ECE3614
Introduction to Communications Systems
Fall 2007

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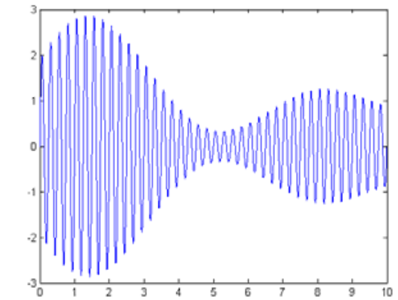
Lecture #4: The Fourier Transform

Part II - Properties



Overview

- Today we continue our discussion of the Fourier Transform (FT) by discussing *properties* of the FT
- In 2704 most of these properties were presented and proven. We will typically present them without proof
- We will also go through examples using tables of pairs and properties to find the Fourier Transform of arbitrary time signals
- Reading
  - Section2 2.2

Lecture Objective

- Today you will learn about properties of the Fourier Transform (FT) that allows you to find FT of a large class of signals without actually going through the FT integral

Fourier Transform Pairs

Rectangular Pulse	$\text{rect}\left(\frac{t}{T}\right)$	$T\left[\text{sinc}(fT)\right]$
Triangular Pulse	$\text{tri}\left(\frac{t}{T}\right)$	$T\left[\text{sinc}(fT)\right]^2$
Unit Step	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
Signum	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at t_o	$\delta(t-t_o)$	$e^{-j2\pi f t_o}$
Sinc	$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
Phasor	$e^{j\omega_o t+\varphi}$	$e^{j\varphi}\delta(f-f_o)$
Sinusoid	$\cos(2\pi ft+\varphi)$	$\frac{1}{2}e^{j\varphi}\delta(f-f_o)+\frac{1}{2}e^{-j\varphi}\delta(f+f_o)$
Gaussian	$e^{-\pi(t/t_o)^2}$	$t_o e^{-\pi(f\,t_o)^2}$

Fourier Transform Properties

Property	
Conjugation	$x^*(t) \rightleftharpoons X^*(-f)$
Linearity	$\alpha x(t) + \beta y(t) \rightleftharpoons \alpha X(f) + \beta Y(f)$
Time-shifting	$x(t - t_o) \rightleftharpoons e^{-j2\pi f t_o} X(f)$
Frequency-shifting	$e^{j2\pi f_o t} x(t) \rightleftharpoons X(f - f_o)$
Time reversal	$x(-t) \rightleftharpoons X(-f)$
Time-differentiation	$\frac{d}{dt}\{x(t)\} \rightleftharpoons (j2\pi f) X(f)$
Time-integration	$\int_{-\infty}^{t} x(\tau) d\tau \overset{*}{\rightleftharpoons} \frac{1}{j2\pi f} X(f)$ *If X(0) = 0
Time/freq-scaling	$x(at) \rightleftharpoons \frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$
Multiplication	$x(t) y(t) \rightleftharpoons X(f) * Y(f)$
Convolution	$x(t) * y(t) \rightleftharpoons X(f) Y(f)$

Linearity

- If $$z(t) = \alpha x(t) + \beta y(t)$$
- Then $$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} \left\{ \alpha x(t) + \beta y(t) \right\} e^{-j2\pi ft} dt$$
$$= \alpha \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt + \beta \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt$$
$$= \alpha X(f) + \beta Y(f)$$
- In other words

$$\boxed{\alpha x(t) + \beta y(t) \rightleftharpoons \alpha X(f) + \beta Y(f)}$$

Frequency Shift

- Let $$z(t) = e^{j2\pi f_o t} x(t)$$

- Then
$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi f t} dt$$
$$= \int_{-\infty}^{\infty} e^{j2\pi f_o t} x(t) e^{-j2\pi f t} dt$$
$$= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f - f_o) t} dt$$
$$= X(f - f_o)$$

$$\boxed{e^{j2\pi f_o t} x(t) \rightleftharpoons X(f - f_o)}$$

Time Shifting

- Let $$z(t) = x(t - t_o)$$

- Then
$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} x(t - t_o) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f(\tau + t_o)} d\tau$$
$$= e^{-j2\pi f t_o} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau$$
$$= e^{-j2\pi f t_o} X(f)$$

