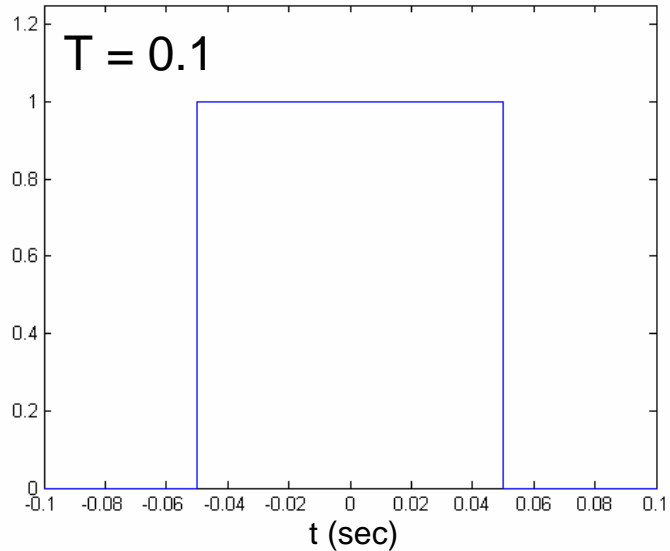
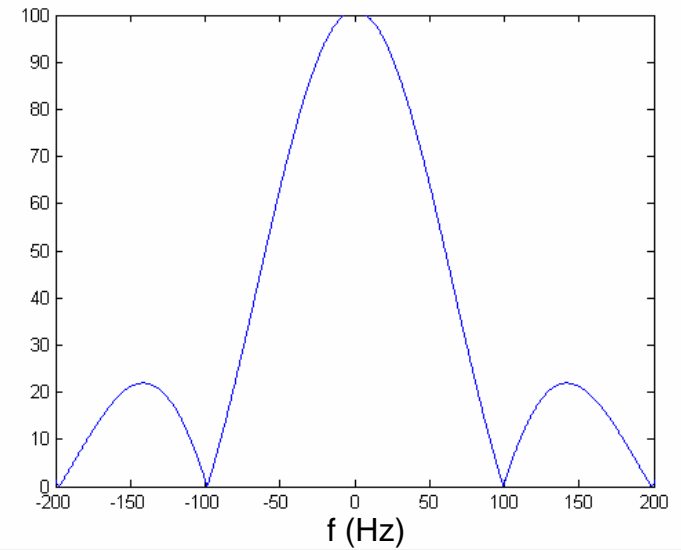


$$T\text{sinc}(fT)$$

Time vs. Frequency



- Expanding a signal in time makes it narrow in frequency
- Faster changes require higher frequency components



