ECE3614
Introduction to Communications Systems
Fall 2007

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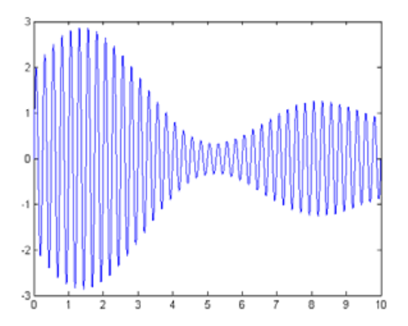
Lecture #6: The Impulse Response and Transfer Function of Linear Systems



Overview

- Today we will apply the Fourier Transform to linear time-invariant systems
- Specifically, we will show that an LTI system can be characterized in the time domain by the impulse response, but an even more powerful way of characterizing an LTI system is in the frequency domain through the use of the transfer function.

- Reading
  - Section 2.6

Lecture Objectives

- To describe and show how to use a system's impulse response and transfer function

Why study LTI systems?

- Several parts of a communication system can be modeled as an LTI system including
  - Filters
  - Equalizers
  - The channel
  - Pulse shaping

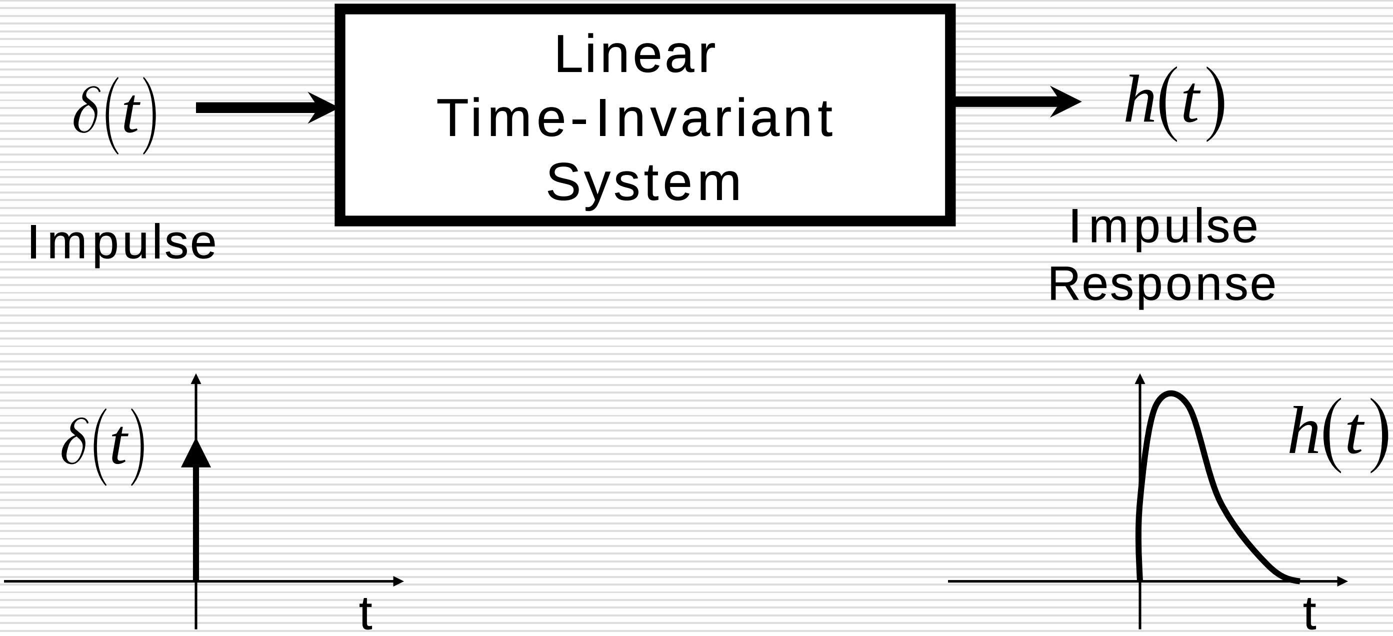
Linear Time-Invariant Systems

- A system is *linear* when superposition holds:

$$y(t) = \Im\left[a_1x_1(t) + a_2x_2(t)\right] = a_1\Im\left[x_1(t)\right] + a_2\Im\left[x_2(t)\right]$$

- A system is *time-invariant* if for a delayed input $x(t-t_o)$ the output is simply delayed by the same amount $y(t-t_o)$. In other words the *shape* of the output is the same no matter when the input is applied.

