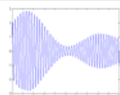
ECE3614
Introduction to
Communications Systems
Fall 2007

Instructor: Dr. R. Michael Buehrer

Lecture #4: The Fourier Transform
Part II - Properties



Overview

- Today we continue our discussion of the Fourier Transform (FT) by discussing *properties* of the FT
- In 2704 most of these properties were presented and proven. We will typically present them without proof
- We will also go through examples using tables of pairs and properties to find the Fourier Transform of arbitrary time signals
- Reading
  - Section2 2.2

Lecture Objective

- Today you will learn about properties of the Fourier Transform (FT) that allows you to find FT of a large class of signals without actually going through the FT integral

Fourier Transform Pairs

Rectangular Pulse	$\text{rect}\left(\frac{t}{T}\right)$	$T\left[\text{sinc}(fT)\right]$
Triangular Pulse	$\text{tri}\left(\frac{t}{T}\right)$	$T\left[\text{sinc}(fT)\right]^2$
Unit Step	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
Signum	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at t_o	$\delta(t-t_o)$	$e^{-j2\pi f t_o}$
Sinc	$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
Phasor	$e^{j\omega_o t+\varphi}$	$e^{j\varphi}\delta(f-f_o)$
Sinusoid	$\cos(2\pi ft+\varphi)$	$\frac{1}{2}e^{j\varphi}\delta(f-f_o)+\frac{1}{2}e^{-j\varphi}\delta(f+f_o)$
Gaussian	$e^{-\pi(t/t_o)^2}$	$t_o e^{-\pi(f t_o)^2}$

Fourier Transform Properties

Property	
Conjugation	$x^*(t) \Longleftrightarrow X^*(-f)$
Linearity	$\alpha x(t)+\beta y(t) \Longleftrightarrow \alpha X(f)+\beta Y(f)$
Time-shifting	$x(t-t_o) \Longleftrightarrow e^{-j2\pi f t_o}X(f)$
Frequency-shifting	$e^{j2\pi f_o t}x(t) \Longleftrightarrow X(f-f_o)$
Time reversal	$x(-t) \Longleftrightarrow X(-f)$
Time-differentiation	$\frac{d}{dt}\{x(t)\} \Longleftrightarrow (j2\pi f)X(f)$
Time-integration	$\int_{-\infty}^{t} x(\tau)d\tau \overset{*}{\Longleftrightarrow} \frac{1}{j2\pi f}X(f)$ *If X(0) = 0
Time/freq-scaling	$x(at) \Longleftrightarrow \frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
Multiplication	$x(t)y(t) \Longleftrightarrow X(f)*Y(f)$
Convolution	$x(t)*y(t) \Longleftrightarrow X(f)Y(f)$

Linearity

- If $z(t)=\alpha x(t)+\beta y(t)$
- Then

$$
\begin{aligned}
Z(f) &= \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \{\alpha x(t)+\beta y(t)\}e^{-j2\pi ft}dt \\
&= \alpha\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt+\beta\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt \\
&= \alpha X(f)+\beta Y(f)
\end{aligned}
$$

- In other words

$$\alpha x(t)+\beta y(t) \Longleftrightarrow \alpha X(f)+\beta Y(f)$$

Frequency Shift

- Let $z(t) = e^{j2\pi f_o t} x(t)$
- Then

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} e^{j2\pi f_o t} x(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f-f_o)t} dt$$
$$= X(f - f_o)$$

$$\boxed{e^{j2\pi f_o t} x(t) \rightleftharpoons X(f - f_o)}$$

Time Shifting

- Let $z(t) = x(t - t_o)$
- Then

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} x(t - t_o) e^{-j2\pi ft} dt$$

let $\tau = t - t_o$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f(\tau + t_o)} d\tau$$
$$= e^{-j2\pi f t_o} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau$$
$$= e^{-j2\pi f t_o} X(f)$$

$$\boxed{x(t - t_o) \rightleftharpoons e^{-j2\pi f t_o} X(f)}$$

Time Scaling

- Let $z(t) = x(at)$
- Then the Fourier Transform is

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt$$

Let $\lambda = at$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f\lambda/a} d\lambda$$
$$= \frac{1}{a} X\left(\frac{f}{a}\right)$$

$$\boxed{x(at) \rightleftharpoons \frac{1}{a} X\left(\frac{f}{a}\right)}$$

Scaling - Interpretation

- Scaling a signal in time by α scales the Fourier transform (i.e., the signal in frequency) by $1/\alpha$.
- Does this make sense? Recall our previous discussion that time and frequency are reciprocal.
- Let's assume that $\alpha > 1$. Scaling a signal in time by α speeds the signal up in time.
  - The resulting transform is scaled by $1/\alpha$ which slows the transform down in frequency – this means that more of the larger frequency values are present to accomplish faster changes.
- Scaling a signal in time by $1/\alpha$ slows the signal down in time.
  - The resulting transform is scaled by α which speeds it up in frequency – this means that more low frequency values are present to account for slower changes.

