ECE3614
Introduction to
Communications Systems
Fall 2007

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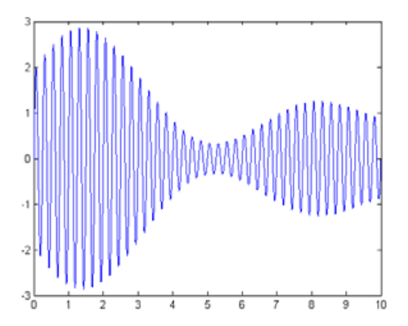
Lecture #5: Singularity Functions and Fourier Transform of Periodic Signals



Overview

- In the previous two weeks the we have examined the Fourier Transform applied to energy signals
- Today we examine the Fourier Transform applied to periodic (i.e., power) signals
- First we must discuss a special family of discontinuous functions known as singularity functions
  - Dirac Delta function
  - Step Function
  - Ramp Function
- Reading
  - 2.4 – Singularity Functions
  - 2.5 – FT of periodic signals

Lecture Objectives

- To review important singularity functions and emphasize their usefulness in communication system analysis and design.
- To show how the delta function allows the Fourier Transform to be applied to power signals

Important Discontinuous Functions

- A very useful set of functions in system analysis have discontinuities or discontinuous derivatives and are related to one another through integrals and derivatives
- Included in this group are
  - Unit step function
  - Unit ramp function
  - Unit Impulse function
  - Signum function
- Useful in creating mathematical descriptions of signals and systems

Unit Step Function

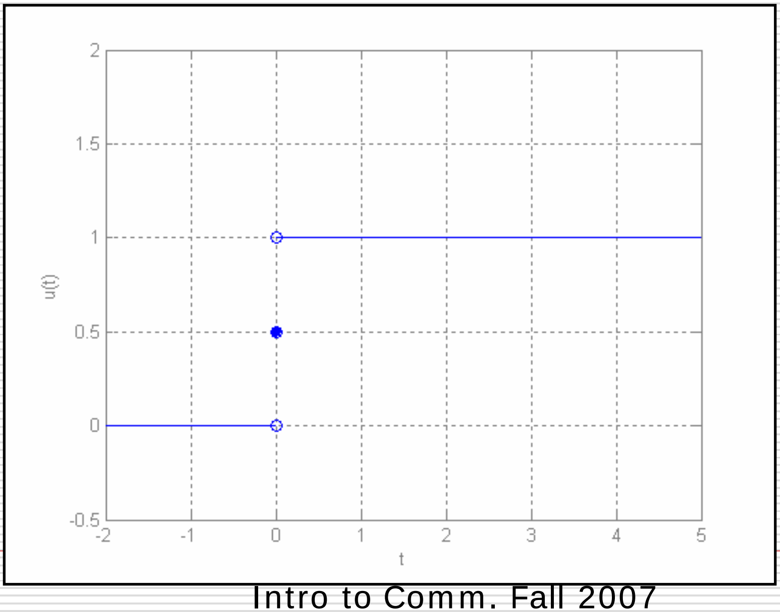
- The *unit step function* is defined as

$$u(t)=\begin{cases}1 & t>0\\ \frac{1}{2} & t=0\\ 0 & t<0\end{cases}$$

*-Note that the definition at t=0 is irrelevant as long as it is finite

- This function is very useful and is commonly used to represent a function or system being switched on

