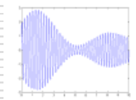
ECE3614
Introduction to
Communications Systems
Fall 2007

Instructor: Dr. R. Michael Buehrer
Lecture #8: The Energy Spectral Density and Power Spectral Density



Overview

- The Fourier Transform provides frequency domain information for both energy signals and power signals.
- The energy spectral density (ESD) provides information of the energy of the signal at each frequency
  - Applies only to energy signals
- The power spectral density (PSD) provides information of the power of a signal at each frequency
  - Applies only to power signals
  - Can be applied to random as well as deterministic signals
- Today we will review the ESD and PSD and their relationship to the autocorrelation function
- Reading
  - 2.8 – 2.9

Developing Energy Spectral Density

- Recall the Convolution/Multiplication property

$$x_1(t)*x_2(t) \Longleftrightarrow X_1(f)X_2(f)$$

- Writing this as an equality

$$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \int_{-\infty}^{\infty} X_1(f)X_2(f)e^{j2\pi ft}df$$

- Reversing time and conjugating we have

$$\int_{-\infty}^{\infty} x_1(\tau)x_2^*(\tau-t)d\tau = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)e^{j2\pi ft}df$$

- Letting $t = 0$ and setting $x_2(t) = x_1(t)$

$$\int_{-\infty}^{\infty} x_1(\tau)x_1^*(\tau)d\tau = \int_{-\infty}^{\infty} X_1(f)X_1^*(f)df$$

ESD – cont.

$$\int_{-\infty}^{\infty} x_1(\tau)x_1^*(\tau)d\tau = \int_{-\infty}^{\infty} X_1(f)X_1^*(f)df$$

- Rewriting

$$\underbrace{\int_{-\infty}^{\infty} |x_1(\tau)|^2 d\tau}_{Energy} = \int_{-\infty}^{\infty} |X_1(f)|^2 df$$

- Since the integral of $|X_1(f)|^2$ over *f* provides the total energy, we can call it the energy spectral density

$$\psi_x(f) = |X_1(f)|^2$$

Autocorrelation

- The autocorrelation function provides a measure of the similarity of a signal with a delayed version of itself

$$R_x(\tau) = \int_{-\infty}^{\infty} x_1(t)x_1^*(t-\tau)dt$$

- Setting the delay τ = 0 we arrive at the energy

$$R_x(0) = \int_{-\infty}^{\infty} x_1(t)x_1^*(t)dt = \int_{-\infty}^{\infty} |x_1(t)|^2 dt$$

Autocorrelation – cont.

- Starting from the correlation relationship we described earlier

$$\int_{-\infty}^{\infty} x_1(t)x_1^*(t-\tau)dt = \int_{-\infty}^{\infty} X_1(f)X_1^*(f)e^{j2\pi f\tau}df$$

- However, the first integral is simply the autocorrelation function

$$R_x(\tau) = \int_{-\infty}^{\infty} X_1(f)X_1^*(f)e^{j2\pi f\tau}df$$

$$= \int_{-\infty}^{\infty} |X_1(f)|^2 e^{j2\pi f\tau}df$$

$$R_x(\tau) = \int_{-\infty}^{\infty} \psi_x(f)e^{j2\pi f\tau}df$$

- Thus, the autocorrelation function and the energy spectral density are Fourier Transform pair

Example 8.1

- Determine the autocorrelation function of a sinc pulse
- Solution: We could use the definition directly, but it doesn't appear to be straightforward

$$R_x(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_1^*(t-\tau)\, dt$$
$$= \int_{-\infty}^{\infty} \text{sinc}(t)\,\text{sinc}(t-\tau)\, dt$$

- How else might we do this problem?

Example 8.1 (cont.)

- Since we know that the autocorrelation function is the Fourier Transform of the ESD, let us examine that approach. Perhaps it is easier.
- The ESD is simply

$$\psi_x(f) = |X(f)|^2$$
$$= |rect(f)|^2$$
$$= rect(f)$$

- Taking the inverse Fourier Transform we have

$$R_x(\tau) = \text{sinc}(\tau)$$

- Thus, the autocorrelation function of a sinc pulse is simply a sinc pulse

Power Spectral Density

- Recall the definition of power as the time-average energy of a signal

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

- Define

$$x_T(t) = rect\left(\frac{t}{2T}\right) x(t)$$

- Thus,

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x_T(t)|^2\, dt$$

- From the Rayleigh energy theorem

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2\, df$$

PSD – cont.