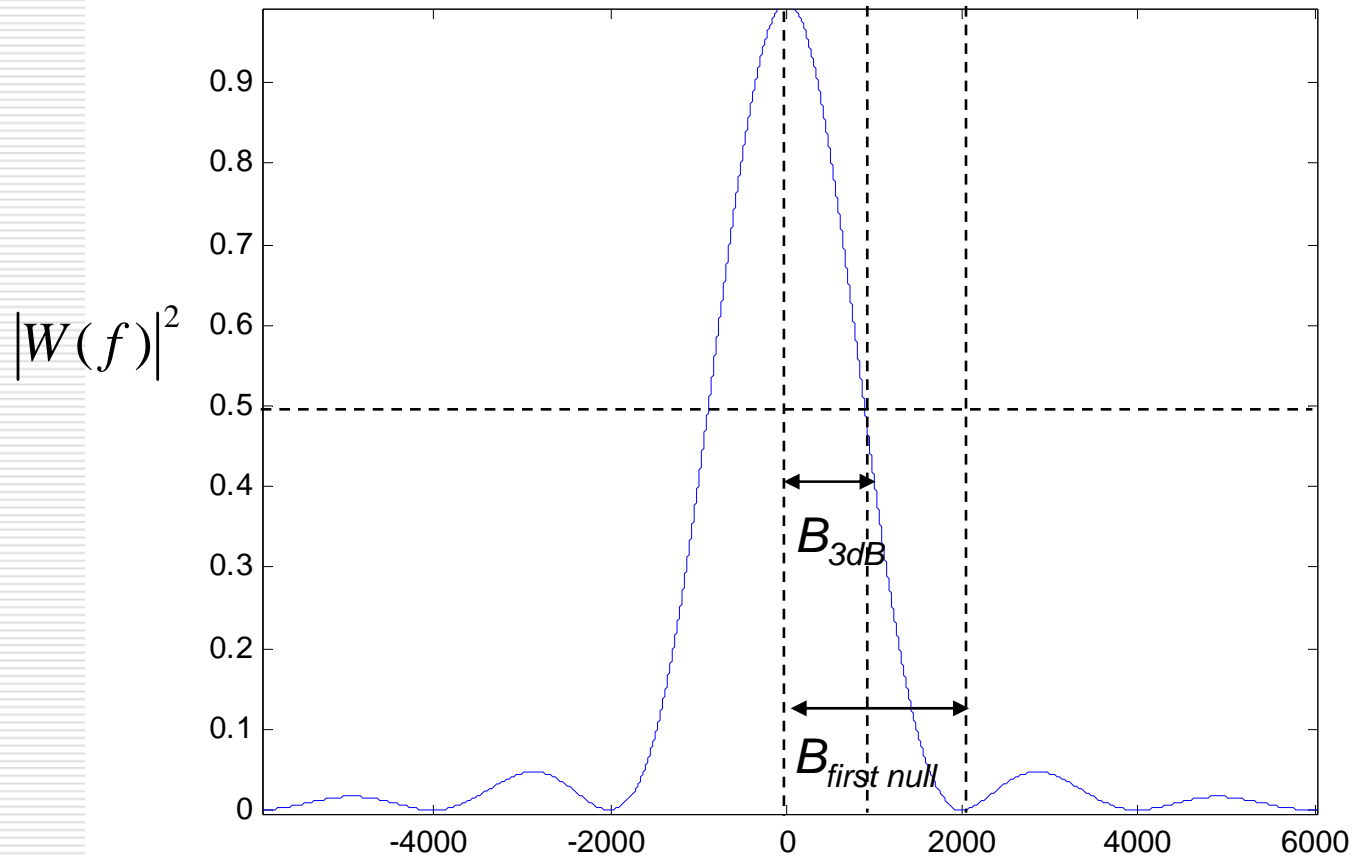
Time vs. Frequency

- As mentioned earlier, time and frequency are reciprocal
- If a function speeds up in time, it slows down in frequency
 - If a signal changes rapidly it requires more high-frequency components
 - Signals which change rapidly in time are said to have a *large bandwidth* (a measure of the frequency content)
- If a function slows down time, it speeds up in frequency
 - If a signal changes slowly in time it requires less high-frequency components and more low-frequency components
 - Signals which change slowly in time are said to have a *small bandwidth*