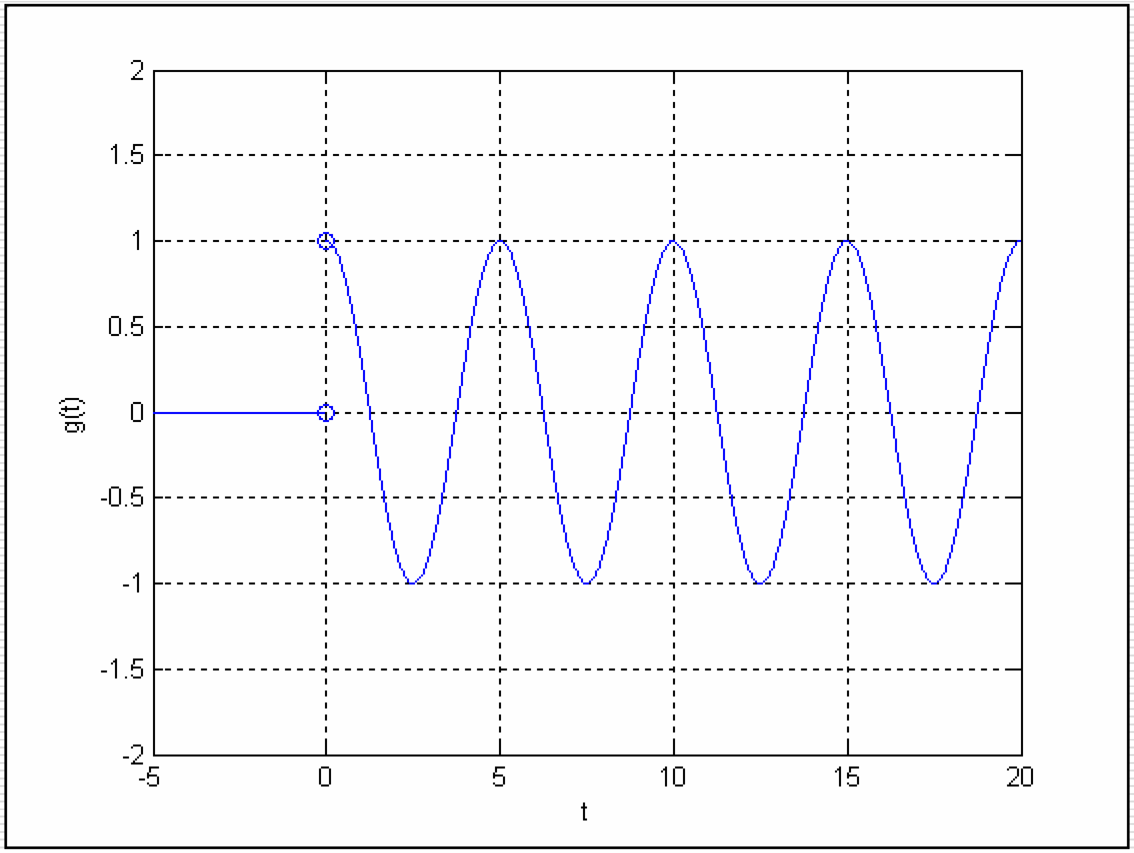
Since the height is “1” we call this the *unit* step function



Note that there is a discontinuity at $t = 0$ which can represents a signal being switched “on”

Example 5.1

- Write a mathematical expression for the following plot



Example 1 (cont.)

- The plot
  - is a sinusoid which starts at $t = 0$
  - is equal to one at $t = 0$ and is thus a cosine
  - Completes one period at t = 5. Thus, frequency is 1/5.
  - Amplitude is 1
- Answer:

$$g(t) = \cos\left(\frac{2\pi}{5}t\right)u(t)$$

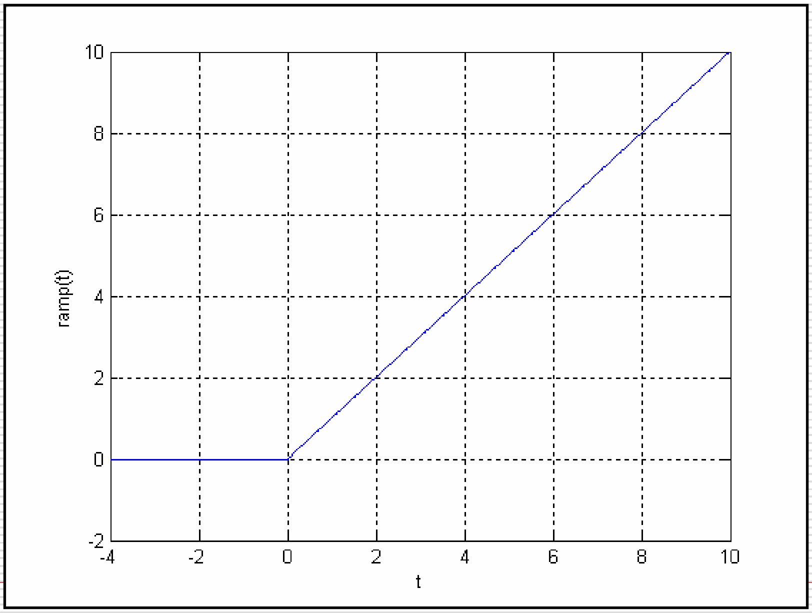
Switches “on” the sinusoid

Unit Ramp Function

- ☐ Another useful function is one which turns on at *t = 0* and increases linearly with time
- ☐ This is termed the *unit ramp function* and is defined as

$$ramp(t) = \begin{cases} t & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Since the slope is "1" we call this the *unit* ramp function