let $\tau = t\text{-}t_o$

$$x(t - t_o) \rightleftharpoons e^{-j2\pi f t_o} X(f)$$

Time Scaling

- Let $$z(t) = x(at)$$
- Then the Fourier Transform is

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt$$

Let λ=at

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f\lambda/a} d\lambda$$

$$= \frac{1}{a} X\left(\frac{f}{a}\right)$$

$$x(at) \rightleftharpoons \frac{1}{a} X\left(\frac{f}{a}\right)$$

Scaling - Interpretation

- Scaling a signal in time by α scales the Fourier transform (i.e., the signal in frequency) by $1/\alpha$.
- Does this make sense? Recall our previous discussion that time and frequency are reciprocal.
- Let's assume that $\alpha > 1$. Scaling a signal in time by α speeds the signal up in time.
  - The resulting transform is scaled by $1/\alpha$ which slows the transform down in frequency – this means that more of the larger frequency values are present to accomplish faster changes.
- Scaling a signal in time by $1/\alpha$ slows the signal down in time.
  - The resulting transform is scaled by α which speeds it up in frequency – this means that more low frequency values are present to account for slower changes.

Convolution and Multiplication

- Convolution

$$x(t) * y(t) \xleftrightarrow{F} X(f)Y(f)$$

- Multiplication

$$x(t)y(t) \xleftrightarrow{F} X(f) * Y(f)$$

- Thus, convolution in the time domain results in multiplication in the frequency domain while multiplication in the time domain results in convolution in the frequency domain.
- This can greatly simplify some system analysis

Time Differentiation

- Using the Fourier Transform representation of *x(t)* and taking the derivative

$$\frac{d}{dt}\{x(t)\} = \frac{d}{dt}\left\{\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df\right\}$$

$$= \int_{-\infty}^{\infty} j2\pi f X(f) e^{j2\pi ft} df$$

$$= F^{-1}\{j2\pi f X(f)\}$$

- Thus,

$$\boxed{\frac{d}{dt}\{x(t)\} \overset{F}{\longleftrightarrow} j2\pi f X(f)}$$

Modulation

- A common operation in communication systems is *modulation* or the multiplication of a signal by a high frequency sinusoid:

$$z(t) = x(t)\cos(2\pi f_c t)$$

- We can find the Fourier Transform of *z(t)* using the multiplication-convolution property

$$Z(f) = X(f) * F\{\cos(2\pi f_c t)\}$$

$$= X(f) * \left\{\frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)\right\}$$

Note: We will more fully discuss the impulse function next class.

- Using the sifting property of the impulse

$$Z(f) = \frac{1}{2}X(f - f_c) + \frac{1}{2}X(f + f_c)$$

Example 4.1

- Find the Fourier Transform of the signal

$$z(t) = rect\left(\frac{t}{T}\right)\cos(2\pi f_o t)$$

- Recall the modulation property:

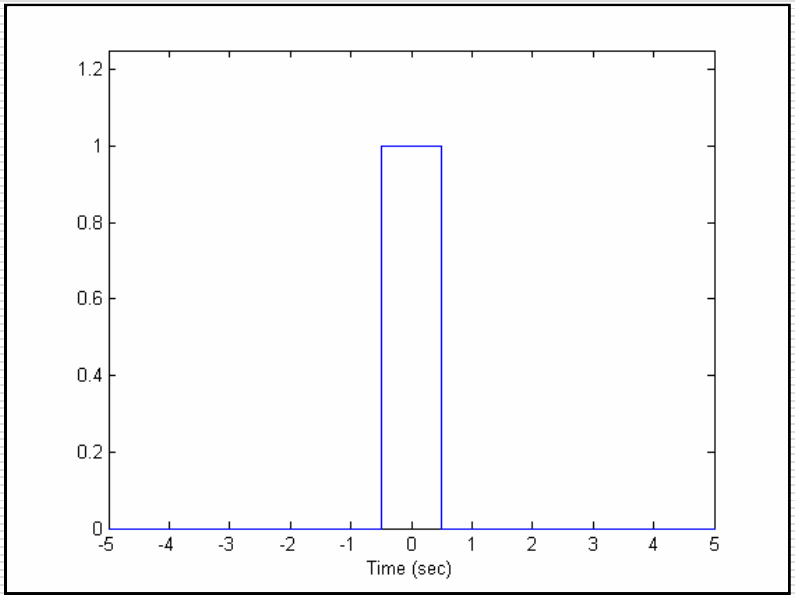
$$Z(f) = \frac{1}{2}X(f - f_c) + \frac{1}{2}X(f + f_c)$$