Impulse Response



$\delta(t)$

Linear
Time-Invariant
System

$h(t)$

Impulse

Impulse
Response

- When the input to a linear time-invariant system is an impulse, we term the output the *impulse response* of the system

Impulse Response

- An LTI system may be characterized by its *impulse response.* If we represent the output and input of an LTI system by $y(t)$ and $x(t)$, respectively we say that the impulse response $h(t)$ is the output $y(t)$ when $x(t)=\delta(t)$
- The impulse response is useful because it can be used to find the output of a system when the input is not an impulse.

Impulse Response (cont.)

- Let us approximate a general input *x(t)* using a series of impulses

$$x(t) \approx \sum_{n=0}^{\infty} x(n\Delta t)[\delta(t - n\Delta t)]\Delta t$$

- Since the system is LTI we can approximate the output as

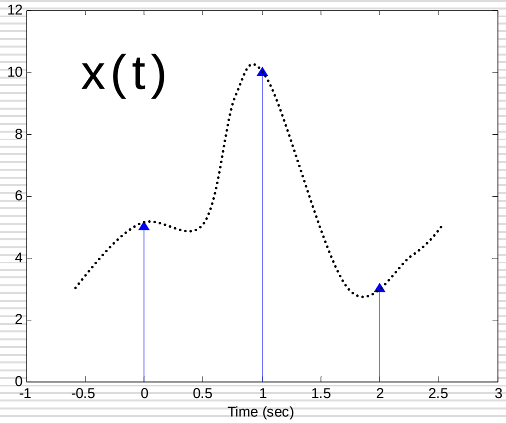
$$y(t) \approx \sum_{n=0}^{\infty} x(n\Delta t)\left[h(t - n\Delta t)\right]\Delta t$$

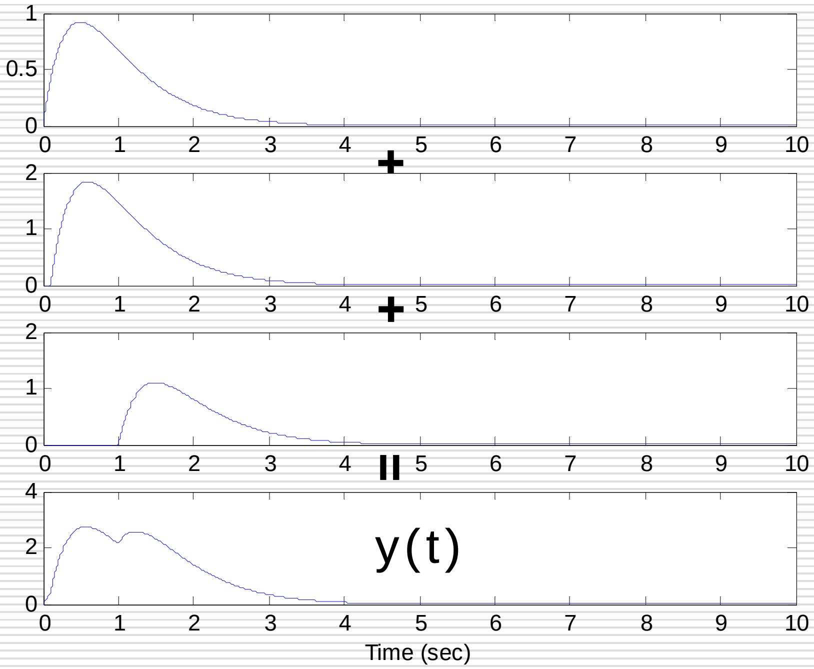
Impulse Response (cont.)

x(t)

Linear
Time-Invariant
System

y(t)





Impulse Response (cont.)

- Now, if we let Δt go to zero we get:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

- This says that the output of an LTI system is the *convolution* of the input and the impulse response. Thus we can find the output based on any input if we know the impulse response.

Impulse Response - Interpretation

- From the convolution integral we can see that the *current output* is determined by the current input and the past input.
- The weighting of the past inputs is equal to the impulse response.
- Thus, the impulse response acts as the memory of the system.
- What if the impulse response were an impulse?

Impulse Response in the Frequency Domain

- Recall the Fourier Transform Property

$$w_1(t) * w_2(t) \rightleftharpoons W_1(f) \cdot W_2(f)$$

- A complicated operation in the time domain can be reduced to a simple operation in the frequency domain
- We can now more simply find the system output

$$y(t) = x(t) * h(t) \Leftrightarrow Y(f) = X(f)H(f)$$

Transfer Function

- Rewriting

$$H(f) = \frac{Y(f)}{X(f)}$$

H(f) is termed the *Transfer Function* of the system.