Relationship between the Step and Ramp functions

- ☐ It is easy to show that

$$ramp(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda$$

- ☐ Mathematically

$$\int_{-\infty}^{t} u(\lambda)\,d\lambda = \begin{cases} \int_{0}^{t} d\lambda & t > 0 \\ 0 & t \le 0 \end{cases}$$

$$= \begin{cases} \lambda\big|_{0}^{t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

$$= \begin{cases} t & t > 0 \\ 0 & t \le 0 \end{cases}$$

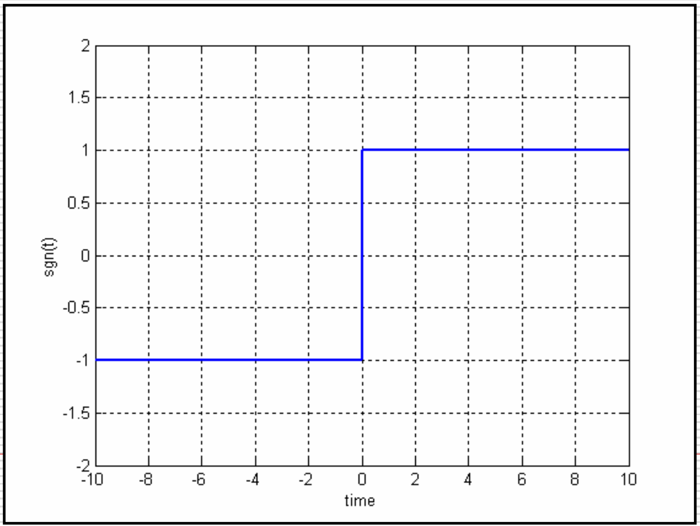
$$= ramp(t)$$

Signum Function

- The signum function is related to the unit step function and is defined as

$$\operatorname{sgn}(t)=\begin{cases}1 & t>0\\ 0 & t=0\\ -1 & t<0\end{cases}$$

- This is also sometimes called the *sign* function since it essentially produces the sign of its argument



The Impulse (Dirac Delta) Function

- The unit impulse (also called the Dirac Delta function) is defined as a function which when multiplied by another function *g(t)* (which is finite and continuous at *t=0*) and the product is integrated between limits which include *t=0*, the result is *g(0)*:

$$g(0) = \int_{-\infty}^{\infty} \delta(t) g(t) dt$$

- The impulse can thus be defined as

$$\delta(t) = 0 \quad t \neq 0 \qquad \int_{t_1}^{t_2} \delta(t) dt = \begin{cases} 1 & t_1 < 0 < t_2 \\ 0 & else \end{cases}$$

Properties of the impulse function

- The strength of an impulse is equal to the area of the impulse.
- The unit impulse has area or strength of one.
- Consider an impulse of strength *k* written as *k* δ(t):

$$\int_{-\infty}^{\infty} k\delta(\lambda)g(\lambda)d\lambda = k\,g(0)$$

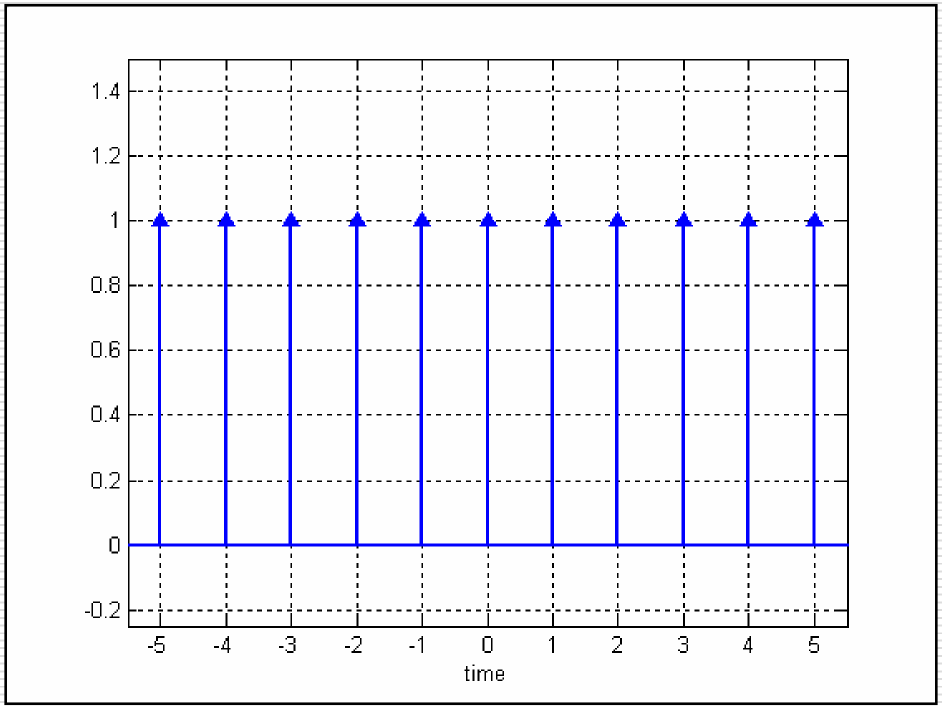
- Equivalence property: $g(t)k\,\delta(t) = k\,g(0)\delta(t)$
- Sampling or sifting property: $\int_{-\infty}^{\infty} \delta(t-t_o)g(t)dt = g(t_o)$
- Replication property: $g(t)*\delta(t-t_o) = g(t-t_o)$