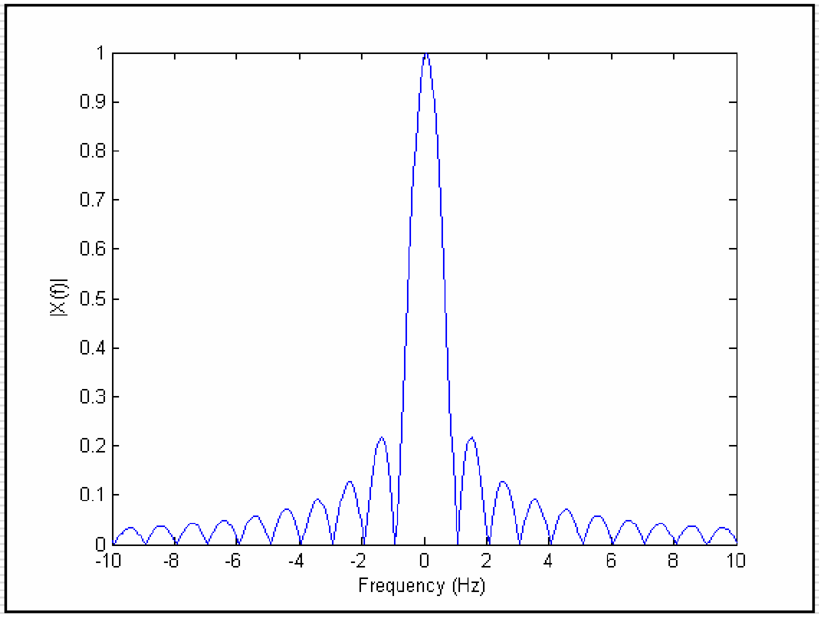
- Thus, we can write directly

$$Z(f) = \frac{T}{2}\mathrm{sinc}\left((f - f_o)T\right) + \frac{T}{2}\mathrm{sinc}\left((f + f_o)T\right)$$

Example 4.1 – cont.