Definitions of Bandwidth for Baseband Signals

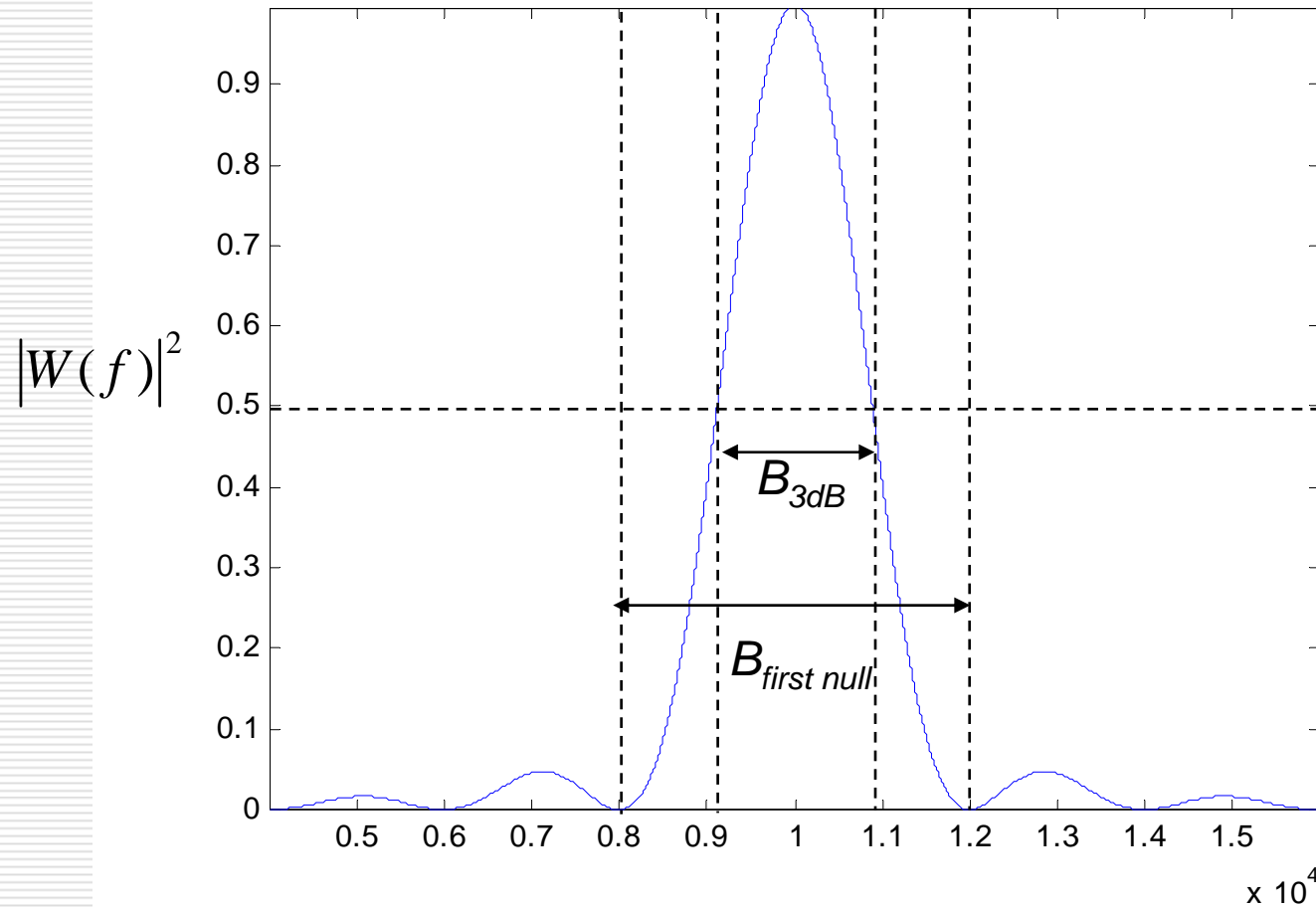
- Bandwidth is a term used to describe a *positive* frequency range over which the signal has significant content. There are various definitions for bandwidth including:
 - Absolute Bandwidth (B_{abs})
 - Defined as \mathbf{B} where $W(f) = 0 \quad f > B$
 - 3-dB Bandwidth (half-power bandwidth - (B_{3dB}))
 - Defined as \mathbf{B} where $|W(f)|^2 < \frac{|W(f)|_{max}^2}{2} \quad f > B$
 - *X-dB Bandwidth*
 - Defined as \mathbf{B} where $20\log_{10}(|W(f)|) < [20\log_{10}(|W(f)|_{max}) - X] \quad f > B$
 - First Null Bandwidth ($B_{first\ null}$)
 - For baseband systems this is equal to the frequency of the first null in the spectrum

Bandwidth - Baseband



$B_{3dB} = 900\text{Hz}$
 $B_{first\ null} = 2\text{kHz}$
 $B_{abs} = \text{infinity}$