Unit Comb

- ☐ The unit comb is a sequence of uniformly spaced unit impulses (sometimes also called an *impulse train*)

$$comb(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

where *n* is an integer



Since the strength of each impulse is "1" and the spacing of the impulses is unity, we call this the *unit* comb function

Unit Comb – Frequency Domain

- The Fourier Transform of the Unit Comb is simply a Unit Comb in the frequency domain:

$$F\left\{\sum_{n=-\infty}^{\infty}\delta(t-n)\right\}=\sum_{n=-\infty}^{\infty}\delta(f-n)$$

- Further, for non-unit spacing (say period of *T*), we scale by *T* in the time domain:

$$comb\left(\frac{t}{T}\right)=\sum_{n=-\infty}^{\infty}\delta\left(\frac{t}{T}-n\right)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$F\left\{comb\left(\frac{t}{T}\right)\right\}=\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(fT-n)$$

Singularity Functions

- The unit impulse, unit step, and unit ramp are part of a larger family of functions termed *singularity functions* written as $u_k(t)$ where k represents the number of times the unit impulse is differentiated
- A negative value of k represents an integral

$$u_0(t) = \delta(t)$$

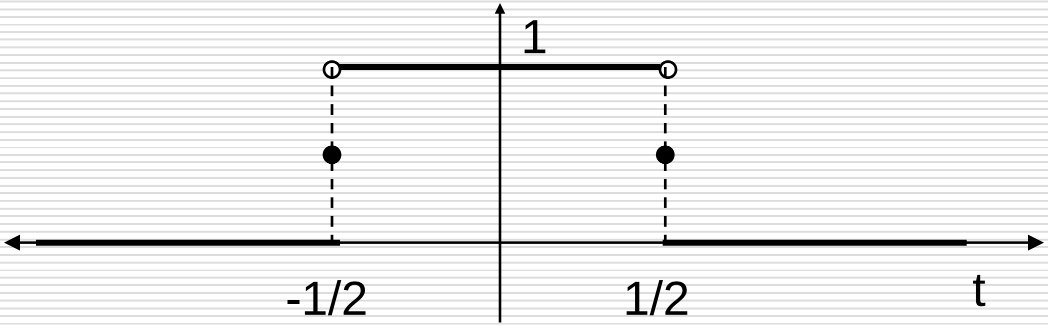
$$u_{-1}(t) = u(t)$$

$$u_{-2}(t) = ramp(t)$$

Other Functions

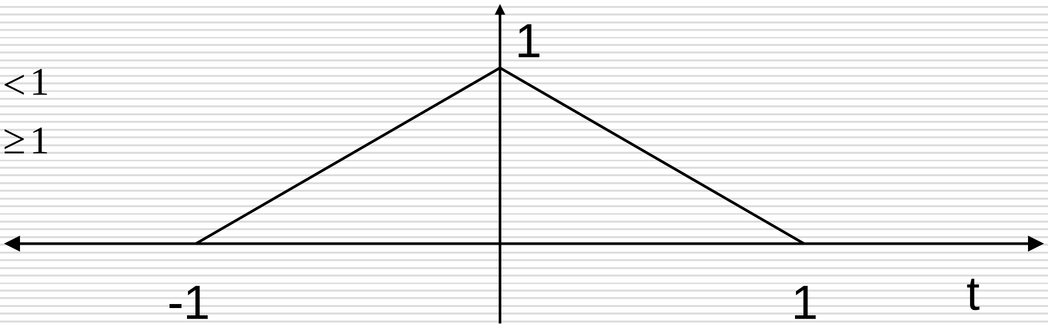
- Unit Rectangle Function [Rectangular Pulse]

$$rect(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ \frac{1}{2} & |t| = \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$