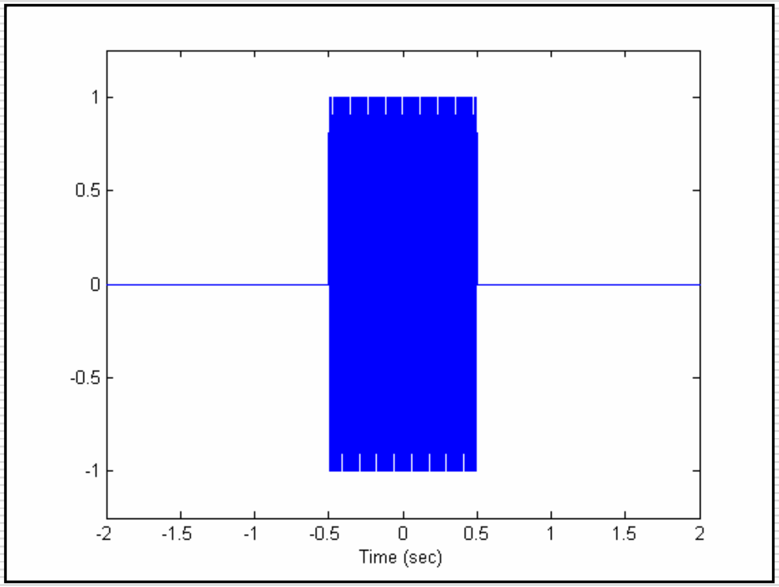
x(t) = rect(t) |X(f)| = |sinc(f)|

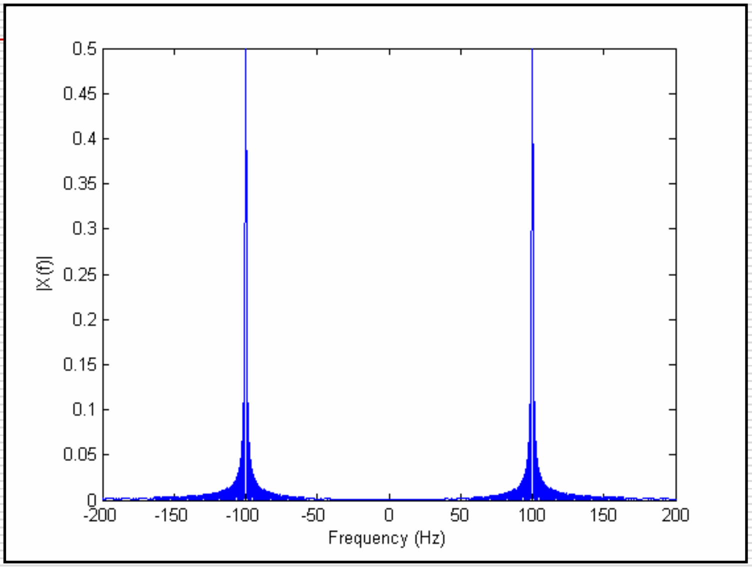




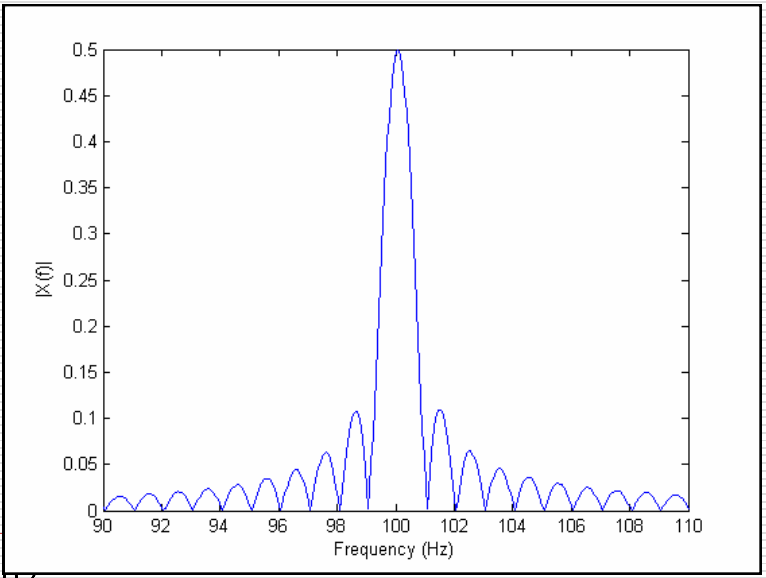
Example 4.1 – cont.

$$z(t) = rect(t)\cos(200\pi t)$$





$$Z(f) = \frac{1}{2}\text{sinc}(f - 100) + \frac{1}{2}\text{sinc}(f + 100)$$



Parseval's Theorem

- While the time domain signal *x(t)* and the frequency domain signal *X(f)* appear quite different they do have the same energy.
- That is

$$E_x = \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| X(f) \right|^2 df$$

- In other words, it doesn't matter whether I calculate the energy of a signal in the time domain or in the frequency domain, I get the same result.
  - This should make sense since the two representations are equivalent

Example 4.2

- Find the energy in the pulse *sinc(t)*
- The energy for any signal can be defined as

$$E = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt$$

- Substituting for this signal gives

$$E = \int_{-\infty}^{\infty} \left|\mathrm{sinc}(t)\right|^2 dt$$

$$= \int_{-\infty}^{\infty} \left|\frac{\sin(\pi t)}{\pi t}\right|^2 dt$$

$$= \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{(\pi t)^2} dt$$

Example 4.2 – cont.

- Continuing

$$E = \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{(\pi t)^2} dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx$$ Making a change of variables x = πt

$$= \frac{2}{\pi} \underbrace{\int_{0}^{\infty} \frac{\sin^2(x)}{x^2} dx}_{\pi/2}$$

$$= \frac{2}{\pi} \frac{\pi}{2}$$

$$= 1$$

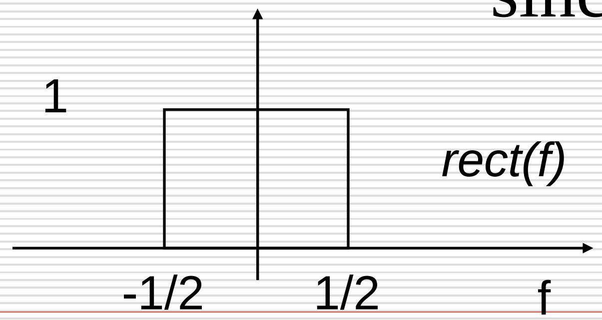
Example 4.2 – cont.

- Using our knowledge of Fourier Transforms and Parseval's Theorem we could have solved this very easily.
- Parseval's Theorem states

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

- Further, the Fourier Transform of *sinc(t)* is

$$\text{sinc}(t) \rightleftharpoons rect(f)$$



The energy can easily be found to be 1 from inspection this plot.