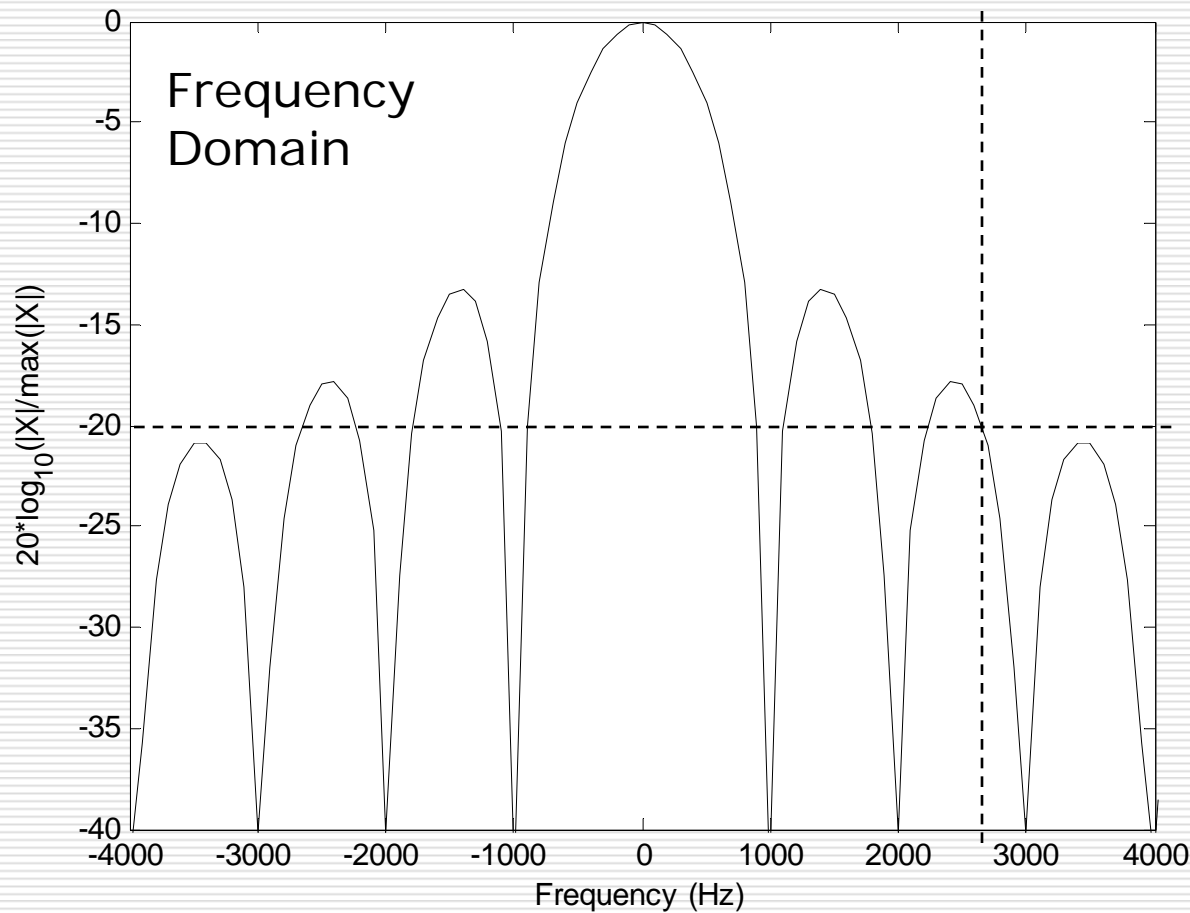
Bandwidth - Bandpass



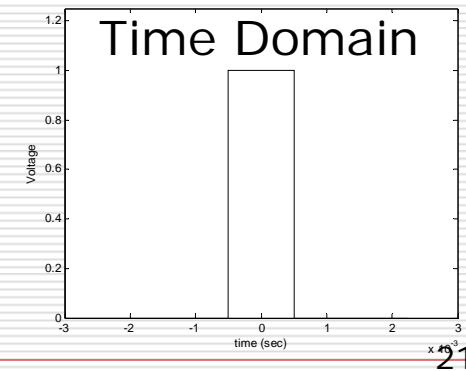
$B_{3dB} = 1.8\text{kHz}$
 $B_{first\ null} = 4\text{kHz}$
 $B_{abs} = \text{infinity}$