- Unit Triangle

$$tri(t) = \begin{cases} 1-|t| & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$$



Energy and Power

- The **Energy** of a signal *g(t)* is defined as:

$$E = \int_{-\infty}^{\infty} g^2(t)\,dt$$

- A signal *g(t)* is classified as an Energy Signal if

$$0 < E < \infty$$

- The **Power** of a signal *g(t)* is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)\,dt$$

- A signal *g(t)* is a Power Signal if $0 < P < \infty$

Generalized Fourier Transform

- Consider the Fourier Transform of a constant *A*

$$x(t) = A$$

$$X(f) = \int_{-\infty}^{\infty} Ae^{-j2\pi ft}dt$$

$$= A\int_{-\infty}^{\infty} e^{-j2\pi ft}dt$$

- Unfortunately, this integral does not converge. Thus, the Fourier Transform does not technically exist. However, we can determine a *generalized Fourier Transform*

Generalized FT (cont.)

- ☐ Consider a signal $x(t) = Ae^{-\sigma|t|}$ for σ > 0

$$X(f) = \int_{-\infty}^{\infty} Ae^{-\sigma|t|} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{0} Ae^{\sigma t} e^{-j2\pi ft} dt + \int_{0}^{\infty} Ae^{-\sigma t} e^{-j2\pi ft} dt$$

$$= A\frac{2\sigma}{\sigma^2 + (2\pi f)^2}$$

- ☐ Now, let σ approach zero

$$\lim_{\sigma \to 0} \left\{ A\frac{2\sigma}{\sigma^2 + (2\pi f)^2} \right\} = \begin{cases} 0 & f \neq 0 \\ 0/0 & f = 0 \end{cases}$$

Generalized FT (cont.)

- Area

$$\int_{-\infty}^{\infty} A\frac{2\sigma}{\sigma^2+(2\pi f)^2}\,df = A\left[\frac{2\sigma}{2\pi\sigma}\tan^{-1}\left(\frac{2\pi f}{\sigma}\right)\right]\Bigg|_{-\infty}^{\infty} = \frac{A}{\pi}\left\{\frac{\pi}{2}+\frac{\pi}{2}\right\} = A$$

- Thus, as σ → 0, the resulting function
  - is zero everywhere except at f=0.
  - has unit area regardless of σ
- This is exactly the definition of the unit impulse.
- Thus, we have the Fourier Transform pair

$$A \overset{F}{\leftrightarrow} A\delta(f)$$

Frequency Shift Property

- ☐ Let

$$z(t) = e^{j2\pi f_o t} x(t)$$

- ☐ Then

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} e^{j2\pi f_o t} x(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f - f_o)t} dt$$

$$= X(f - f_o)$$

$$\boxed{e^{j2\pi f_o t} x(t) \xleftrightarrow{FS} X(f - f_o)}$$

FT of Complex Exponentials

- Using the generalized Fourier Transform of a constant and the frequency shift property we can find the FT of a complex exponential:

$$z(t) = e^{j2\pi f_o t}$$

- This is just a “frequency shift” of a constant, thus

$$Z(f) = \delta(f - f_o)$$

- This results in

$$e^{j2\pi f_o t} \overset{=}{\rightleftharpoons} \delta(f - f_o)$$

$$\sin(2\pi f_o t) \rightleftharpoons \frac{1}{2j}\delta(f - f_o) - \frac{1}{2j}\delta(f + f_o)$$

$$\cos(2\pi f_o t) \rightleftharpoons \frac{1}{2}\delta(f - f_o) + \frac{1}{2}\delta(f + f_o)$$

Fourier Transform of Periodic Signals

- Using this same approach we can develop the Fourier Transform for any periodic signal.
- Specifically, we can define a Fourier Series for any periodic signal that is valid over all time by making $T_F = T_o$.
- Using the linearity property of the Fourier Transform we can write the FT of any periodic signal:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_o t}$$

$$X(f) = F\left\{ \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_o t} \right\}$$

$$= \sum_{k=-\infty}^{\infty} X[k] \delta(f - k f_o)$$

Example 5.2

- Find the Fourier Transform of a 50% duty cycle square wave with period T_o, amplitude 1 and average value 1/2 .
- The signal can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t - nT_o}{T_o/2}\right)$$

- Clearly this signal is not integrable and thus does not have a Fourier Transform in the strict sense.
- However, we can write this in terms of its Fourier Series coefficients as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_F t}$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_F t}$$

$$c_k = f_o T_w \text{sinc}(k f_o T_w)$$

$$= \frac{1}{2}\text{sinc}\left(\frac{k}{2}\right)$$

Example 5.2 (cont.)