Duality

- Due to the similar nature of the Fourier Transform and the Inverse Fourier Transform, the CTFT exhibits the *duality property.*
- The duality property says that if we have the Fourier Transform pair

$$x(t) \rightleftharpoons X(f)$$

then we also have the Fourier Transform pair

$$X(t) \rightleftharpoons x(-f)$$

Example 4.3

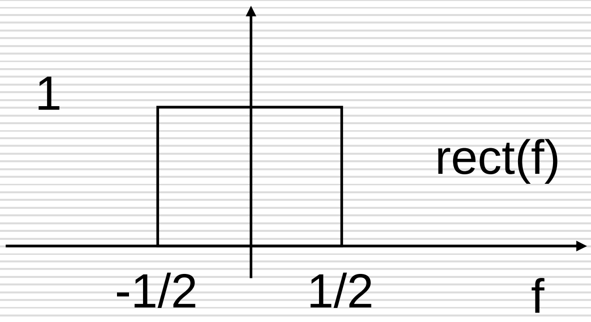
- From our previous development we know the following FT pair

$$rect(t) \stackrel{F}{\longleftrightarrow} \text{sinc}(f)$$

- The duality property says that

$$\text{sinc}(t) \stackrel{F}{\longleftrightarrow} rect(-f) = rect(f)$$

Check: Find the Inverse Fourier Transform for



Example 4.3 (cont.)

- Check:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$= \int_{-1/2}^{1/2} e^{j2\pi ft} df$$

$$= \left. \frac{e^{j2\pi ft}}{j2\pi t} \right|_{-1/2}^{1/2}$$

$$= \frac{e^{j\pi t}}{j2\pi t} - \frac{e^{-j\pi t}}{j2\pi t}$$

$$= \frac{1}{\pi t} \frac{e^{j\pi t} - e^{-j\pi t}}{2j}$$

$$= \frac{\sin(\pi t)}{\pi t}$$

$$= \operatorname{sinc}(t)$$

Example 4.3 (cont.)

More generally, consider the function *X(f) = rect(f/B)*:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\
&= \int_{-B/2}^{B/2} e^{j2\pi ft} df \\
&= \left. \frac{e^{j2\pi ft}}{j2\pi t} \right|_{-B/2}^{B/2} \\
&= \frac{e^{j\pi tB}}{j2\pi t} - \frac{e^{-j\pi tB}}{j2\pi t} \\
&= \frac{1}{\pi t} \frac{e^{j\pi tB} - e^{-j\pi tB}}{2j} \\
&= \frac{\sin(\pi tB)}{\pi t} \\
&= B \operatorname{sinc}(tB)
\end{aligned}
$$

Example 4.4

- From the table we know the following FT pair

$$1 \rightleftharpoons \delta(f)$$

- The duality property says that

$$\delta(t) \rightleftharpoons 1$$

- Check: Find the Fourier Transform for δ(t)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt$$

$$= e^{-j2\pi ft}\Big|_{t=0}$$

$$= 1$$

Example 4.5

- Using Parseval's Theorem, determine the energy of

$$x(t) = \cos(2\pi f_o t)$$

- Solution: From Parseval's Theorem:

$$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$= \int_{-\infty}^{\infty} \left| \frac{1}{2}\delta(f - f_o) + \frac{1}{2}\delta(f + f_o) \right|^2 df$$

$$= \int_{-\infty}^{\infty} \frac{1}{4}\delta^2(f - f_o) + \frac{1}{4}\delta^2(f + f_o) df$$

$$= \infty$$

Side Bar – δ²(t)

- How do we evaluate *δ²(t)* ?

$$\int_{-\infty}^{\infty}\delta^2(t)dt = \lim_{a\to 0}\int_{-\infty}^{\infty}\left\{\frac{1}{a}rect\left(\frac{t}{a}\right)\right\}^2 dt$$

$$= \lim_{a\to 0}\frac{1}{a^2}\int_{-a/2}^{a/2}dt$$

$$= \lim_{a\to 0}\frac{1}{a^2}a$$

$$= \lim_{a\to 0}\frac{1}{a}$$

$$= \infty$$

Thus we have the following:

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_o)dt = g(t_o)$$

$$\int_{-\infty}^{\infty} g(t)\delta^2(t-t_o)dt = \infty$$

Example 4.5 – cont.