Convolution and Multiplication

- Convolution

$$x(t) * y(t) \overset{F}{\longleftrightarrow} X(f)Y(f)$$

- Multiplication

$$x(t)y(t) \overset{F}{\longleftrightarrow} X(f) * Y(f)$$

- Thus, convolution in the time domain results in multiplication in the frequency domain while multiplication in the time domain results in convolution in the frequency domain.
- This can greatly simplify some system analysis

Time Differentiation

- Using the Fourier Transform representation of $x(t)$ and taking the derivative

$$\frac{d}{dt}\{x(t)\} = \frac{d}{dt}\left\{\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df\right\}$$

$$= \int_{-\infty}^{\infty} j2\pi fX(f)e^{j2\pi ft}df$$

$$= F^{-1}\{j2\pi fX(f)\}$$

- Thus,

$$\frac{d}{dt}\{x(t)\} \overset{F}{\longleftrightarrow} j2\pi fX(f)$$

Modulation

- A common operation in communication systems is *modulation* or the multiplication of a signal by a high frequency sinusoid:

$$z(t) = x(t)\cos(2\pi f_c t)$$

- We can find the Fourier Transform of *z(t)* using the multiplication-convolution property

$$Z(f) = X(f) * F\{\cos(2\pi f_c t)\}$$
$$= X(f) * \left\{\frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)\right\}$$

Note: We will more fully discuss the impulse function next class.

- Using the sifting property of the impulse

$$Z(f) = \frac{1}{2}X(f - f_c) + \frac{1}{2}X(f + f_c)$$

Example 4.1

- Find the Fourier Transform of the signal

$$z(t) = rect\left(\frac{t}{T}\right)\cos(2\pi f_o t)$$

- Recall the modulation property:

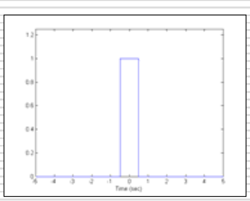
$$Z(f) = \frac{1}{2}X(f - f_c) + \frac{1}{2}X(f + f_c)$$

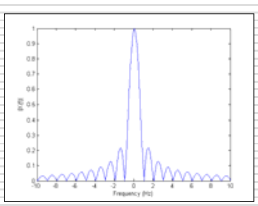
- Thus, we can write directly

$$Z(f) = \frac{T}{2}\mathrm{sinc}((f - f_o)T) + \frac{T}{2}\mathrm{sinc}((f + f_o)T)$$

Example 4.1 – cont.

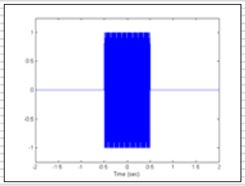
x(t) = rect(t) |X(f)| = |sinc(f)|

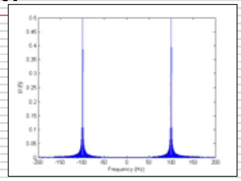


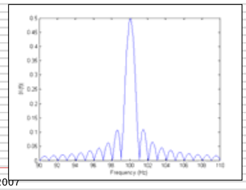


Example 4.1 – cont.

$$z(t) = rect(t)\cos(200\pi t)$$







$$Z(f) = \frac{1}{2}\text{sinc}(f-100) + \frac{1}{2}\text{sinc}(f+100)$$

Parseval's Theorem

- While the time domain signal *x(t)* and the frequency domain signal *X(f)* appear quite different they do have the same energy.
- That is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

- In other words, it doesn't matter whether I calculate the energy of a signal in the time domain or in the frequency domain, I get the same result.
  - This should make sense since the two representations are equivalent

Example 4.2

- Find the energy in the pulse *sinc(t)*
- The energy for any signal can be defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

- Substituting for this signal gives

$$E = \int_{-\infty}^{\infty} |\text{sinc}(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} \left|\frac{\sin(\pi t)}{\pi t}\right|^2 dt$$

$$= \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{(\pi t)^2} dt$$

Example 4.2 – cont.