- Writing the signal in terms of its Fourier Series coefficients (using $T_F = T_o$):

$$x(t) = \frac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_o t}$$

- Using the linearity property and the FT for a complex exponential:

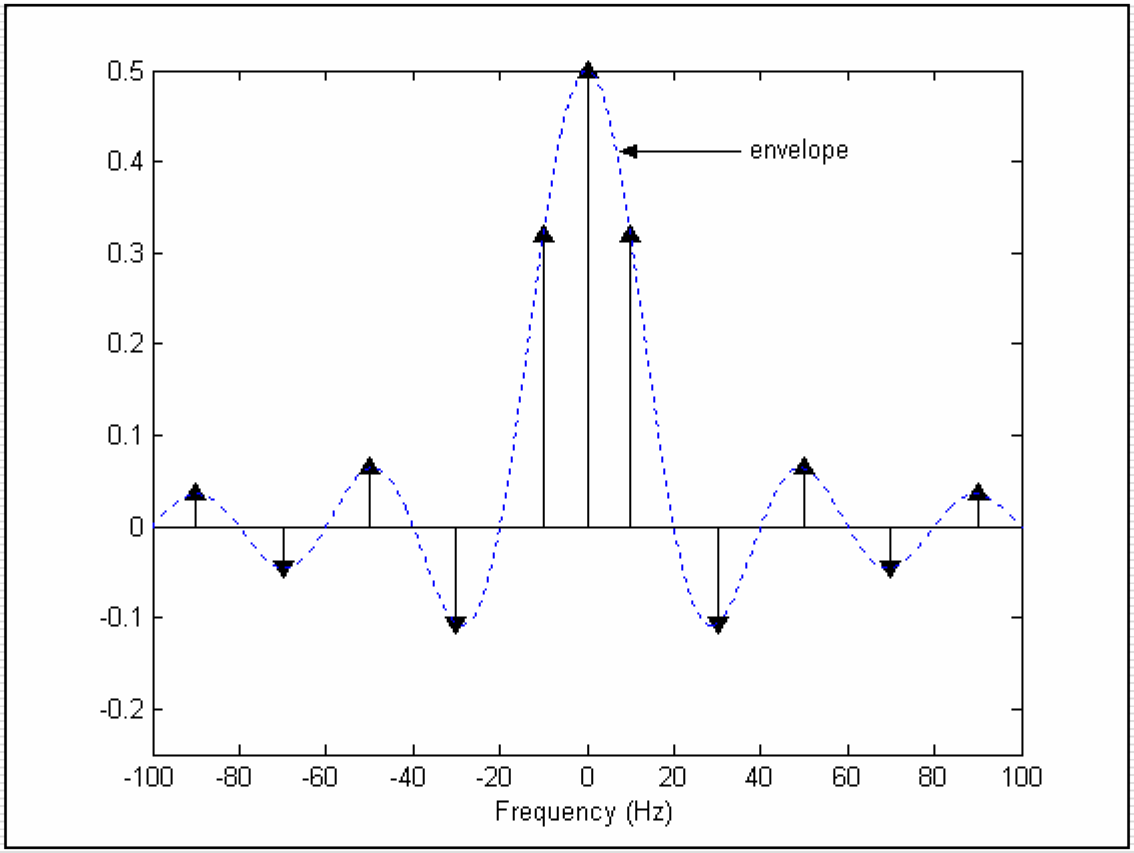
$$F\left\{e^{j2\pi f_o t}\right\} = \delta\left(f - f_o\right)$$

$$\begin{aligned} X(f) &= F\left\{\frac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{k}{2}\right) e^{j2\pi k f_o t}\right\} \\ &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{k}{2}\right) F\left\{e^{j2\pi k f_o t}\right\} \\ &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{k}{2}\right) \delta\left(f - k f_o\right) \end{aligned}$$

Note that since the signal is *periodic* the spectrum (i.e., Fourier Transform) is *discrete*

Example 5.2 (cont.)

- Plotting the spectrum:



Example 5.3

- ☐ Determine the Fourier Transform of the signum function

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

- ☐ Technically, this does not satisfy the Dirichlet conditions and thus does not have a Fourier Transform. However consider the function

$$g(t) = \begin{cases} \exp(-at) & t > 0 \\ 0 & t = 0 \\ -\exp(at) & t < 0 \end{cases}$$

Example 5.3 (cont.)