- Thus, we can find the transfer function of an LTI system by dividing the output signal's Fourier Transform by the Fourier Transform of the input

Alternate View

- Consider the input of a linear time-invariant system characterized by impulse response *h(t)* with input

$$x(t) = e^{j2\pi f_o t}$$

- The output can be determined as

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \\ &= \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda \\ &= \int_{-\infty}^{\infty} h(\lambda)e^{j2\pi f_o (t-\lambda)}d\lambda \end{aligned}$$

Transfer Function (cont.)

- Continuing

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\lambda) e^{j2\pi f_o (t-\lambda)} d\lambda \\
&= e^{j2\pi f_o t} \int_{-\infty}^{\infty} h(\lambda) e^{-j2\pi f_o \lambda} d\lambda \\
&= e^{j2\pi f_o t} H\left(f_o\right)
\end{aligned}
$$

- Thus, when the input to a linear time-invariant system is a complex sinusoid of frequency f_o, the output is also a complex sinusoid of frequency f_o weighted by the transfer function at that frequency, $H(f_o)$

Transfer Function (cont.)

In general, a signal *x(t)* can be expressed in terms of the inverse Fourier Transform

$$x(t)=\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

which can be written in the limiting form

$$x(t)=\lim_{\Delta f\to 0}\sum_{k=-\infty}^{\infty} X(k\Delta f)e^{j2\pi k\Delta ft}\Delta f$$

Thus, from the previous development we can write

$$y(t)=\lim_{\Delta f\to 0}\sum_{k=-\infty}^{\infty} H(k\Delta f)X(k\Delta f)e^{j2\pi k\Delta ft}\Delta f$$

$$=\int_{-\infty}^{\infty} \underbrace{H(f)X(f)}_{Y(f)}e^{j2\pi ft}df$$

Thus

$$Y(f)=H(f)X(f)$$

Magnitude/Phase Response

- The transfer function *H(f)* is a characteristic property of an LTI system
- We are interested both in the *amplitude response or magnitude response* |*H(f)*| and the phase response *angle(H(f))*
- If the impulse response is real, then the magnitude response is even

$$\left|H(f)\right| = \left|H(-f)\right|$$

and the phase response is odd

$$\angle H(f) = -\angle H(-f)$$

Proof - Magnitude

- Taking the magnitude

$$
\begin{aligned}
\left|H(f)\right| &= \left|\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt\right| \\
&= \left|\int_{-\infty}^{\infty} h(t)\cos(2\pi ft)dt + j\int_{-\infty}^{\infty} h(t)\sin(2\pi ft)dt\right| \\
&= \sqrt{\left(\int_{-\infty}^{\infty} h(t)\cos(2\pi ft)dt\right)^2 + \left(\int_{-\infty}^{\infty} h(t)\sin(2\pi ft)dt\right)^2}
\end{aligned}
$$

- The values for *f* and –*f* are the same

Proof - Angle of *H(f)*

- Now, we can take the angle

$$\begin{aligned}\angle H(f) &= \angle\left(\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt\right)\\ &= \angle\left(\int_{-\infty}^{\infty} h(t)\cos(2\pi ft)dt + j\int_{-\infty}^{\infty} h(t)\sin(2\pi ft)dt\right)\\ &= \tan^{-1}\left(\frac{\int_{-\infty}^{\infty} h(t)\sin(2\pi ft)dt}{\int_{-\infty}^{\infty} h(t)\cos(2\pi ft)dt}\right)\end{aligned}$$

where we assumed a real signal *h(t)* to make the first step

Proof - Angle (cont.)

- ☐ Substituting for *–f* we have

$$\angle H(-f) = \tan^{-1}\left(\frac{\int_{-\infty}^{\infty} h(t)\sin\left(2\pi(-f)t\right)dt}{\int_{-\infty}^{\infty} h(t)\cos\left(2\pi(-f)t\right)dt}\right) = \tan^{-1}\left(\frac{-\int_{-\infty}^{\infty} h(t)\sin\left(2\pi ft\right)dt}{\int_{-\infty}^{\infty} h(t)\cos\left(2\pi ft\right)dt}\right)$$

$$= -\tan^{-1}\left(\frac{\int_{-\infty}^{\infty} h(t)\sin\left(2\pi ft\right)dt}{\int_{-\infty}^{\infty} h(t)\cos\left(2\pi ft\right)dt}\right) = -\angle H(f)$$

Example 6.1

- Consider a system with impulse response

$$h(t) = e^{-5t}u(t)$$

- Determine the output when the input is

$$x(t) = \cos(200\pi t) + \cos(20\pi t)$$

- What does the system do to the input?