- We can double check the result using time domain integration:

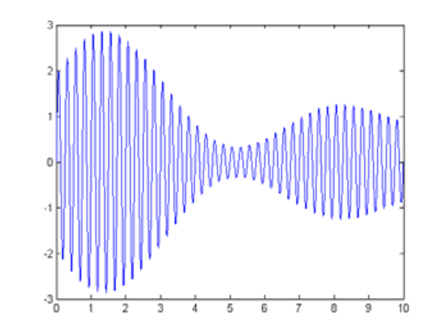
$$
\begin{aligned}
E_x &= \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} \left| \cos(2\pi f_o t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} \cos^2(2\pi f_o t) dt \\
&= \infty
\end{aligned}
$$

Summary

- In this lecture we have examined several properties of the Fourier Transform and examples which demonstrate their usefulness
- The properties are very similar to those for the Fourier Series
- We will find these properties very useful in determining the Fourier Transform of arbitrary signals.
  - Using a simple table of Fourier Transforms and FT properties, we can determine the FT of most signals of interest.

Supplemental Slides

Convolution and Multiplication Properties



Convolution in Time

$$z(t) = x(t) * y(t)$$

- Let

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$

- Then

$$= \int_{-\infty}^{\infty} \left\{ x(t) * y(t) \right\} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \right\} e^{-j2\pi ft} dt$$

- Changing the order of integration:

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\left\{ \int_{-\infty}^{\infty} y(t-\tau) e^{-j2\pi ft} dt \right\}}_{F\{y(t-\tau)\}} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} Y(f) d\tau$$

Convolution (cont.)

- Finishing .... $Z(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} Y(f)\, d\tau$

$$= Y(f) \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau}\, d\tau$$

$$= X(f)Y(f)$$

$$\boxed{x(t) * y(t) \overset{F}{\longleftrightarrow} X(f)Y(f)}$$

Multiplication in Time

$$z(t) = x(t)y(t)$$

- Now let

$$Z(f) = \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} x(t)\left\{\int_{-\infty}^{\infty} Y(\lambda)e^{j2\pi\lambda t}d\lambda\right\}e^{-j2\pi ft}dt$$

- Changing the order of integration

$$= \int_{-\infty}^{\infty} Y(\lambda)\underbrace{\left\{\int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-\lambda)t}dt\right\}}_{X(f-\lambda)}d\lambda$$

Multiplication (cont.)

- Continuing ...

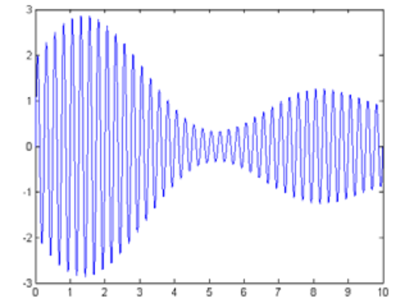
$$Z(f) = \int_{-\infty}^{\infty} Y(\lambda) X(f-\lambda) d\lambda$$
$$= X(f) * Y(f)$$

$$x(t)y(t) \xleftrightarrow{F} X(f) * Y(f)$$

- Thus, convolution in the time domain results in multiplication in the frequency domain while multiplication in the time domain results in convolution in the frequency domain.
- This can greatly simplify some system analysis

Supplemental Slides

Additional Example



Example 4.6

- Use the integration property to determine the Fourier Transform of *z(t) = tri(t)*
- Taking the derivative of *tri(t)* twice we have

$$\frac{d}{dt}\left\{tri(t)\right\} = rect\left(t+\frac{1}{2}\right) - rect\left(t-\frac{1}{2}\right)$$

$$\frac{d}{dt}\left\{rect\left(t+\frac{1}{2}\right) - rect\left(t-\frac{1}{2}\right)\right\} = \delta(t+1) - \delta(t) - \left(\delta(t) - \delta(t-1)\right)$$

$$= \delta(t+1) - 2\delta(t) + \delta(t-1)$$

$$F\left\{\delta(t+1) - 2\delta(t) + \delta(t-1)\right\} = \exp(j2\pi f) - 2 + \exp(-j2\pi f)$$

$$= \underbrace{2\cos(2\pi f) - 2}_{X(f)}$$

Example 4.6 – cont.

- From the integration property we have

$$
\begin{aligned}
Z(f) &= \frac{1}{(j2\pi f)^2} X(f) \\
&= \frac{1}{(j2\pi f)^2}\left\{2\cos(2\pi f) - 2\right\} \\
&= \frac{-1}{(j\pi f)^2}\sin^2(\pi f) \\
&= \frac{\sin^2(\pi f)}{(\pi f)^2} \\
&= \operatorname{sinc}^2(f)
\end{aligned}
$$

Note X(0) = 0

$$\sin^2(x) = \frac{1}{2}\left[1 - \cos(2x)\right]$$