- Continuing

$$E=\int_{-\infty}^{\infty}\frac{\sin^2(\pi t)}{(\pi t)^2}dt$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}dx \quad \text{Making a change of variables } x=\pi t$$
$$=\frac{2}{\pi}\underbrace{\int_{0}^{\infty}\frac{\sin^2(x)}{x^2}dx}_{\pi/2}$$
$$=\frac{2}{\pi}\frac{\pi}{2}$$
$$=1$$

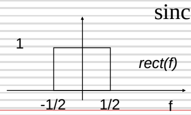
Example 4.2 – cont.

- Using our knowledge of Fourier Transforms and Parseval's Theorem we could have solved this very easily.
- Parseval's Theorem states

$$E_x=\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_{-\infty}^{\infty}|X(f)|^2\,df$$

- Further, the Fourier Transform of *sinc(t)* is

$$\mathrm{sinc}(t)\rightleftharpoons rect(f)$$



The energy can easily be found to be 1 from inspection this plot.

Duality

- Due to the similar nature of the Fourier Transform and the Inverse Fourier Transform, the CTFT exhibits the *duality property.*
- The duality property says that if we have the Fourier Transform pair

$$x(t)\rightleftharpoons X(f)$$

then we also have the Fourier Transform pair

$$X(t)\rightleftharpoons x(-f)$$

Example 4.3

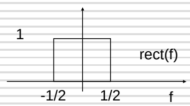
- From our previous development we know the following FT pair

$$rect(t) \overset{F}{\longleftrightarrow} \text{sinc}(f)$$

- The duality property says that

$$\text{sinc}(t) \overset{F}{\longleftrightarrow} rect(-f) = rect(f)$$

Check: Find the Inverse Fourier Transform for



Example 4.3 (cont.)

- Check:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$
$$= \int_{-1/2}^{1/2} e^{j2\pi ft} df$$
$$= \left. \frac{e^{j2\pi ft}}{j2\pi t} \right|_{-1/2}^{1/2}$$
$$= \frac{e^{j\pi t}}{j2\pi t} - \frac{e^{-j\pi t}}{j2\pi t}$$
$$= \frac{1}{\pi t} \frac{e^{j\pi t} - e^{-j\pi t}}{2j}$$
$$= \frac{\sin(\pi t)}{\pi t}$$
$$= \text{sinc}(t)$$

Example 4.3 (cont.)

More generally, consider the function *X(f) = rect(f/B)*:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$
$$= \int_{-B/2}^{B/2} e^{j2\pi ft} df$$
$$= \left. \frac{e^{j2\pi ft}}{j2\pi t} \right|_{-B/2}^{B/2}$$
$$= \frac{e^{j\pi tB}}{j2\pi t} - \frac{e^{-j\pi tB}}{j2\pi t}$$
$$= \frac{1}{\pi t} \frac{e^{j\pi tB} - e^{-j\pi tB}}{2j}$$
$$= \frac{\sin(\pi tB)}{\pi t}$$
$$= B\,\text{sinc}(tB)$$

Example 4.4

- From the table we know the following FT pair

$$1 \rightleftharpoons \delta(f)$$

- The duality property says that

$$\delta(t) \rightleftharpoons 1$$

- Check: Find the Fourier Transform for δ(t)

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt \\ &= e^{-j2\pi ft}\Big|_{t=0} \\ &= 1 \end{aligned}$$

Example 4.5

- Using Parseval's Theorem, determine the energy of

$$x(t) = \cos(2\pi f_o t)$$

- Solution: From Parseval's Theorem:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |X(f)|^2 df \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{2}\delta(f - f_o) + \frac{1}{2}\delta(f + f_o) \right|^2 df \\ &= \int_{-\infty}^{\infty} \frac{1}{4}\delta^2(f - f_o) + \frac{1}{4}\delta^2(f + f_o) df \\ &= \infty \end{aligned}$$

Side Bar – $\delta^2(t)$

- How do we evaluate $\delta^2(t)$?

$$\begin{aligned} \int_{-\infty}^{\infty} \delta^2(t) dt &= \lim_{a \to 0} \int_{-\infty}^{\infty} \left\{ \frac{1}{a} rect\left(\frac{t}{a}\right) \right\}^2 dt \\ &= \lim_{a \to 0} \frac{1}{a^2} \int_{-a/2}^{a/2} dt \\ &= \lim_{a \to 0} \frac{1}{a^2} a \\ &= \lim_{a \to 0} \frac{1}{a} \\ &= \infty \end{aligned}$$

Thus we have the following:

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_o) dt = g(t_o)$$

$$\int_{-\infty}^{\infty} g(t)\delta^2(t - t_o) dt = \infty$$

Example 4.5 – cont.