Example 3.2

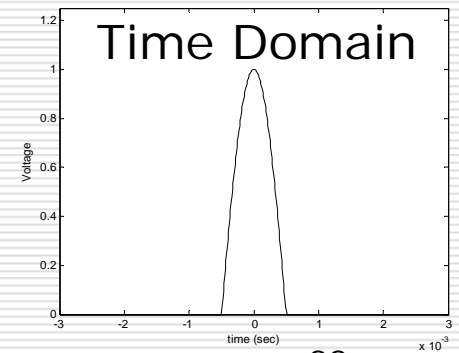
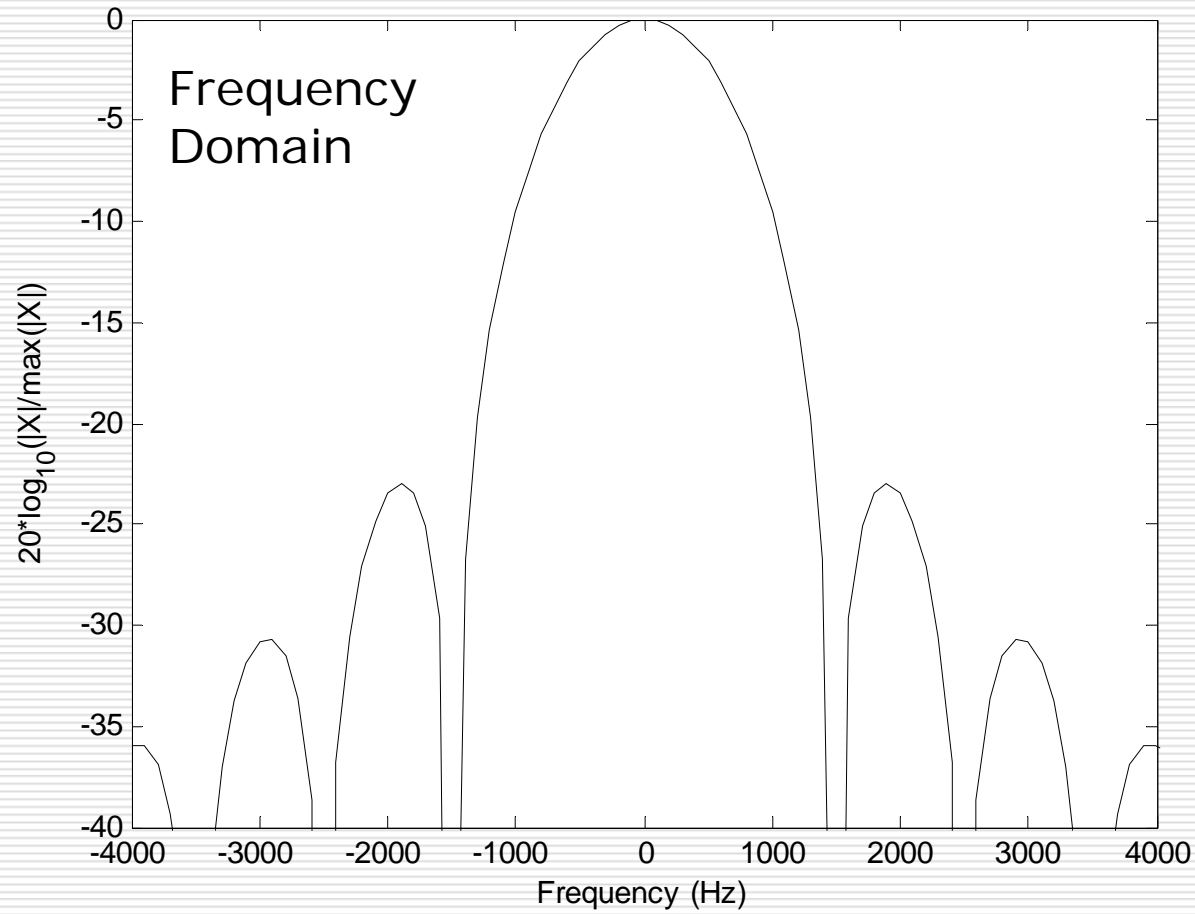
- Find the 20dB bandwidth, the absolute bandwidth and the first-null bandwidth



$$B_{20\text{dB}} = 2.75\text{kHz}$$
$$B_{\text{abs}} = \text{infinite}$$
$$B_{\text{null}} = 1\text{kHz}$$