- ☐ We can easily find the Fourier Transform of this function as

$$g(t) \rightleftharpoons G(f) = \left(\frac{-j4\pi f}{a^2 + (2\pi f)^2} \right)$$

- ☐ Now, if we let a → 0

$$\lim_{a\to 0} g(t) = \operatorname{sgn}(t)$$

$$\lim_{a\to 0} G(f) = \left(\frac{-j4\pi f}{(2\pi f)^2} \right)$$

$$= \frac{1}{j\pi f}$$

- ☐ Thus,

$$\operatorname{sgn}(t) \rightleftharpoons \frac{1}{j\pi f}$$

Example 5.4

- Determine the Fourier Transform of the unit step

$$u(t)=\begin{cases}1 & t>0\\ 1/2 & t=0\\ 0 & t<0\end{cases}$$

- We can easily see that

$$u(t)=\frac{1}{2}\left[\operatorname{sgn}(t)+1\right]$$

- Using linearity

$$u(t)\rightleftharpoons\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$$

Integration

- ☐ Previously we stated that $\int_{-\infty}^{t} x(\tau)\,d\tau \overset{*}{\rightleftharpoons} \frac{1}{j2\pi f} X(f)$ which assumed that *X(0)=0.*
- ☐ The integral of any function to time *t* is equal to the convolution with a unit step function

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\,d\tau$$

- ☐ Since convolution in the time domain is multiplication in the frequency domain

$$\begin{aligned} Y(f) &= X(f)U(f) \\ &= X(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \\ &= \frac{X(f)}{j2\pi f} + \frac{X(0)}{2}\delta(f) \end{aligned}$$

In-Class Drill

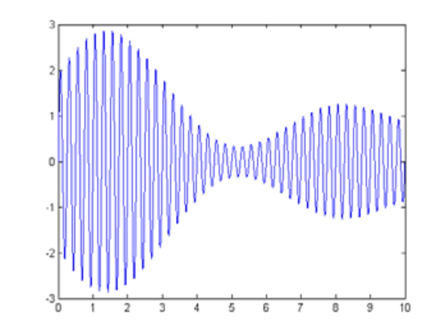
- Find the Fourier Transform of the following signal

Summary

- In this class we have discussed a very important class of discontinuous functions including the Dirac delta function
- The Dirac delta function (or impulse) can be used to define a generalized Fourier Transform for periodic signals
  - In general the Fourier Series can be determined for any periodic signal. When combined with the Dirac delta function, the Fourier Series coefficients allow us to determine the Fourier Transform

Supplemental Slides

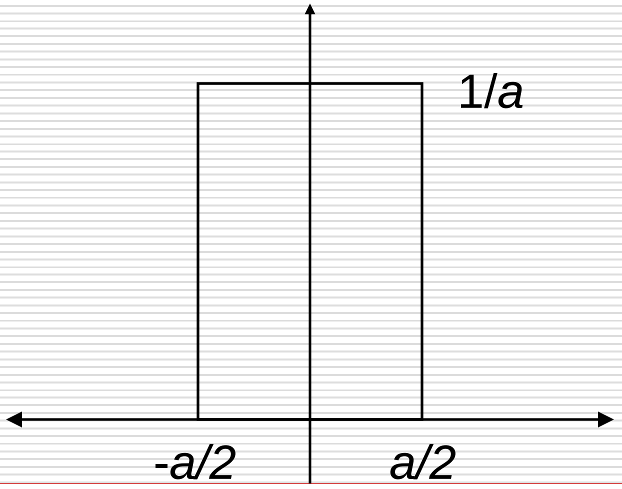
The Dirac Delta Function



Impulse (Dirac Delta) Function

- One of the most useful, yet strange, functions that we will encounter in this class is the unit impulse function, $\delta(t)$ (sometimes also called the Dirac delta function).
- To understand the unit impulse function consider a unit area pulse $\delta_a(t)$ which has width *a* and height *1/a*:

$$\delta_a(t) = \begin{cases} \frac{1}{a} & |t| < \frac{a}{2} \\ 0 & else \end{cases}$$