- We can double check the result using time domain integration:

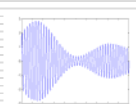
$$\begin{aligned}E_x &= \int_{-\infty}^{\infty} |x(t)|^2\, dt \\ &= \int_{-\infty}^{\infty} |\cos(2\pi f_o t)|^2\, dt \\ &= \int_{-\infty}^{\infty} \cos^2(2\pi f_o t)\, dt \\ &= \infty\end{aligned}$$

Summary

- In this lecture we have examined several properties of the Fourier Transform and examples which demonstrate their usefulness
- The properties are very similar to those for the Fourier Series
- We will find these properties very useful in determining the Fourier Transform of arbitrary signals.
  - Using a simple table of Fourier Transforms and FT properties, we can determine the FT of most signals of interest.

Supplemental Slides

Convolution and Multiplication Properties



Convolution in Time

$$z(t) = x(t) * y(t)$$

- Let

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$

- Then

$$= \int_{-\infty}^{\infty} \{x(t) * y(t)\} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \right\} e^{-j2\pi ft} dt$$

- Changing the order of integration:

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\left\{ \int_{-\infty}^{\infty} y(t-\tau) e^{-j2\pi ft} dt \right\}}_{F\{y(t-\tau)\}} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} Y(f) d\tau$$

Convolution (cont.)

- Finishing ....

$$Z(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} Y(f) d\tau$$

$$= Y(f) \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau$$

$$= X(f) Y(f)$$

$$\boxed{x(t) * y(t) \xleftrightarrow{F} X(f) Y(f)}$$

Multiplication in Time

$$z(t) = x(t) y(t)$$

- Now let

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} x(t) y(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} x(t) \left\{ \int_{-\infty}^{\infty} Y(\lambda) e^{j2\pi\lambda t} d\lambda \right\} e^{-j2\pi ft} dt$$

- Changing the order of integration

$$= \int_{-\infty}^{\infty} Y(\lambda) \underbrace{\left\{ \int_{-\infty}^{\infty} x(t) e^{-j2\pi(f-\lambda)t} dt \right\}}_{X(f-\lambda)} d\lambda$$

Multiplication (cont.)

- Continuing …

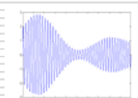
$$Z(f) = \int_{-\infty}^{\infty} Y(\lambda)X(f-\lambda)d\lambda$$
$$= X(f)*Y(f)$$

$$\boxed{x(t)y(t) \xleftrightarrow{F} X(f)*Y(f)}$$

- Thus, convolution in the time domain results in multiplication in the frequency domain while multiplication in the time domain results in convolution in the frequency domain.
- This can greatly simplify some system analysis

Supplemental Slides

Additional Example



Example 4.6

- Use the integration property to determine the Fourier Transform of *z(t) = tri(t)*
- Taking the derivative of *tri(t)* twice we have

$$\frac{d}{dt}\{tri(t)\} = rect\left(t+\frac{1}{2}\right) - rect\left(t-\frac{1}{2}\right)$$

$$\frac{d}{dt}\left\{rect\left(t+\frac{1}{2}\right) - rect\left(t-\frac{1}{2}\right)\right\} = \delta(t+1)-\delta(t)-\left(\delta(t)-\delta(t-1)\right)$$
$$= \delta(t+1)-2\delta(t)+\delta(t-1)$$

$$F\{\delta(t+1)-2\delta(t)+\delta(t-1)\} = \exp(j2\pi f)-2+\exp(-j2\pi f)$$
$$= \underbrace{2\cos(2\pi f)-2}_{X(f)}$$

Example 4.6 – cont.

- From the integration property we have

$$\begin{aligned} Z(f) &= \frac{1}{(j2\pi f)^2} X(f) \\ &= \frac{1}{(j2\pi f)^2}\{2\cos(2\pi f) - 2\} \\ &= \frac{-1}{(j\pi f)^2}\sin^2(\pi f) \\ &= \frac{\sin^2(\pi f)}{(\pi f)^2} \\ &= \mathrm{sinc}^2(f) \end{aligned}$$

Note X(0) = 0

$\sin^2(x) = \frac{1}{2}[1-\cos(2x)]$