In-class drill



Example 3.3: FT of Exponential

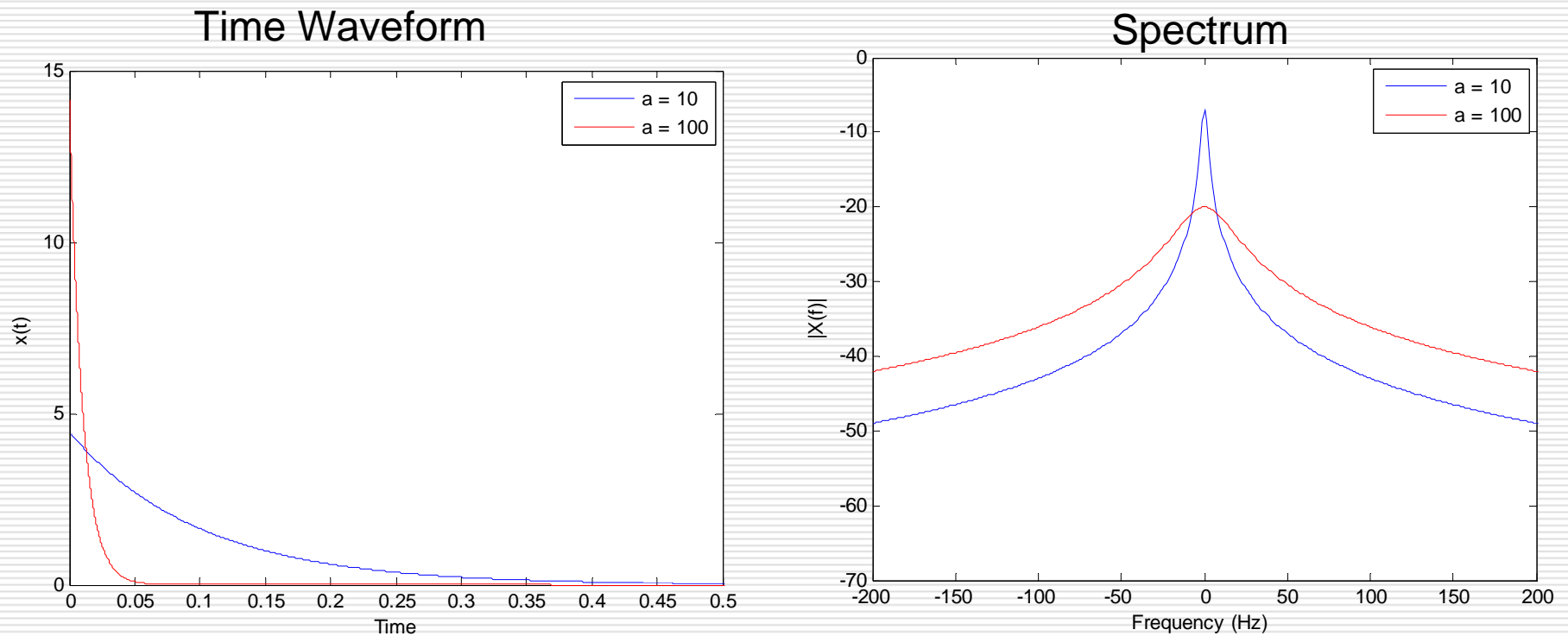
$$w(t) = u(t)e^{-at} = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$W(f) = \int_{-\infty}^{\infty} u(t)e^{-at} e^{-j2\pi ft} dt = \int_0^{\infty} e^{-(a+j2f\pi)t} dt$$

$$= \left[\frac{e^{-(a+j2f\pi)t}}{a+j2f\pi} \right]_0^{\infty} = \frac{1}{a+j2f\pi}$$

$$|W(f)| = \frac{1}{\sqrt{a^2 + (2f\pi)^2}} \quad \arg(W(f)) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

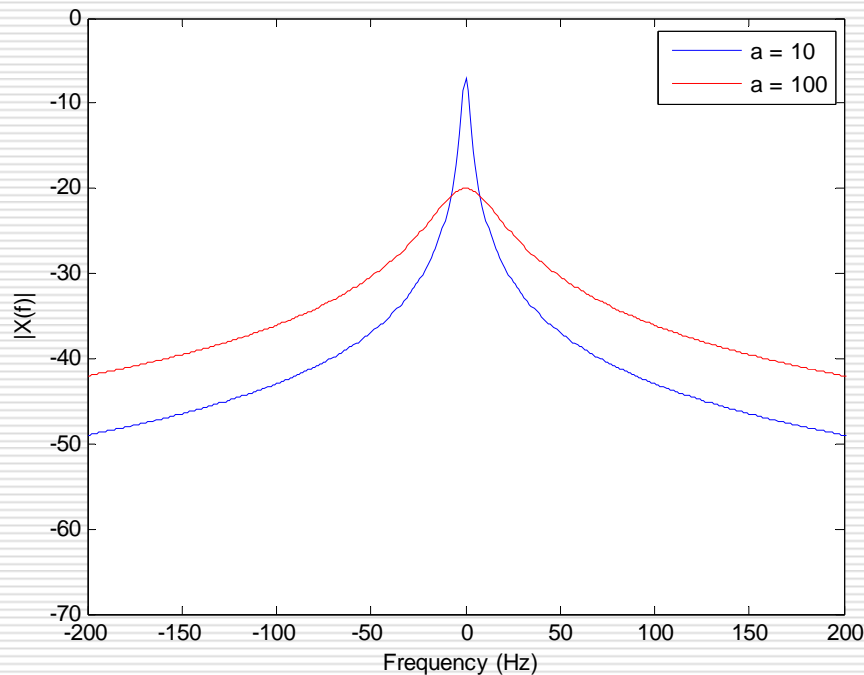
Example 3.3 (cont.)