Example 6.1 (solution)

- We can solve this in the time domain or in the frequency domain
- In the time domain we have

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} \left(\cos\left(200\pi\lambda\right)+\cos\left(20\pi\lambda\right)\right)\exp(-5\left(t-\lambda\right))d\lambda$$

- This doesn’t appear to be a fun convolution! (Although manageable)

Example 6.1 (cont.)

- Let's try this in the frequency domain instead.
- In the frequency domain, we have

$$H(f)=\frac{1}{5+j2\pi f}$$

$$X(f)=\frac{1}{2}\left[\delta\left(f-100\right)+\delta\left(f+100\right)+\delta\left(f-10\right)+\delta\left(f+10\right)\right]$$

$$Y(f)=X(f)H(f)$$

Example 6.1

☐ Calculating the output spectrum

$$Y(f) = \frac{1}{2}\Big[H(100)\delta(f-100) + H(-100)\delta(f+100)$$

$$\ldots + H(10)\delta(f-10) + H(-10)\delta(f+10)\Big]$$

$$= \frac{1}{2}\left[\frac{1}{5+j200\pi}\delta(f-100) + \frac{1}{5-j200\pi}\delta(f+100)\right.$$

$$\left.\ldots + \frac{1}{5+j20\pi}\delta(f-10) + \frac{1}{5-j20\pi}\delta(f+10)\right]$$

Continuing...

$$Y(f)=\frac{1}{2}\left[\frac{1}{5+j200\pi}\delta\left(f-100\right)+\frac{1}{5-j200\pi}\delta\left(f+100\right)\right.$$

$$\left.\ldots+\frac{1}{5+j20\pi}\delta\left(f-10\right)+\frac{1}{5-j20\pi}\delta\left(f+10\right)\right]$$

$$=\frac{1}{2}\left[\frac{5-j200\pi}{25+\left(200\pi\right)^2}\delta\left(f-100\right)+\frac{5+j200\pi}{25+\left(200\pi\right)^2}\delta\left(f+100\right)\right.$$

$$\left.\ldots+\frac{5-j20\pi}{25+\left(20\pi\right)^2}\delta\left(f-10\right)+\frac{5+j20\pi}{25+\left(20\pi\right)^2}\delta\left(f+10\right)\right]$$

Continuing...

- Collecting cos and sin terms

$$Y(f)=\frac{1}{2}\left[\frac{5}{25+(200\pi)^2}\left[\delta(f-100)+\delta(f+100)\right]\right.$$

$$\ldots+\frac{200\pi}{25+(200\pi)^2}\frac{1}{j}\left[\delta(f-100)-\delta(f+100)\right]$$

$$\ldots+\frac{5}{25+(20\pi)^2}\left[\delta(f-10)+\delta(f+10)\right]$$

$$\left.\ldots+\frac{20\pi}{25+(20\pi)^2}\frac{1}{j}\left[\delta(f-10)-\delta(f+10)\right]\right]$$

Example – time domain

- In the time domain we have

$$y(t)=\frac{5}{25+\left(200\pi\right)^{2}}\cos\left(200\pi t\right)+\frac{200\pi}{25+\left(200\pi\right)^{2}}\sin\left(200\pi t\right)$$

$$\ldots+\frac{5}{25+\left(20\pi\right)^{2}}\cos\left(20\pi t\right)+\frac{20\pi}{25+\left(20\pi\right)^{2}}\sin\left(20\pi t\right)$$

- Now

$$\frac{a}{a^{2}+b^{2}}\cos\left(2\pi f_{o}t\right)+\frac{b}{a^{2}+b^{2}}\sin\left(2\pi f_{o}t\right)=\frac{1}{\sqrt{a^{2}+b^{2}}}\cos\left(2\pi f_{o}t+\theta\right)$$

$$\theta=\tan^{-1}\left(\frac{b}{a}\right)$$

Example 6.1 (cont.)