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \left|X_T(f)\right|^2 df$$

- Interchanging the limiting and integration operations

$$P = \int_{-\infty}^{\infty} \left[\lim_{T\to\infty} \frac{1}{2T} \left|X_T(f)\right|^2\right] df$$

$$= \int_{-\infty}^{\infty} S_x(f)\, df$$

where $S_x(f) = \lim_{T\to\infty} \frac{1}{2T} \left|X_T(f)\right|^2$ is termed the Power Spectral Density

Autocorrelation Function

- The autocorrelation function of a power signal is

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x^*(t-\tau)\, dt$$

- Further, just as with energy signals we can show that the autocorrelation function of power signals and the power spectral density are a Fourier Transform pair

$$R_x(\tau) \rightleftharpoons S_x(f)$$

Periodic Signals

- While the previous definitions are helpful for general power signals, for the important subclass of power signals, periodic signals we can find a more direct expression for the power spectral density
- Recall that we can write a periodic signal in terms of the Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_o t}$$

- Further, the autocorrelation function is

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x^*(t-\tau)\, dt$$

Periodic Signals (cont.)

- Substituting the Fourier Series representation into the autocorrelation function definition

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_o t}\right]\left[\sum_{m=-\infty}^{\infty} c_m^* e^{-j2\pi m f_o (t-\tau)}\right]dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n e^{j2\pi n f_o t} c_m^* e^{-j2\pi m f_o (t-\tau)}\right]dt$$

$$= \left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n c_m^* \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{j2\pi n f_o t} e^{-j2\pi m f_o (t-\tau)}dt\right)\right]$$

$$= \left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n c_m^* e^{j2\pi m f_o \tau}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{j2\pi (n-m) f_o t}dt\right)\right]$$

$R_x(\tau)$ of Periodic Signals – cont.

- Continuing…

$$R_x(\tau) = \left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n c_m^* e^{j2\pi m f_o \tau}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{j2\pi (n-m) f_o t}dt\right)\right]$$

$$= \left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n c_m^* e^{j2\pi m f_o \tau}\delta(n-m)\right]$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 e^{j2\pi n f_o \tau}$$

- The Power Spectral Density can then be found using the Fourier Transform

PSD of Periodic Signals

- We can find the PSD using the Fourier Transform

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{j2\pi n f_o \tau}$$

$$S_x(f) = F\{R_x(\tau)\}$$

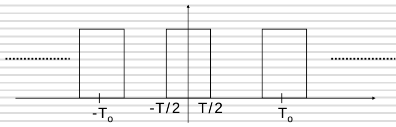
$$= F\left\{\sum_{n=-\infty}^{\infty} |c_n|^2 e^{j2\pi n f_o \tau}\right\}$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 F\left\{e^{j2\pi n f_o \tau}\right\}$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - n f_o)$$

Example 8.2

- Find the PSD of the pulse train



Example 8.2 (cont.)

- We know that the PSD of a periodic signal can be written as

$$S_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_o)$$

- Further, we can determine the Fourier Series coefficients as

$$c_n = \frac{1}{T_o}\int_{-T/2}^{T/2} e^{-j2\pi n f_o t}dt$$
$$= \frac{T}{T_o}\text{sinc}(nf_oT)$$

Example 8.2 (final)

- Thus, we have

$$S_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_o)$$
$$= \sum_{n=-\infty}^{\infty} \left|\frac{T}{T_o}\text{sinc}(nf_oT)\right|^2 \delta(f - nf_o)$$

Filtering Energy Signals

- We know from system theory that the Fourier Transform of the output of a system can be written as

$$Y(f) = H(f)X(f)$$

- The ESD can be written as

$$\begin{aligned}\psi_Y(f) &= |Y(f)|^2 \\ &= |H(f)X(f)|^2 \\ &= |H(f)|^2 |X(f)|^2 \\ &= |H(f)|^2 \psi_X(f)\end{aligned}$$

- Which tells us that the ESD at the output of a system is the ESD of the input multiplied by the magnitude of the transfer function squared

Filtering Power Signals

- The Fourier Transform of the output of a system when the input is periodic can be written as

$$\begin{aligned}Y(f) &= H(f)X(f) \\ &= H(f)\sum_{n=-\infty}^{\infty} c_n\delta(f-nf_o) \\ &= \sum_{n=-\infty}^{\infty} H(nf_o)c_n\delta(f-nf_o)\end{aligned}$$

- The PSD can be written as

$$\begin{aligned}S_Y(f) &= \sum_{n=-\infty}^{\infty} |H(nf_o)c_n|^2 \delta(f-nf_o) \\ &= \sum_{n=-\infty}^{\infty} |H(nf_o)|^2 |c_n|^2 \delta(f-nf_o)\end{aligned}$$

Summary

- In this lecture we have briefly reviewed the concepts of Energy Spectral Density and Power Spectral Density
- The ESD or PSD of the output of a LTI system can be readily found using the ESD or PSD of the input and the transfer function of the system
- The autocorrelation function and the power spectral density (energy spectral density) form a Fourier Transform pair for power (energy) signals