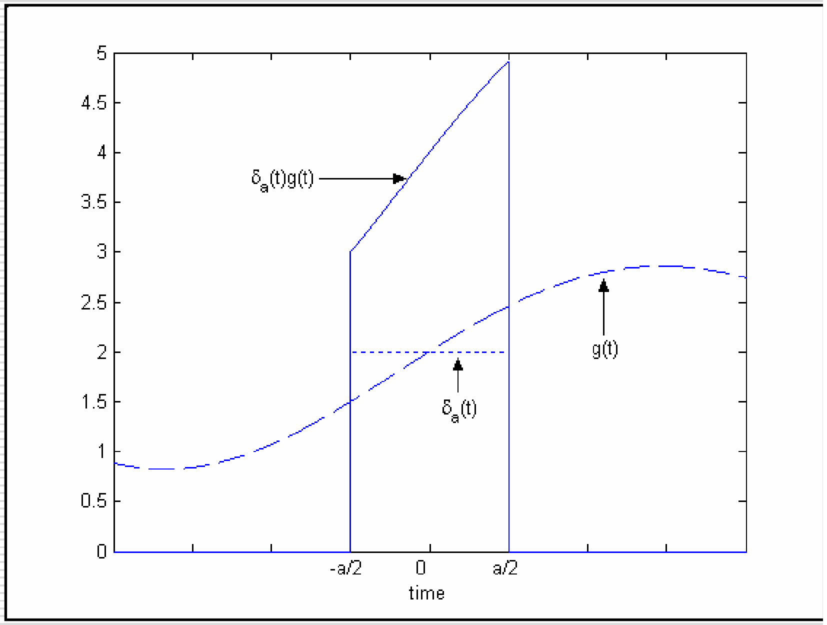
Area = a*1/a = 1

Unit Impulse Function (cont.)

☐ Now, consider a the integral of the unit pulse times a function g(t):

$$A = \int_{-\infty}^{\infty} \delta_a(t) g(t)\,dt$$

$$= \frac{1}{a} \int_{-a/2}^{a/2} g(t)\,dt$$



If we let the interval, *a*, get very small:

$$\lim_{a\to 0} A = \lim_{a\to 0}\left\{ \int_{-\infty}^{\infty} \delta_a(t) g(t)\,dt \right\}$$

$$= \lim_{a\to 0}\left\{ \frac{1}{a} \int_{-a/2}^{a/2} g(t)\,dt \right\}$$

$$= g(0) \lim_{a\to 0}\left\{ \frac{1}{a} \int_{-a/2}^{a/2} dt \right\}$$

$$= g(0) \lim_{a\to 0} \frac{1}{a} a$$

$$= g(0)$$

Thus, in the limit integrating the multiplication of unit pulse with a function results in the value of the function at zero.

Unit Impulse Function (cont.)

- Thus, in the limit as *a* → 0, the function $\delta_a(t)$ has the property that it extracts the value of the function at time equal 0 when their product is integrated over any limits which include $t=0$.
- Note that we could arrive at this same property with an entirely different function:

$$\delta_a(t)=\begin{cases}\frac{1}{a}\left(1-\frac{|t|}{a}\right) & |t|<a\\ 0 & |t|>a\end{cases}$$

$$\begin{aligned}\lim_{a\to 0} A &= \lim_{a\to 0}\left\{\frac{1}{a}\int_{-a}^{a}\left(1-\frac{|t|}{a}\right)g(t)\,dt\right\}\\ &= g(0)\lim_{a\to 0}\left\{\frac{2}{a}\int_{0}^{a}\left(1-\frac{t}{a}\right)dt\right\}\\ &= g(0)\lim_{a\to 0}\frac{2}{a}\left[t-\frac{t^2}{2a}\right]_0^a\\ &= g(0)\end{aligned}$$

- The key to this property is that the function has unit area. The shape of the function is irrelevant.

Relationship between the Impulse and Step Functions

- We saw previously that

$$ramp(t)=\int_{-\infty}^{t}u(\lambda)d\lambda$$

- What is the derivative of the unit step?
- For functions with discontinuities, we must use the generalized derivative:

$$\frac{d}{dt}\{g(t)\}=\frac{d}{dt}\{g(t)\}_{t\neq t_o}+\lim_{\varepsilon\to 0}\left[g(t+\varepsilon)-g(t-\varepsilon)\right]\delta(t-t_o)$$

where t_o is the point of the discontinuity

- Let’s apply this to the unit step function

Derivative of the Unit Step

- Taking the derivative:

$$\frac{d}{dt}\left\{u(t)\right\} = \frac{d}{dt}\left\{u(t)\right\}_{t\neq 0} + \lim_{\varepsilon\to 0}\left[u(t+\varepsilon)-u(t-\varepsilon)\right]\delta(t)$$

- The derivative for t< 0 is zero. The derivative for t > 0 is also zero.
- Thus we have

$$\frac{d}{dt}\left\{u(t)\right\} = 0+\left[1-0\right]\delta(t)$$

$$=\delta(t)$$

- The unit impulse function is the generalized derivative of the unit step function.
- Further

$$u(t)=\int_{-\infty}^{t}\delta(\lambda)\,d\lambda$$