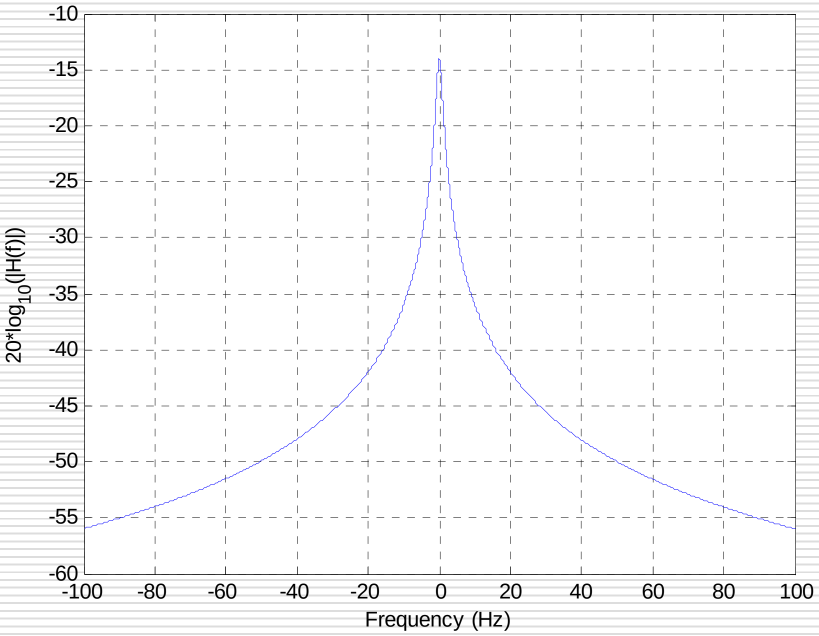
- Finally, we have

$$y(t)=\frac{1}{\sqrt{25+(200\pi)^2}}\cos\left(200\pi t+\theta_1\right)+\frac{1}{\sqrt{25+(20\pi)^2}}\cos\left(20\pi t+\theta_2\right)$$

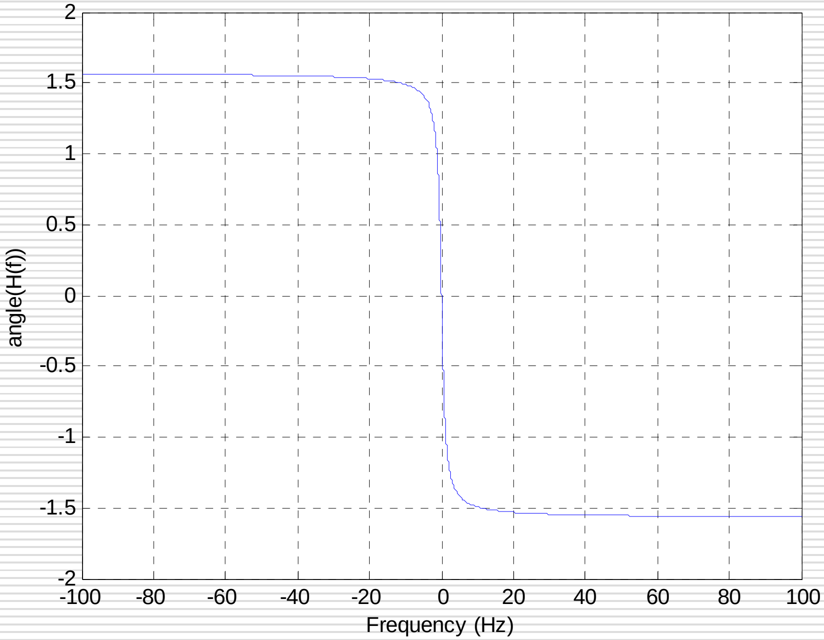
- How do we interpret this?
- The input was a pair of sinusoids with frequencies f_1 = 100 and f_2 = 10.
- The output was also a pair of sinusoids with frequencies f_1 = 100 and f_2 = 10.
- However, the two sinusoids are both attenuated with the first being attenuated more.
- Additionally, both were delayed in time (ie, phase shifted) – θ_1 = 1.56 radians, θ_2 = 1.49 radians (both nearly $\pi/2$)

Transfer Function of the System

Magnitude Response



Phase Response



Summary

- LTI Systems can be characterized in the time domain by their impulse response.
- LTI Systems can also be characterized in the frequency domain by their transfer function which is the Fourier Transform of the impulse response
- The transfer function tells us how input frequencies will be attenuated (from the magnitude response) and how they will be phase shifted (by the phase response).