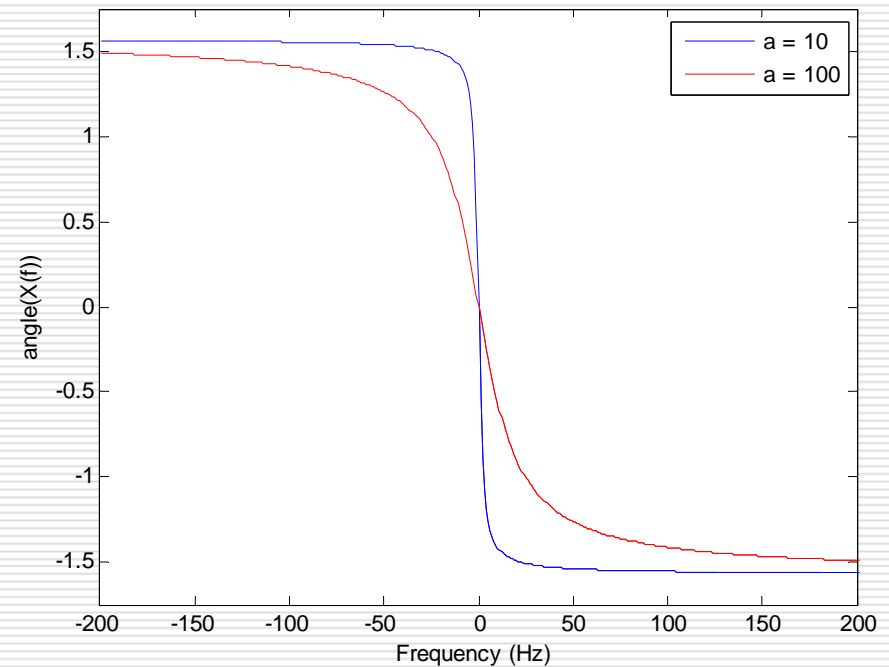
As time waveform decreases more slowly, the *more low frequency content in the wave.*

Example 3.3 (cont.)

Magnitude



Phase



$$|W(f)| = \frac{1}{\sqrt{a^2 + (2f\pi)^2}}$$

$$\arg(W(f)) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

Approach to Finding Fourier Transform Pairs

- ❑ We could continue to find transform pairs according to the definition, but this is inefficient
- ❑ In general, we compile a table of known transform pairs
- ❑ We also compile a table of simple rules for modifying transform pairs (i.e., properties).
- ❑ We will study properties next class.
- ❑ Using the known pairs and transform properties we can find most transforms needed.

Fourier Transform Pairs

Rectangular Pulse	$\text{rect}\left(\frac{t}{T}\right)$	$T [\text{sinc}(fT)]$
Triangular Pulse	$\text{tri}\left(\frac{t}{T}\right)$	$T [\text{sinc}(fT)]^2$
Unit Step	$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
Signum	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at t_o	$\delta(t - t_o)$	$e^{-j2\pi f t_o}$
Sinc	$\text{sinc}(2Wt)$	$\frac{1}{2W} \text{rect}\left(\frac{f}{2W}\right)$
Phasor	$e^{j\omega_o t + \varphi}$	$e^{j\varphi} \delta(f - f_o)$
Sinusoid	$\cos(2\pi f t + \varphi)$	$\frac{1}{2} e^{j\varphi} \delta(f - f_o) + \frac{1}{2} e^{-j\varphi} \delta(f + f_o)$
Gaussian	$e^{-\pi(t/t_o)^2}$	$t_o e^{-\pi(f t_o)^2}$

Note: Think of a constant as a sinusoid with an infinite period ($f = 0$). Does the transform make sense?

In-class drill

- Determine the first-null bandwidth, absolute bandwidth and 30dB bandwidth of the signal

Summary

- In this lecture we have discussed a vital tool in communication system analysis termed the *Fourier Transform*.
- The Fourier Transform is useful for providing a *frequency domain* representation of periodic and aperiodic signals that is valid for *all time*.
 - We will examine periodic signals next week.
- The Fourier Transform is an incredibly useful tool in many fields of engineering.
- Understanding the relationship between time and frequency is perhaps one of the most important concepts in this course.